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Hille–Nehari type criteria and conditionally oscillatory half-linear differential equations

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Abstract. We study perturbations of the generalized conditionally oscillatory half-linear equation of the Riemann–Weber type. We formulate new oscillation and nonoscillation criteria for this equation and find a perturbation such that the perturbed Riemann–Weber type equation is conditionally oscillatory.

Keywords: half-linear differential equation, generalized Riemann–Weber equation, non(oscillation) criteria, perturbation principle.

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1 Introduction

In the paper we study oscillatory properties of the half-linear equation

$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1.1}$$

where the coefficients r, c are continuous functions, $r(t) > 0$ on the interval under consideration, which is a neighbourhood of infinity. In the special case when $p = 2$ this equation becomes the linear Sturm–Liouville equation. If $p \neq 2$, equation (1.1) is called half-linear since it has one half of the properties that characterize linearity: the solution space is homogeneous, but is generally not additive. Despite the missing additivity, the classical linear Sturmian theory has been extended to half-linear equations. We refer to the book [8] for the overview of the methods and results concerning half-linear equations up to year 2005. Concerning the recent results on half-linear differential equtions, see, e.g., [14–20] and the references therein.

Recall that equation (1.1) is said to be *oscillatory* if all its solutions are oscillatory, i.e., all the solutions have infinitely many zeros tending to infinity. In the opposite case equation (1.1) is said to be *nonoscillatory*. Note also that oscillatory and nonoscillatory solutions of (1.1) cannot coexist and this means that this equation is nonoscillatory if all solutions have constant sign eventually.

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Throughout this paper we suppose that equation (1.1) is nonoscillatory and we study the influence of perturbations of the coefficient c on the oscillatory behaviour of equation (1.1), i.e., we study equations of the form

$$(r(t)\Phi(x'))' + (c(t) + \tilde{c}(t))\Phi(x) = 0. \tag{1.2}$$

It is known from the Sturmian theory, that if the perturbation $\tilde{c}$ is "sufficiently positive", the equation becomes oscillatory, if it is "not too much positive", the equation remains nonoscillatory. If we find a positive function d and a constant λ_0 such that the equation

$$(r(t)\Phi(x'))' + (c(t) + \lambda d(t))\Phi(x) = 0 \tag{1.3}$$

is nonoscillatory for $\lambda < \lambda_0$ and oscillatory for $\lambda > \lambda_0$, we say that equation (1.3) is *conditionally oscillatory* with the *oscillation constant* λ_0. Examples of conditionally oscillatory equations (written below with the oscillation constants) are the Euler type equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p \tag{1.4}$$

and the perturbed Euler type equations, such as the Riemann–Weber type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right)\Phi(x) = 0, \quad \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \tag{1.5}$$

or equations with arbitrary number of perturbation terms of the form

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t}\right)\Phi(x) = 0, \tag{1.6}$$

where $n \in \mathbb{N}$, $\log_1 t = \log t$, $\log_k t = \log_{k-1}(\log t)$, $k \geq 2$, $\operatorname{Log}_j t = \prod_{k=1}^{j} \log_k t$. All these equations are nonoscillatory also in the critical case with the oscillation constants. The appropriate results concerning the Euler type equation and its perturbations in the coefficient $\frac{\gamma_p}{t^p}$ including the asymptotic formulas for nonoscillatory solutions of these equations can be found in the paper of Elbert and Schneider [11]. Note that the result of Elbert and Schneider has been generalized to the case when also perturbations in the term with derivative are allowed and also to the case of equations with non-constant coefficients, see, e.g., [4,6,7,14] and the references therein.

In this paper we study perturbations of general nonoscillatory equation (1.1). We suppose that h is a solution of (1.1) such that $h(t) > 0$ and $h'(t) \neq 0$, for $t \geq t_0$, where t_0 is a real number from the interval of consideration of (1.1). Moreover, we suppose that

$$\int^\infty R^{-1}(t)\,\mathrm{d}t = \infty \quad \text{and} \quad \liminf_{t\to\infty} |G(t)| > 0, \tag{1.7}$$

where

$$R(t) = r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) = r(t)h(t)\Phi(h'(t)). \tag{1.8}$$

Note that we follow the notation used in [10] and wherever we consider the integral $\int^\infty R^{-1}(t)\,\mathrm{d}t$, its lower limit is omitted, as it can be a constant greater or equal to t_0 such that all relevant conditions hold.

The motivation for our research comes from paper [10]. In that paper, under assumptions (1.7), it is shown that if $d(t) = \big(h^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2\big)^{-1}$ in (1.3), then (1.3) is conditionally oscillatory equation and the oscillation constant is $\frac{1}{2q}$, where q is a conjugate number to p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. It is also shown that the equation in the critical case with the oscillation constant $\frac{1}{2q}$

$$\hat{L}[x] := (r(t)\Phi(x'))' + \left[c(t) + \frac{1}{2qh^p(t)R(t)\left(\int^t R^{-1}(s)\,ds\right)^2}\right]\Phi(x) = 0 \tag{1.9}$$

is nonoscillatory and the asymptotic formula for one of solutions of (1.9) is established. Consequently, the perturbed equation

$$\tilde{L}[x] := (r(t)\Phi(x'))' + \left[c(t) + \frac{1}{2qh^p(t)R(t)\left(\int^t R^{-1}(s)\;ds\right)^2} + g(t)\right]\Phi(x) = 0 \tag{1.10}$$

is studied. In particular, a nonoscillation criterion of the Hille–Nehari type for (1.10), where limits inferior and superior of the expression

$$\log\left(\int^t R^{-1}(s)\,ds\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,d\tau\,ds$$

are compared with certain constants, is proved, see [10, Theorem 5]. The crucial role in the proof of this criterion plays the fact that the asymptotic formula for a solution of (1.9) is known.

The aim of our paper is to improve the above mentioned nonoscillation criterion for (1.10), to formulate a relevant oscillation criterion for (1.10) and to find a perturbation g in (1.10) such that (1.10) becomes conditionally oscillatory. We also formulate a version of a nonoscillatory Hille–Nehari type criterion for (1.10) in the case when we handle the asymptotic formula for the second solution of (1.9), which has been found recently in [3].

The paper is organized as follows. In the next section we formulate auxiliary results and technical lemmas which are important in our proofs. The main results, oscillation and nonoscillation criteria for (1.10), are presented in Section 3. The last section is devoted to remarks.

2 Auxiliary results

The proofs of our main results are based on the following theorems which can be found in [5] and [12]. For a positive and differentiable function $\tilde{x}$ denote

$$\tilde{R}(t) := r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}, \quad \tilde{G}(t) := r(t)\tilde{x}(t)\Phi(\tilde{x}'(t)). \tag{2.1}$$

Theorem A ([12, Theorem 3.2]). *Let $\tilde{x}$ be a function such that $\tilde{x}(t) > 0$ and $\tilde{x}'(t) \neq 0$, both for large t. Suppose that*

$$\int^\infty \tilde{x}(t)L[\tilde{x}](t)\,dt \quad \textit{is convergent and}$$

$$\lim_{t\to\infty} |\tilde{G}(t)| \int_T^t \frac{ds}{\tilde{R}(s)} = \infty, \tag{2.2}$$

where $T \in \mathbb{R}$ is sufficiently large. If

$$\limsup_{t\to\infty} \int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s \int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{2.3}$$

$$\liminf_{t\to\infty} \int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s \int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{2.4}$$

for some $\alpha > 0$, then (1.1) is nonoscillatory.

Theorem B ([5, Theorem 1]). *Let $\tilde{x}$ be a continuously differentiable function satisfying conditions*

$$\tilde{x}(t)L[\tilde{x}](t) \geq 0 \quad \textit{for large } t, \qquad \int^\infty \tilde{x}(t)L[\tilde{x}](t)\,dt < \infty, \tag{2.5}$$

$$\int^\infty \frac{dt}{\tilde{R}(t)} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \tilde{G}(t) = \infty. \tag{2.6}$$

If

$$\liminf_{t\to\infty} \int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s \int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\,\mathrm{d}s > \frac{1}{2q}, \tag{2.7}$$

where $T \in \mathbb{R}$ is sufficiently large, then (1.1) is oscillatory.

Theorem C ([12, Theorem 3.1]). *Let $\tilde{x}$ be a function such that $\tilde{x}(t) > 0$ and $\tilde{x}'(t) \neq 0$, both for large t. Suppose that the following conditions hold:*

$$\int^\infty \tilde{R}^{-1}(t)\ \mathrm{d}t < \infty, \quad \lim_{t\to\infty} |\tilde{G}(t)| \int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s = \infty. \tag{2.8}$$

If

$$\limsup_{t\to\infty} \int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s \int_T^t \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{2.9}$$

$$\liminf_{t\to\infty} \int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s \int_T^t \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{2.10}$$

for some $\alpha > 0$, $T \in \mathbb{R}$ sufficiently large, then (1.1) is nonoscillatory.

Theorem D ([5, Theorem 2]). *Let $\tilde{x}$ be a positive continuously differentiable function satisfying the following conditions:*

$$\tilde{x}(t)L[\tilde{x}](t) \geq 0 \ \textit{ for large } t, \qquad \int^\infty \tilde{x}(t)L[\tilde{x}](t)\ \mathrm{d}t = \infty, \tag{2.11}$$

$$\int^\infty \tilde{R}^{-1}(t)\ \mathrm{d}t = \infty \quad \textit{and} \quad \lim_{t\to\infty} \tilde{G}(t) = \infty. \tag{2.12}$$

Then (1.1) is oscillatory.

In the next lemma we collect some technical facts which are frequently used in the proofs of our main results.

Lemma 2.1. *Suppose that conditions (1.7) hold.*

(i) Let $j \in \mathbb{Z}$ be arbitrary and $k, l \in \mathbb{Z}$ be such that $k > 0$, $l \geq 0$. Then

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{G^l(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^k} = 0. \tag{2.13}$$

(ii) The integrals

$$\int_T^\infty \frac{G'(t)\log^j(\int^t R^{-1}(s)\,ds)}{G^2(t)\int^t R^{-1}(s)\,ds}, \quad \int_T^\infty \frac{\log^j(\int^t R^{-1}(s)\,ds)}{G(t)R(t)(\int^t R^{-1}(s)\,ds)^2} \tag{2.14}$$

are convergent for arbitrary $j \in \mathbb{Z}$, $T \in \mathbb{R}$ sufficiently large.

Proof. (i) Assumptions (1.7) imply that there exists a constant K such that $\frac{1}{|G(t)|} \le K$ for sufficiently large t and

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,ds)}{(\int^t R^{-1}(s)\,ds)^k} = 0 \tag{2.15}$$

which can be shown by L'Hospital's Rule as follows. If $j \le 0$, then (2.15) is evident. If $j > 0$, then we apply L'Hospital's Rule j times to obtain

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,ds)}{(\int^t R^{-1}(s)\,ds)^k} = \frac{j!}{k^j}\lim_{t\to\infty} \frac{1}{(\int^t R^{-1}(s)\,ds)^k} = 0.$$

Therefore also (2.13) holds.

(ii) The integrals are convergent by the comparison test for improper integrals. The first integral in (2.14) is convergent, because the integral

$$\int_T^\infty \frac{G'(t)}{G^2(t)}\,dt = \frac{1}{G(T)} - \lim_{t\to\infty}\frac{1}{G(t)}$$

is convergent and

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,ds)}{\int^t R^{-1}(s)\,ds} = 0.$$

Concerning the second integral in (2.14) we show that the integral

$$\int_T^\infty \frac{\log^j(\int^t R^{-1}(s)\,ds)}{R(t)(\int^t R^{-1}(s)\,ds)^2}\,dt \tag{2.16}$$

is convergent. If $j = 0$, the convergence follows immediately from (1.7), since in this case

$$\int_T^\infty \frac{1}{R(t)(\int^t R^{-1}(s)\,ds)^2}\,dt = \frac{1}{\int^T R^{-1}(s)\,ds} - \lim_{t\to\infty}\frac{1}{\int^t R^{-1}(s)\,ds}.$$

By induction, suppose that integral in (2.16) is convergent for a positive integer j and consider the case $j+1$. Using integration by parts we obtain

$$\begin{aligned}\int_T^\infty \frac{\log^{j+1}(\int^t R^{-1}(s)\,ds)}{R(t)(\int^t R^{-1}(s)\,ds)^2}\,dt &= \frac{\log^{j+1}(\int^T R^{-1}(s)\,ds)}{\int^T R^{-1}(s)\,ds} - \lim_{t\to\infty}\frac{\log^{j+1}(\int^t R^{-1}(s)\,ds)}{\int^t R^{-1}(s)\,ds} \\ &\quad + (j+1)\int_T^\infty \frac{\log^j(\int^t R^{-1}(s)\,ds)}{R(t)(\int^t R^{-1}(s)\,ds)^2}\,dt.\end{aligned}$$

This implies the convergence of the integral in (2.16) for any positive integer j. If j is negative, the convergence is evident. The convergence of the second integral in (2.14) follows then from the fact that $\frac{1}{|G(t)|}$ is bounded for large t. □

In the last part of this section we evaluate $\tilde{x}\hat{L}[\tilde{x}]$ from (1.9) for some particular functions $\tilde{x}$. The first of the following statements comes from [10]. The identity (2.17) follows from [10, Theorem 3], where we use the fact that $h'/h = G/R$ and fix the constant in the leading term. The convergence of the corresponding integral is shown in the proof of [10, Theorem 4]. It follows also from Lemma 2.1.

Lemma 2.2. *Let h be a positive solution of (1.1) such that $h'(t) \neq 0$ for large t and (1.7) holds. Set $\tilde{x}(t) := h(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^{\frac{1}{p}}$. Then*

$$\begin{aligned}\tilde{x}(t)\hat{L}[\tilde{x}](t) = {}& \frac{(p-2)(1-p)G'(t)}{2p^2G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}(1+o(1)) \\ &+ \frac{2(1-p)(p-2)}{3p^2G(t)R(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^2}(1+o(1))\end{aligned} \tag{2.17}$$

as $t \to \infty$ and the integral

$$\int^\infty \tilde{x}(t)\hat{L}[\tilde{x}](t)\,\mathrm{d}t$$

converges.

In the proofs of the following two statements we use the notation

$$\varphi(t) := \int^t R^{-1}(s)\,\mathrm{d}s. \tag{2.18}$$

Lemma 2.3. *Let h be a positive solution of (1.1) such that $h'(t) \neq 0$ for large t and (1.7) holds. Set $\tilde{x}(t) := h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\log^{\frac{1}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)$. Then*

$$\begin{aligned}&\tilde{x}(t)\hat{L}[\tilde{x}](t) + \frac{1}{2qR(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)\log\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)} \\ &\quad= \frac{(p-2)(1-p)G'(t)\log\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)}{2p^2G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}(1+o(1)) \\ &\qquad+ \frac{2(1-p)(p-2)\log\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)}{3p^2G(t)R(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^2}(1+o(1))\end{aligned} \tag{2.19}$$

as $t \to \infty$.

Proof. We use notation (2.18). By a direct computation and using the fact that $hG = h'R$ we obtain

$$\begin{aligned}\tilde{x}'(t) &= h'(t)\varphi^{\frac{1}{p}}(t)\log^{\frac{1}{p}}\varphi(t) + \frac{1}{p}h(t)R^{-1}(t)\varphi^{\frac{1}{p}-1}(t)\log^{\frac{1}{p}}\varphi(t) \\ &\quad+ \frac{1}{p}h(t)R^{-1}(t)\varphi^{\frac{1}{p}-1}(t)\log^{\frac{1}{p}-1}\varphi(t) \\ &= h'(t)\varphi^{\frac{1}{p}}(t)\log^{\frac{1}{p}}\varphi(t)\left[1 + \frac{h(t)}{ph'(t)R(t)\varphi(t)} + \frac{h(t)}{ph'(t)R(t)\varphi(t)\log\varphi(t)}\right] \\ &= h'(t)\varphi^{\frac{1}{p}}(t)\log^{\frac{1}{p}}\varphi(t)\left[1 + \frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}\right].\end{aligned} \tag{2.20}$$

Let us denote

$$A(t) := 1 + \frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}.$$

Then

$$r(t)\Phi(\tilde{x}'(t)) = r(t)\Phi(h'(t))\varphi^{\frac{p-1}{p}}(t)\big(\log\varphi(t)\big)^{\frac{p-1}{p}}A^{p-1}(t)$$

and hence,

$$\begin{aligned}\big(r(t)\Phi(\tilde{x}'(t))\big)' = {}& \big(r(t)\Phi(h'(t))\big)'\varphi^{\frac{p-1}{p}}(t)\big(\log\varphi(t)\big)^{\frac{p-1}{p}}A^{p-1}(t)\\ &+ \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)\varphi^{\frac{-1}{p}}(t)(\log\varphi(t))^{\frac{p-1}{p}}A^{p-1}(t)\\ &+ \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)\varphi^{\frac{-1}{p}}(t)\big(\log\varphi(t)\big)^{-\frac{1}{p}}A^{p-1}(t)\\ &+ (p-1)r(t)\Phi(h'(t))\varphi^{\frac{p-1}{p}}(t)\big(\log\varphi(t)\big)^{\frac{p-1}{p}}A^{p-2}(t)A'(t).\end{aligned}$$

Consequently,

$$\tilde{x}(t)\big(r(t)\Phi(\tilde{x}'(t))\big)' = h(t)A^{p-2}(t)B(t),$$

where

$$\begin{aligned}B(t) := {}& \big(r(t)\Phi(h'(t))\big)'\varphi(t)\log\varphi(t)A(t) + \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)\log\varphi(t)A(t)\\ &+ \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)A(t) + (p-1)r(t)\Phi(h'(t))\varphi(t)\log\varphi(t)A'(t).\end{aligned}$$

Next, for the derivative of $A(t)$ we have

$$\begin{aligned}A'(t) = {}& -\frac{G'(t)\varphi(t) + G(t)R^{-1}(t)}{pG^2(t)\varphi^2(t)}\\ &- \frac{G'(t)\varphi(t)\log\varphi(t) + G(t)R^{-1}(t)\log\varphi(t) + G(t)R^{-1}(t)}{pG^2(t)\varphi^2(t)\log^2\varphi(t)}\\ = {}& -\frac{G'(t)}{pG^2(t)\varphi(t)} - \frac{1}{pG(t)R(t)\varphi^2(t)} - \frac{G'(t)}{pG^2(t)\varphi(t)\log\varphi(t)}\\ &- \frac{1}{pG(t)R(t)\varphi^2(t)\log\varphi(t)} - \frac{1}{pG(t)R(t)\varphi^2(t)\log^2\varphi(t)},\end{aligned}$$

hence, substituting formulas for $A(t)$ and $A'(t)$ in $B(t)$, we obtain

$$\begin{aligned}B(t) = {}& (r\Phi(h'(t)))'\varphi(t)\log\varphi(t) + \frac{(r(t)\Phi(h'(t)))'\log\varphi(t)}{pG(t)} + \frac{(r(t)\Phi(h'(t)))'}{pG(t)}\\ &+ \frac{(p-1)r(t)\Phi(h'(t))\log\varphi(t)}{pR(t)} - \frac{(p-1)^2 r(t)\Phi(h'(t))\log\varphi(t)}{p^2G(t)R(t)\varphi(t)}\\ &- \frac{(p-1)(p-2)r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)} + \frac{(p-1)r\Phi(h')}{pR(t)} - \frac{(p-1)^2r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)\log\varphi(t)}\\ &- \frac{(p-1)r(t)\Phi(h'(t))G'(t)\log\varphi(t)}{pG^2(t)} - \frac{(p-1)r(t)\Phi(h'(t))G'(t)}{pG^2(t)}.\end{aligned}$$

Using the fact that $G' = h(r\Phi(h'))' + h'r\Phi(h')$ and $hG = h'R$, we simplify the previous formula as follows

$$\begin{aligned}
B(t) = {} & (r(t)(\Phi(h'(t)))'\varphi(t)\log\varphi(t) + \frac{(2-p)(r(t)(\Phi(h'(t)))'\log\varphi(t)}{pG(t)} \\
& + \frac{(2-p)(r(t)(\Phi(h'(t)))'}{pG(t)} - \frac{(p-1)^2 r(t)\Phi(h'(t))\log\varphi(t)}{p^2G(t)R(t)\varphi(t)} \\
& - \frac{(p-1)(p-2)r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)} - \frac{(p-1)^2 r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)\log\varphi(t)}.
\end{aligned}$$

To express $A^{p-2}(t)$ we use the power expansion

$$(1+x)^s = \sum_{j=0}^{\infty}\binom{s}{j}x^j, \quad |x|<1,\ s\in\mathbb{R} \tag{2.21}$$

with

$$x = \frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}.$$

Note that the applicability of this power expansion is guaranteed by conditions (1.7). Hence

$$\begin{aligned}
A^{p-2}(t) &= \sum_{j=0}^{\infty}\binom{p-2}{j}\left[\frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}\right]^j \\
&= 1 + \frac{p-2}{pG(t)\varphi(t)} + \frac{p-2}{pG(t)\varphi(t)\log\varphi(t)} \\
&\quad + \frac{(p-2)(p-3)}{2p^2G^2(t)\varphi^2(t)} + \frac{(p-2)(p-3)}{p^2G^2(t)\varphi^2(t)\log\varphi(t)} + \frac{(p-2)(p-3)}{2p^2G^2(t)\varphi^2(t)\log^2\varphi(t)} \\
&\quad + \frac{(p-2)(p-3)(p-4)}{6p^3G^3(t)\varphi^3(t)} + o\big(\varphi^{-3}(t)\big) \quad \text{as } t\to\infty.
\end{aligned}$$

By a direct computation we obtain

$$\begin{aligned}
A^{p-2}(t)B(t) = {} & (r(t)(\Phi(h'(t)))'\varphi(t)\log\varphi(t) \\
& + \frac{(p-2)(1-p)\log\varphi(t)}{2p^2G^2(t)\varphi(t)}(r(t)(\Phi(h'(t)))' - \frac{(p-1)^2\log\varphi(t)}{p^2G(t)R(t)\varphi(t)}r(t)\Phi(h'(t)) \\
& + \frac{(p-2)(1-p)}{p^2G^2(t)\varphi(t)}(r(t)(\Phi(h'(t)))' - \frac{(p-1)(p-2)}{p^2G(t)R(t)\varphi(t)}r(t)\Phi(h'(t)) \\
& + \frac{(p-2)(1-p)}{2p^2G^2(t)\varphi(t)\log\varphi(t)}(r(t)(\Phi(h'(t)))' - \frac{(p-1)^2}{p^2G(t)R(t)\varphi(t)\log\varphi(t)}r(t)\Phi(h'(t)) \\
& + \frac{(p-2)(1-p)(p-3)\log\varphi(t)}{3p^3G^3(t)\varphi^2(t)}(r(t)(\Phi(h'(t)))'(1+o(1)) \\
& - \frac{(p-1)^2(p-2)\log\varphi(t)}{p^3G(t)R(t)\varphi^2(t)}r(t)\Phi(h'(t))(1+o(1)) \quad \text{as } t\to\infty.
\end{aligned}$$

Now, using the identities

$$\frac{(r\Phi(h'))'}{G} = \frac{G'}{hG} - \frac{h'}{h^2} \quad \text{and} \quad \frac{r\Phi(h')}{R} = \frac{h'}{h^2},$$

which follow from the definitions of R, G in (1.8), we get

$$\begin{aligned}\tilde{x}(t)\big(r(t)\Phi(\tilde{x}'(t))\big)' &= h(t)A^{p-2}(t)B(t)\\ &= h(t)(r(t)\big(\Phi(h'(t))\big)'\varphi(t)\log\varphi(t)\\ &\quad+\frac{\log\varphi(t)}{G(t)\varphi(t)}\left[\frac{(p-2)(1-p)}{2p^2}\frac{G'(t)}{G(t)}-\frac{p-1}{2p}\frac{h'(t)}{h(t)}\right]\\ &\quad+\frac{1}{G(t)\varphi(t)}\left[\frac{(p-2)(1-p)}{p^2}\frac{G'(t)}{G(t)}\right]\\ &\quad+\frac{1}{G(t)\varphi(t)\log\varphi(t)}\left[\frac{(p-2)(1-p)}{2p^2}\frac{G'(t)}{G(t)}-\frac{p-1}{2p}\frac{h'(t)}{h(t)}\right]\\ &\quad+\frac{\log\varphi(t)}{G^2(t)\varphi^2(t)}\left[\frac{(p-2)(1-p)(p-3)}{3p^3}\frac{G'(t)}{G(t)}-\frac{2(p-1)(p-2)}{3p^2}\frac{h'(t)}{h(t)}\right]\\ &\quad+\frac{G'(t)}{G^2(t)}o\left(\frac{\log\varphi(t)}{\varphi^2(t)}\right)+\frac{h'(t)}{h(t)G^2(t)}o\left(\frac{\log\varphi(t)}{\varphi^2(t)}\right)\quad \text{as } t\to\infty.\end{aligned}$$

Finally, we have

$$\tilde{x}(t)\hat{L}[\tilde{x}](t)=\tilde{x}(t)\big(r(t)\Phi(\tilde{x}'(t))\big)'+c(t)h^p(t)\varphi(t)\log\varphi(t)+\frac{\log\varphi(t)}{2qR(t)\varphi(t)}.$$

Using the facts that h is a solution of (1.1), $h'/h=G/R$ and $q=p/(p-1)$, the last two formulas lead to (2.19). □

Lemma 2.4. *Let h be a positive solution of* (1.1) *such that $h'(t)\neq 0$ for large t and that* (1.7) *holds. Further let $\tilde{x}(t):=h(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{\frac{1}{p}}\log^{\frac{2}{p}}(\int^t R^{-1}(s)\,\mathrm{d}s)$. Then*

$$\begin{aligned}\tilde{x}\hat{L}[\tilde{x}]&=\frac{(p-2)(1-p)G'(t)\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}{2p^2G^2(t)(\int^t R^{-1}(s)\,\mathrm{d}s)}(1+o(1))\\ &\quad+\frac{2(p-2)(1-p)\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}{3p^2G(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}(1+o(1))\end{aligned}\tag{2.22}$$

as $t\to\infty$.

Proof. We use notation (2.18) and, suppressing the argument t, we proceed similarly as in the proof of Lemma 2.3. By a direct differentiation of $\tilde{x}$ and since $hG=h'R$, we obtain

$$\begin{aligned}\tilde{x}'&=h'\varphi^{\frac{1}{p}}\log^{\frac{2}{p}}\varphi+\frac{1}{p}hR^{-1}\varphi^{\frac{1}{p}-1}\log^{\frac{2}{p}}\varphi+\frac{2}{p}hR^{-1}\varphi^{\frac{1}{p}-1}\log^{\frac{2}{p}-1}\varphi\\ &=h'\varphi^{\frac{1}{p}}\log^{\frac{2}{p}}\varphi\left[1+\frac{1}{pG\varphi}+\frac{2}{pG\varphi\log\varphi}\right].\end{aligned}\tag{2.23}$$

Let us denote

$$\bar{A}:=1+\frac{1}{pG\varphi}+\frac{2}{pG\varphi\log\varphi}.$$

Then

$$r\Phi(\tilde{x}')=r\Phi(h')\varphi^{\frac{p-1}{p}}\log^{\frac{2p-2}{p}}\varphi\bar{A}^{p-1}$$

and its differentiation gives

$$(r\Phi(\tilde{x}'))' = (r\Phi(h'))'\varphi^{\frac{p-1}{p}} \log^{\frac{2p-2}{p}} \varphi \bar{A}^{p-1} + \frac{p-1}{p} r\Phi(h')R^{-1}\varphi^{\frac{-1}{p}} \log^{\frac{2p-2}{p}} \varphi \bar{A}^{p-1}$$
$$+ \frac{2p-2}{p} r\Phi(h')R^{-1}\varphi^{\frac{-1}{p}} \log^{1-\frac{2}{p}} \varphi \bar{A}^{p-1} + (p-1) r\Phi(h')\varphi^{\frac{p-1}{p}} \log^{\frac{2p-2}{p}} \varphi \bar{A}^{p-2}\bar{A}'$$
$$= \varphi^{-\frac{1}{p}} \log^{1-\frac{2}{p}} \varphi \bar{A}^{p-2} \Big\{ (r\Phi(h'))'\varphi \log\varphi \bar{A} + \frac{p-1}{p} r\Phi(h')R^{-1} \log\varphi \bar{A}$$
$$+ \frac{2(p-1)}{p} r\Phi(h')R^{-1}\bar{A} + (p-1) r\Phi(h')\varphi \log\varphi \bar{A}' \Big\},$$

where

$$\bar{A}' = -\frac{G'}{pG^2\varphi} - \frac{1}{pGR\varphi^2} - \frac{2G'}{pG^2\varphi\log\varphi} - \frac{2}{pGR\varphi^2\log\varphi} - \frac{2}{pGR\varphi^2\log^2\varphi}.$$

Denote the inside of the last curly brackets as $\bar{B}$. With the use of formulas for $\bar{A}$ and $\bar{A}'$ followed by the fact that $G' = h(r\Phi(h'))' + h' r\Phi(h')$ and $hG = h'R$ we get

$$\bar{B} = (r\Phi(h'))'\varphi\log\varphi + \frac{(r\Phi(h'))'\log\varphi}{pG} + \frac{2(r\Phi(h'))'}{pG} + \frac{(p-1)r\Phi(h')\log\varphi}{pR}$$
$$- \frac{(p-1)^2 r\Phi(h')\log\varphi}{p^2GR\varphi} + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi} + \frac{2(p-1)r\Phi(h')}{pR}$$
$$+ \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi\log\varphi} - \frac{(p-1)r\Phi(h')G'\log\varphi}{pG^2} - \frac{2(p-1)r\Phi(h')G'}{pG^2}$$
$$= (r(\Phi(h'))'\varphi\log\varphi + \frac{(2-p)(r(\Phi(h'))'\log\varphi}{pG} + \frac{2(2-p)(r(\Phi(h'))'}{pG}$$
$$- \frac{(p-1)^2 r\Phi(h')\log\varphi}{p^2GR\varphi} + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi} + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi\log\varphi}.$$

Next, since conditions (1.7) hold, we can use the power expansion (2.21) with

$$x = \frac{1}{pG\varphi} + \frac{2}{pG\varphi\log\varphi}$$

and we obtain

$$\bar{A}^{p-2} = 1 + \frac{p-2}{pG\varphi} + \frac{2(p-2)}{pG\varphi\log\varphi} + \frac{(p-2)(p-3)}{2p^2G^2\varphi^2} + \frac{2(p-2)(p-3)}{p^2G^2\varphi^2\log\varphi}$$
$$+ \frac{2(p-2)(p-3)}{p^2G^2\varphi^2\log^2\varphi} + \frac{(p-2)(p-3)(p-4)}{6p^3G^3\varphi^3} + o(\varphi^{-3}) \quad \text{as } t \to \infty.$$

Expanding $\bar{A}^{p-2}\bar{B}$ and joining the terms together with respect to φ yields

$$\bar{A}^{p-2}\bar{B} = (r\Phi(h'))'\varphi\log\varphi$$
$$+ \frac{\log\varphi}{G\varphi}\left(-\frac{(p-1)^2}{p^2}\frac{r\Phi(h')}{R} + \frac{(p-2)(1-p)}{2p^2}\frac{(r\Phi(h'))'}{G} \right)$$
$$+ \frac{1}{G\varphi}\left(\frac{2(p-1)(2-p)}{p^2}\frac{r\Phi(h')}{R} + \frac{2(p-2)(1-p)}{p^2}\frac{(r\Phi(h'))'}{G} \right)$$
$$+ \frac{1}{G\varphi\log\varphi}\left(\frac{2(p-1)(2-p)}{p^2}\frac{r\Phi(h')}{R} + \frac{2(p-2)(1-p)}{p^2}\frac{(r\Phi(h'))'}{G} \right)$$

$$
+\frac{\log\varphi}{G^2\varphi^2}\left(\frac{(p-1)^2(2-p)}{p^3}\frac{r\Phi(h')}{R}+\frac{(p-2)(1-p)(p-3)}{3p^2}\frac{(r\Phi(h'))'}{G}\right)
+\frac{r\Phi(h')}{R}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)+\frac{(r\Phi(h'))'}{G}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)\quad\text{as } t\to\infty.
$$

Using the identities

$$
\frac{r\Phi(h')}{R}=\frac{h'}{h^2},\qquad \frac{(r\Phi(h'))'}{G}=\frac{G'}{hG}-\frac{h'}{h^2},
$$

the above product $\bar{A}^{p-2}\bar{B}$ simplifies to

$$
\begin{aligned}
\bar{A}^{p-2}\bar{B}&=(r\Phi(h'))'\varphi\log+\frac{\log\varphi}{G\varphi}\left(-\frac{(p-1)}{2p}\frac{h'}{h^2}+\frac{(p-2)(1-p)}{2p^2}\frac{G'}{hG}\right)\\
&\quad+\frac{1}{G\varphi}\left(\frac{2(p-2)(1-p)}{p^2}\frac{G'}{hG}\right)+\frac{1}{G\varphi\log\varphi}\left(\frac{2(p-2)(1-p)}{p^2}\frac{G'}{hG}\right)\\
&\quad+\frac{\log\varphi}{G^2\varphi^2}\left(\frac{2(p-1)(2-p)}{3p^2}\frac{h'}{h^2}+\frac{(p-2)(1-p)(p-3)}{3p^2}\frac{G'}{hG}\right)\\
&\quad+\frac{h'}{h^2}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)+\frac{G'}{hG}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)\quad\text{as } t\to\infty.
\end{aligned}
$$

Altogether we have

$$
\begin{aligned}
\tilde{x}\hat{L}[\tilde{x}]&=\tilde{x}(r\Phi(\tilde{x}'))'+c\tilde{x}^p+\frac{\tilde{x}^p}{2qh^pR\varphi^2}\\
&=h\log\varphi\bar{A}^{p-2}\bar{B}+ch^p\varphi\log^2\varphi+\frac{\log^2\varphi}{2qR\varphi}.
\end{aligned}
$$

Since h solves the equation $(r\Phi(h'))'+c\Phi(h)=0$, $\frac{1}{q}=\frac{p-1}{p}$ and $\frac{1}{R}=\frac{h'}{Gh}$, we finally obtain

$$
\begin{aligned}
\tilde{x}\hat{L}[\tilde{x}]&=\frac{(p-2)(1-p)}{2p^2}\frac{G'\log^2\varphi}{G^2\varphi}+\frac{2(p-2)(1-p)}{p^2}\frac{G'\log\varphi}{G^2\varphi}\\
&\quad+\frac{2(p-2)(1-p)}{p^2}\frac{G'}{G^2\varphi}(1+o(1))+\frac{2(p-1)(2-p)}{3p^2}\frac{\log^2\varphi}{GR\varphi^2}(1+o(1)).
\end{aligned}
$$

as $t\to\infty$. This means that $\tilde{x}\hat{L}[\tilde{x}]$ can be written in the form (2.22). □

3 Oscillation and nonoscillation criteria for (1.10)

The following theorem is an improved version of [10, Theorem 5]. In contrast to that result, we do not need the condition

$$
\lim_{t\to\infty}\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)R(t)G'(t)=0
$$

considered in [10] and we have generalized the statement to $\alpha\neq\frac{1}{2}$.

Theorem 3.1. *Suppose that h is a positive solution of (1.1) such that $h'(t)\neq 0$ for large t, (1.7) holds and the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ converges. If*

$$
\limsup_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s<\frac{1}{q}(-\alpha+\sqrt{2\alpha}),\tag{3.1}
$$

$$
\liminf_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s>\frac{1}{q}(-\alpha-\sqrt{2\alpha})\tag{3.2}
$$

for some $\alpha>0$, then (1.10) is nonoscillatory.

Proof. The idea of the proof is to apply Theorem A to equation (1.10), i.e., $L := \tilde{L}$. We take $\tilde{x}(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}$. By a direct differentiation and using the fact that $h'R = hG$, we get

$$\tilde{x}'(t) = h'(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left[1 + \frac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right].$$

Now we express the functions $\tilde{R}$ and $\tilde{G}$ defined in (2.1) for this concrete $\tilde{x}$ and use (1.8) and (1.7) to obtain

$$\begin{aligned}\tilde{R}(t) &= r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}\\ &= r(t)h^2(t)|h'(t)|^{p-2}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\left[1 + \frac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right]^{p-2}\\ &= R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)(1+o(1)) \quad \text{as } t \to \infty\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\tilde{G}(t) &= r(t)\tilde{x}(t)\Phi(\tilde{x}'(t))\\ &= r(t)h(t)\Phi(h'(t))\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\left[1 + \frac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right]^{p-1}\\ &= G(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)(1+o(1)) \quad \text{as } t \to \infty.\end{aligned} \tag{3.4}$$

It follows from (3.3) that

$$\int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s = (1+o(1))\log\left(\int^t R^{-1}\right) - K, \quad K \in \mathbb{R},$$

hence conditions (1.7) and (3.4) imply that (2.2) is fulfilled.

Since

$$\tilde{x}(t)\tilde{L}[\tilde{x}](t) = \tilde{x}(t)\hat{L}[\tilde{x}](t) + g(t)|\tilde{x}(t)|^p = \tilde{x}(t)\hat{L}[\tilde{x}](t) + g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s,$$

Lemma 2.2 and the condition for the convergence of $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ guarantee that the integral $\int^\infty \tilde{x}(t)\tilde{L}[\tilde{x}](t)\,\mathrm{d}t$ is convergent and we have

$$\begin{aligned}&\int_T^t \tilde{R}^{-1}(s)\;\mathrm{d}s \int_t^\infty \tilde{x}(s)\tilde{L}[\tilde{x}](s)\;\mathrm{d}s\\ &\quad\sim \log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty\left(\tilde{x}(s)\hat{L}[\tilde{x}](s) + g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s\end{aligned} \tag{3.5}$$

as $t \to \infty$. Now we show that

$$\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \tilde{x}\hat{L}[\tilde{x}](s)\,\mathrm{d}s = 0. \tag{3.6}$$

By (2.17), it is sufficient to show that

$$\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \frac{1}{G(s)R(s)\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2}\,\mathrm{d}s = 0 \tag{3.7}$$

and

$$\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \frac{G'(s)}{G^2(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s = 0. \tag{3.8}$$

Since $\lim_{t\to\infty}\int_t^\infty \frac{1}{GR(\int^t R^{-1})^2}\,\mathrm{d}s = 0$, using L'Hospital's rule and (2.13) we have

$$\begin{aligned}
&\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \frac{1}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s\\
&\quad=\lim_{t\to\infty}\frac{-G^{-1}(t)R^{-1}(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-2}}{-\log^{-2}(\int^t R^{-1}(s)\,\mathrm{d}s)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-1}R^{-1}(t)}\\
&\quad=\lim_{t\to\infty}\frac{\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}=0,
\end{aligned}$$

hence (3.7) holds. To show (3.8), we use integration by parts

$$\int_t^\infty \frac{G'(s)}{G^2(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s=\frac{1}{G(t)\int^t R^{-1}(t)\,\mathrm{d}t}-\int_t^\infty\frac{1}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s,$$

which, together with (2.13) and (3.7), yields to (3.8). Hence (3.6) is proved. Consequently, by (3.5), we obtain

$$\begin{aligned}
&\int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s\int_t^\infty \tilde{x}(s)\tilde{L}[\tilde{x}](s)\ \mathrm{d}s\\
&\quad\sim\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}\tag{3.9}$$

as $t\to\infty$. This means that conditions (2.3), (2.4) follow from (3.1), (3.2). All the assumptions of Theorem A are fulfilled, hence (1.10) is nonoscillatory. □

The next statement is an oscillatory counterpart of Theorem 3.1.

Theorem 3.2. *Suppose that h is a positive solution of (1.1) such that $h'(t)\neq 0$ for large t, (1.7) holds, the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ converges and let there exist constants γ_1 and γ_2 such that*

$$g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\geq\frac{\gamma_1|G'(t)|}{G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}+\frac{\gamma_2}{|G(t)|R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\tag{3.10}$$

for large t, where

$$\gamma_1>\frac{(p-1)(p-2)}{2p^2}\operatorname{sgn}G',\qquad\gamma_2>\frac{2(p-1)(p-2)}{3p^2}\operatorname{sgn}G.\tag{3.11}$$

If

$$\liminf_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s>\frac{1}{2q}\tag{3.12}$$

then (1.10) is oscillatory.

Proof. We apply Theorem B with $L:=\tilde{L}$. Taking $\tilde{x}(t):=h(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{\frac{1}{p}}$ we obtain (3.3) and (3.4). Consequently, conditions (1.7) imply that both conditions in (2.6) are satisfied. Similarly to the proof of Theorem 3.1, we conclude that the second condition in (2.5) holds due to Lemma 2.2, since

$$\tilde{x}(t)\tilde{L}[\tilde{x}](t)=\tilde{x}(t)\hat{L}[\tilde{x}](t)+g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s$$

and condition (2.7) follows from (3.9) and (3.12). Concerning the first condition in (2.5), we have from Lemma 2.2 that

$$\begin{aligned}\tilde{x}(t)\tilde{L}[\tilde{x}](t) &= \frac{(p-2)(1-p)G'(t)}{2p^2G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}(1+o(1))\\ &\quad+\frac{2(1-p)(p-2)}{3p^2G(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}(1+o(1))\\ &\quad+g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\end{aligned}$$

as $t\to\infty$. Hence, the first condition in (2.5) is ensured by (3.10). Equation (1.10) is oscillatory by Theorem B. □

In the next theorem we handle equation (1.10) in the case, when the perturbation $g(t)$ is of the form

$$g(t) := \frac{\lambda}{h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}, \quad \lambda\in\mathbb{R}. \tag{3.13}$$

In this special case equation (1.10) becomes conditionally oscillatory.

Theorem 3.3. *Suppose that h is a positive solution of (1.1) such that $h'(t)\neq 0$ for large t and (1.7) holds and consider the equation*

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{1}{h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\left(\frac{1}{2q}+\frac{\lambda}{\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\right)\right]\Phi(x) = 0. \tag{3.14}$$

If $\lambda \le \frac{1}{2q}$, then (3.14) is nonoscillatory. If $\lambda > \frac{1}{2q}$ and there exists a constant γ such that

$$\frac{1}{R(t)\log^2(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)} \ge \frac{\gamma|G'(t)|}{G^2(t)}, \quad \gamma > \frac{p-2}{p}\operatorname{sgn} G'(t) \tag{3.15}$$

holds for large t, then (3.14) is oscillatory.

Proof. If $\lambda\neq\frac{1}{2q}$, then the statement follows from Theorem 3.1 and Theorem 3.2. Indeed, if $g(t)$ is given by (3.13), then

$$\begin{aligned}\int_T^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t &= \int_T^\infty \frac{\lambda}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\,\mathrm{d}t\\ &= \frac{\lambda}{\log(\int^T R^{-1}(s)\,\mathrm{d}s)} - \lim_{t\to\infty}\frac{\lambda}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)},\end{aligned}$$

so the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ is convergent. Consequently, concerning conditions (3.1), (3.2) and (3.12), we have

$$\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s = \lambda.$$

Hence, if $-\frac{3}{2q} < \lambda < \frac{1}{2q}$, then (3.14) is nonoscillatory by Theorem 3.1, where we take $\alpha = \frac{1}{2}$ in (3.1) and (3.2). If $\lambda \le -\frac{3}{2q}$, the nonoscillation of (3.14) follows form the well-known Sturm comparison theorem. If $\lambda > \frac{1}{2q}$, we use Theorem 3.2. It remains to show that condition (3.15) is sufficient for (3.10). Since $\lambda > \frac{1}{2q}$, there exists $\varepsilon > 0$ such that $\lambda = \frac{1}{2q} + \varepsilon$ and condition (3.10) with $g(t)$ defined in (3.13) can be written as

$$\frac{\frac{1}{2q} + \varepsilon}{R(t)\log^2(\int^t R^{-1}(s)\,ds)} \geq \frac{\gamma_1 |G'(t)|}{G^2(t)} + \frac{\gamma_2}{|G(t)|R(t)\int^t R^{-1}(s)\,ds},$$

where γ_1, γ_2 satisfy (3.11). If we set $\gamma_1 := \frac{\gamma}{2q}$, then

$$\gamma_1 > \frac{(p-2)\operatorname{sgn} G'(t)}{2pq} = \frac{(p-1)(p-2)}{2p^2}\operatorname{sgn} G'$$

and condition (3.15) implies that

$$\frac{1}{2qR(t)\log^2(\int^t R^{-1}(s)\,ds)} \geq \frac{\gamma_1|G'(t)|}{G^2(t)} \quad \text{for large } t.$$

Hence, it remains to show that

$$\frac{\varepsilon}{\log^2(\int^t R^{-1}(s)\,ds)} \geq \frac{\gamma_2}{|G(t)|\int^t R^{-1}(s)\,ds},$$

i.e.,

$$\varepsilon \geq \frac{\gamma_2 \log^2(\int^t R^{-1}(s)\,ds)}{|G(t)|\int^t R^{-1}(s)\,ds}$$

for large t. This inequality is satisfied for any $\gamma_2 \in \mathbb{R}$ since the limit of the function on the right-hand side of this inequality is zero as $t \to \infty$. Oscillation of (3.14) follows from Theorem 3.2.

In the remaining part of the proof we deal with the critical case $\lambda = \frac{1}{2q}$. We use Theorem A with $\tilde{x}(t) := h(t)\big(\int^t R^{-1}(s)\,ds\big)^{\frac{1}{p}} \log^{\frac{1}{p}}\big(\int^t R^{-1}(s)\,ds\big)$ and

$$L(t) := \hat{L}(t) + \frac{1}{2qh^p(t)R(t)\big(\int^t R^{-1}(s)\,ds\big)^2 \log^2\big(\int^t R^{-1}(s)\,ds\big)}\Phi(x(t)). \tag{3.16}$$

In this case we have from (2.20)

$$\begin{aligned}\tilde{x}'(t) &= h'(t)\big(\int^t R^{-1}(s)\,ds\big)^{\frac{1}{p}} \log^{\frac{1}{p}}\big(\int^t R^{-1}(s)\,ds\big)\\ &\quad\times\left[1 + \frac{1}{pG(t)\int^t R^{-1}(s)\,ds} + \frac{1}{pG(t)\int^t R^{-1}(s)\,ds \log\big(\int^t R^{-1}(s)\,ds\big)}\right],\end{aligned}$$

hence, according to (2.1) (suppressing the arguments) we have

$$\begin{aligned}\tilde{R} &= rh^2|h'|^{p-2}\big(\textstyle\int^t R^{-1}\big)\log\big(\int^t R^{-1}\big)\left[1 + \frac{1}{pG(\int^t R^{-1})} + \frac{2}{pG(\int^t R^{-1})\log\big(\int^t R^{-1}\big)}\right]^{p-2}\\ &= R\big(\textstyle\int^t R^{-1}\big)\log\big(\int^t R^{-1}\big)\big(1+o(1)\big) \quad \text{as } t \to \infty\end{aligned}$$

and

$$\begin{aligned}\tilde{G} &= rh\Phi(h')\big(\textstyle\int^t R^{-1}\big)\log\big(\int^t R^{-1}\big)\left[1 + \frac{1}{pG(\int^t R^{-1})} + \frac{2}{pG(\int^t R^{-1})\log\big(\int^t R^{-1}\big)}\right]^{p-1}\\ &= rh\Phi(h')\big(\textstyle\int^t R^{-1}\big)\log\big(\int^t R^{-1}\big)\big(1+o(1)\big) \quad \text{as } t \to \infty.\end{aligned} \tag{3.17}$$

These computations and conditions (1.7) imply that

$$\int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s \sim \log\left(\log \int^t R^{-1}(s)\,\mathrm{d}s\right) \quad \text{as } t \to \infty \tag{3.18}$$

and thus condition (2.2) is satisfied.

Next, since $\frac{p}{p-1} = q$, from (3.16) and Lemma 2.3 we obtain

$$\begin{aligned}\tilde{x}(t)L[\tilde{x}](t) &= \tilde{x}(t)\hat{L}[\tilde{x}](t) + \frac{1}{2qR(t)(\int^t R^{-1}(s)\,\mathrm{d}s)\log(\int^t R^{-1}(s)\,\mathrm{d}s)}\\ &= \frac{G'(t)\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G^2(t)\int^t (R^{-1}(s)\,\mathrm{d}s)}\left[\frac{(p-2)(1-p)}{2p^2} + o(1)\right]\\ &\quad + \frac{\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}\left[\frac{2(p-2)(1-p)}{3p^2} + o(1)\right]\end{aligned} \tag{3.19}$$

as $t \to \infty$. By Lemma 2.1 we have that $\int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t$ is convergent. Using L'Hospital's rule we obtain

$$\begin{aligned}&\lim_{t\to\infty} \log\left(\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)\int_t^\infty \frac{\log(\int^s (R^{-1}(\tau)\,\mathrm{d}\tau))}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s\\ &= \lim_{t\to\infty} \frac{-\log(\int^t R^{-1}(s)\,\mathrm{d}s)G^{-1}(t)R^{-1}(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-2}}{-\log^{-2}(\log(\int^t R^{-1}(s)\,\mathrm{d}s))\log^{-1}(\int^t R^{-1}(s)\,\mathrm{d}s)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-1}R^{-1}(t)}\\ &= \lim_{t\to\infty} \frac{\log^2(\log(\int^t R^{-1}(s)\,\mathrm{d}s))\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s} = 0.\end{aligned} \tag{3.20}$$

Integration by parts gives

$$\int_t^\infty \frac{G'(s)\log(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}{G^2(s)\int^s (R^{-1}(\tau)\,\mathrm{d}\tau)}\,\mathrm{d}s = \frac{\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \int_t^\infty \frac{1-\log(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s,$$

hence, by Lemma 2.1 and (3.20), we obtain

$$\lim_{t\to\infty} \log\left(\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)\int_t^\infty \frac{G'(s)\log(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}{G^2(s)\int^s (R^{-1}(\tau)\,\mathrm{d}\tau)}\,\mathrm{d}s = 0. \tag{3.21}$$

Consequently, (3.20) and (3.21) imply

$$\lim_{t\to\infty}\int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s\int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\,\mathrm{d}s = \lim_{t\to\infty}\log\left(\log\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\,\mathrm{d}s = 0.$$

Hence, conditions (2.3) and (2.4) are satisfied with $\alpha = \frac{1}{2}$ and this means that (3.14) with $\lambda = \frac{1}{2q}$ is nonoscillatory by Theorem A. □

The next result is a nonoscillatory criterion for (1.10) based on Theorem C.

Theorem 3.4. *Let h be a positive solution of (1.1) such that $h'(t) \neq 0$ for large t and (1.7) holds. If*

$$\limsup_{t\to\infty} \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)} < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{3.22}$$

$$\liminf_{t\to\infty} \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)} > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{3.23}$$

for some $\alpha > 0$, $T \in \mathbb{R}$ sufficiently large, then equation (1.10) is nonoscillatory.

Proof. Take $\tilde{x}(t) = h(t)(\int^t R^{-1}(s)\,ds)^{\frac{1}{p}} \log^{\frac{2}{p}}(\int^t R^{-1}(s)\,ds)$ and $L := \tilde{L}$ in Theorem C.

Using (2.1) and (2.23) we express (suppressing the arguments)

$$\begin{aligned}\tilde{R} &= rh^2|h'|^{p-2}\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)\left[1+\frac{1}{pG\left(\int^t R^{-1}\right)}+\frac{2}{pG\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)}\right]^{p-2}\\ &= R\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)(1+o(1)) \quad \text{as } t\to\infty\end{aligned}$$

and

$$\begin{aligned}\tilde{G} &= rh\Phi(h')\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)\left[1+\frac{1}{pG\left(\int^t R^{-1}\right)}+\frac{2}{pG\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)}\right]^{p-1}\\ &= G\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)(1+o(1)) \quad \text{as } t\to\infty.\end{aligned}$$

From these formulas we have that the integral $\int^\infty \tilde{R}^{-1}(t)\,dt$ is convergent since

$$\begin{aligned}\int_T^\infty \frac{1}{\tilde{R}(s)}\,ds &= \int_T^\infty \frac{R^{-1}(s)}{\int^s R^{-1}(\tau)\,d\tau \log^2(\int^s R^{-1}(\tau)\,d\tau)}(1+o(1))\,ds\\ &= \frac{1}{\log \int^T R^{-1}(t)\,dt}(1+o(1)) < \infty.\end{aligned}$$

Next, let us observe that

$$|\tilde{G}(t)|\int_t^\infty \frac{1}{\tilde{R}(s)}\,ds = |G(t)|\left(\int^t R^{-1}(s)\,ds\right)\log\left(\int^t R^{-1}(s)\,ds\right)(1+o(1)) \to \infty$$

as $t\to\infty$, hence conditions in (2.8) hold.

Further we are interested in the expression

$$\begin{aligned}&\int_t^\infty \tilde{R}^{-1}(s)\ ds \int_T^t \tilde{x}(s)L[\tilde{x}](s)\ ds\\ &\sim \frac{\int_T^t \left(\tilde{x}(s)\hat{L}[\tilde{x}](s) + g(s)h^p(s)(\int^s R^{-1}(\tau)\,d\tau)\log^2\left(\int^s R^{-1}(\tau)\,d\tau\right)\right)ds}{\log(\int^t R^{-1}(s)\,ds)}\end{aligned}$$

as $t\to\infty$. Since, by Lemma 2.1 and (2.22), the integral $\int_T^\infty \tilde{x}\hat{L}[\tilde{x}]\,ds$ is convergent, property (3.22) is sufficient for (2.9) and (3.23) is sufficient for (2.10). □

To formulate the oscillatory version of Theorem 3.3 we first prove the following oscillation criterion.

Theorem 3.5. *Suppose that there exist constants γ_1, γ_2 satisfying (3.11) such that*

$$\tilde{g}(t)h^p(t)\int^t R^{-1}(s)\,ds \geq \frac{\gamma_1|G'(t)|}{G^2(t)\int^t R^{-1}(s)\,ds} + \frac{\gamma_2}{|G(t)|R(t)(\int^t R^{-1}(s)\,ds)^2} \tag{3.24}$$

for large t. If

$$\int^\infty \tilde{g}(t)h^p(t)(\textstyle\int^t R^{-1}(s)\,ds)\log(\int^t R^{-1}(s)\,ds)\,dt = \infty, \tag{3.25}$$

then equation

$$\begin{aligned}\bar{L}[x] := (r(t)\Phi(x'))' + \Bigg(c(t) &+ \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2}\\ &+\frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2\log^2(\int^t R^{-1}(s)\,ds)} + \tilde{g}(t)\Bigg)\Phi(x) = 0.\end{aligned} \tag{3.26}$$

is oscillatory.

Proof. Let us take $\tilde{x}(t) := h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{1}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)$ and verify conditions (2.11) and (2.12) in Theorem D. Let us remark that $\tilde{x}$ is chosen to be the same as in the second part of the proof of Theorem 3.3. Condition (2.12) is a direct consequence of (3.17) and (3.18) together with (1.7).

Next, let us consider the operator $L[x]$ given by (3.16). According to the proof of Theorem 3.3, $\int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t$ converges. Since $\bar{L}[x] = L[x] + \tilde{g}(t)\Phi(x)$, we have

$$\begin{aligned}\int^\infty \tilde{x}(t)\bar{L}[\tilde{x}](t)\,\mathrm{d}t &= \int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t + \int^\infty \tilde{g}(t)|\tilde{x}(t)|^p\,\mathrm{d}t \\ &= \int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t + \int^\infty \tilde{g}(t)h^p(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty\end{aligned}$$

thanks to (3.25).

Finally, using (3.19), we see that

$$\begin{aligned}\tilde{x}(t)\bar{L}[\tilde{x}](t) = {}& \frac{G'(t)\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)}{G^2(t)\int^t (R^{-1}(s)\,\mathrm{d}s)}\left[\frac{(p-2)(1-p)}{2p^2} + o(1)\right] \\ &+ \frac{\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)}{G(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}\left[\frac{2(p-2)(1-p)}{3p^2} + o(1)\right] \\ &+ \tilde{g}(t)h^p(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\end{aligned}$$

which is, provided (3.11) and (3.24), nonnegative for large t. Hence both the parts of (2.11) hold and the statement follows from Theorem D. □

The oscillatory counterpart of Theorem 3.3 reads as follows. Here, equation (1.10) is seen as a perturbation of (3.14) with $\lambda = \frac{1}{2q}$, i.e., (1.10) is considered as an equation of the form (3.26).

Theorem 3.6. *Suppose that there exist constants γ_1, γ_2 satisfying (3.11) such that*

$$\begin{aligned}&\left(g(t) - \frac{1}{2qh^p(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)}\right)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s \\ &\quad\geq \frac{\gamma_1|G'(t)|}{G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{\gamma_2}{|G(t)|R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}\end{aligned} \tag{3.27}$$

for large t. If

$$\liminf_{t\to\infty} \frac{\int_T^t g(s)h^p(s)\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s}{\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)} > \frac{1}{2q}, \tag{3.28}$$

then equation (1.10) is oscillatory.

Proof. Equation (1.10) can be rewritten in the form of (3.26) with

$$\tilde{g}(t) = g(t) - \frac{1}{2qh^p(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)},$$

on which we apply Theorem 3.5. Condition (3.24) follows from (3.27). Next we show that (3.25) holds. From (3.28) we have that there exists $\varepsilon > 0$ and $\tilde{T} > T$ such that

$$\frac{\int_T^t g(s)h^p(s)\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s}{\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)} > \frac{1}{2q} + \varepsilon, \quad t > \tilde{T}. \tag{3.29}$$

Let $b > \tilde{T}$, then (suppressing some unnecessary arguments)

$$
\begin{aligned}
I &:= \int_T^b \left(g(t) - \frac{1}{2qh^p(t)R(\int^t R^{-1})^2 \log^2(\int^t R^{-1})} \right) h^p(t)(\textstyle\int^t R^{-1}) \log(\int^t R^{-1})\, \mathrm{d}t \\
&= \int_T^b g(t)h^p(t)(\textstyle\int^t R^{-1}) \log(\int^t R^{-1})\, \mathrm{d}t - \int_T^b \frac{1}{2qR(\int^t R^{-1}) \log(\int^t R^{-1})}\, \mathrm{d}t \\
&= \int_T^b \frac{1}{\log(\int^t R^{-1})} g(t)h^p(t)(\textstyle\int^t R^{-1}) \log^2(\int^t R^{-1})\, \mathrm{d}t - \int_T^b \frac{1}{2qR(\int^t R^{-1}) \log(\int^t R^{-1})}\, \mathrm{d}t.
\end{aligned}
$$

With the use of integration by parts and the notation

$$
K_1 := \int_T^{\tilde{T}} \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}) \log^2(\int^s R^{-1})\, \mathrm{d}s}{R(\int^t R^{-1}) \log^2(\int^t R^{-1})}\, \mathrm{d}t
$$

we have

$$
\begin{aligned}
I = &\left[\frac{1}{\log(\int^t R^{-1})} \int_T^t g(s)h^p(s)(\textstyle\int^s R^{-1}) \log^2(\int^s R^{-1})\, \mathrm{d}s \right]_T^b + K_1 \\
&+ \int_{\tilde{T}}^b \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}) \log^2(\int^s R^{-1})\, \mathrm{d}s}{R(\int^t R^{-1}) \log^2(\int^t R^{-1})}\, \mathrm{d}t - \frac{1}{2q} \left[\log(\log(\textstyle\int^t R^{-1})) \right]_T^b .
\end{aligned}
$$

With respect to (3.29), we can estimate:

$$
\begin{aligned}
I \geq &\frac{1}{\log(\int^b R^{-1})} \int_T^b g(t)h^p(t)(\textstyle\int^t R^{-1}) \log^2(\int^t R^{-1})\, \mathrm{d}t + K_1 \\
&+ \int_{\tilde{T}}^b \frac{\frac{1}{2q} + \varepsilon}{R(\int^t R^{-1}) \log^2(\int^t R^{-1})}\, \mathrm{d}t - \frac{1}{2q} \left[\log(\log(\textstyle\int^t R^{-1})) \right]_T^b \geq \frac{1}{2q} + \varepsilon + K_1 \\
&+ \left(\frac{1}{2q} + \varepsilon \right) \left[\log(\log(\textstyle\int^t R^{-1})) \right]_{\tilde{T}}^b - \frac{1}{2q} \log(\log(\int^b R^{-1})) + \frac{1}{2q} \log(\log(\int^T R^{-1})) \\
= &\frac{1}{2q} + \varepsilon + K_1 + \varepsilon \log(\log(\textstyle\int^b R^{-1})) + K_2,
\end{aligned}
$$

where $K_2 = -\left(\frac{1}{2q} + \varepsilon\right) \log(\log(\int^{\tilde{T}} R^{-1})) + \frac{1}{2q} \log(\log(\int^T R^{-1}))$ is constant and therefore integral I tends to infinity as $b \to \infty$. □

4 Remarks

Remark 4.1. Let us consider the nonoscillatory Euler type equation with the oscillation constant (1.4). It is known that the function $t^{\frac{p-1}{p}}$ is a solution of this equation. To show how our results apply to perturbations of (1.4), consider the interval $[1, \infty)$ and take the solution

$$
h(t) := \left(\frac{p}{p-1} \right)^{\frac{p-2}{p}} t^{\frac{p-1}{p}}.
$$

Then $R(t) = t$, $G = \frac{p-1}{p}$ and $\int_1^t R^{-1}(s)\, \mathrm{d}s = \log t$, hence conditions (1.7) hold. Consequently,

$$
\frac{1}{2qh^p(t)R(t)\left(\int_1^t R^{-1}(s)\, \mathrm{d}s\right)^2} = \frac{\mu_p}{t^p \log^2 t}
$$

and this means that equation (1.9) becomes the Riemann–Weber type equation (1.5) and results of the previous section reduce to those for the perturbed Riemann–Weber type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + g(t)\right)\Phi(x) = 0. \tag{4.1}$$

In particular, Theorem 3.1 with $\alpha = \frac{1}{2}$ reduces to [1, Corollary 2], see also [2, Theorem 3.3] in the case $n = 1$. Theorem 3.2 is a generalized version of [9, Corollary 1] and also of [2, Theorem 3.3] with $n = 1$. Note that, since $G' = 0$, condition (3.10) simplifies to

$$g(t)t^p \log^3 t \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2}, \quad \gamma := \gamma_2 \left(\frac{p-1}{p}\right)^{p-3},$$

which is condition (3.15) from [2]. Concerning Theorem 3.3, observe that condition (3.15) is satisfied and equation (3.14) with $\lambda = \frac{1}{2q}$ is equation (1.6) with $n = 2$. Hence, Theorem 3.3 generalizes results of [11] for $n = 2$. Finally, Theorem 3.4 applied to (4.1) is [13, Theorem 3.1] in the case $n = 1$.

Remark 4.2. Based on the results of this paper and their comparison with those for the perturbed Euler type equation discussed in Remark 4.1, we suppose that we can study perturbations of equation (3.14) with $\lambda = \frac{1}{2q}$ and find a perturbation such that the obtained perturbed equation is conditionally oscillatory. More generally, we conjecture that the equation with arbitrary number of iterated logarithmic terms

$$(\Phi(x'))' + \left(c(t) + \sum_{j=0}^{n} \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2 \operatorname{Log}_j^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\right)\Phi(x) = 0 \tag{4.2}$$

is conditionally oscillatory (here $\operatorname{Log}_0 t := 1$). This would generalize the result of [11] concerning equation (1.6) and give us the possibility to generalize the oscillation and nonoscillation criteria of this paper to the case when we study perturbations of (4.2), similarly as in [2, 13], where perturbations of (1.6) are studied.

Remark 4.3. Let us comment the particular choice of the functions $\tilde{x}$ in the proofs of our results. Consider the operators $\hat{L}$ and $\tilde{L}$ defined in (1.9) and (1.10), respectively. If $\tilde{x}$ is a solution of (1.9), then $\tilde{x}(t)\tilde{L}[\tilde{x}](t) = g(t)\tilde{x}^p(t)$ and hence, when applying one of the Theorems A, B, C to equation (1.10), the expression $\tilde{x}(s)\tilde{L}[\tilde{x}](s)$ appearing in conditions (2.3), (2.4), (2.7), (2.9), (2.10) is replaced by $g(s)\tilde{x}^p(s)$. It has been shown in [3, 10] that equation (1.9) has a pair of linearly independent solutions that are asymptotically close (as $t \to \infty$) to the functions

$$x_1(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}},$$

$$x_2(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{2}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right).$$

We have taken $\tilde{x} := x_1$ in the proofs of Theorem 3.1 and Theorem 3.2 and $\tilde{x} := x_2$ in the proof of Theorem 3.4. Computations in proofs of these theorems together with Lemma 2.2 and Lemma 2.4 show that in both the cases $\tilde{x} := x_1$ and $\tilde{x} := x_2$, the expression $\tilde{x}\hat{L}[\tilde{x}]$ is small enough such that it does not have an influence on the limits superior and inferior in conditions

(2.3), (2.4), (2.7), (2.9), (2.10), hence the expression $g(s)x_1^p(s)$ appears in (3.1), (3.2) and (3.12), and $g(s)x_2^p(s)$ appears in (3.22) and (3.23).

In the proofs of Theorem 3.3 and Theorem 3.5 we have taken $\tilde{x} := h(\int^t R^{-1})^{\frac{1}{p}} \log^{\frac{1}{p}}(\int^t R^{-1})$, since we conjecture that this function is asymptotically close to one of the solutions of (3.14) with $\lambda = \frac{1}{2q}$. This conjecture is supported by Lemma 2.3 (observe that the left-hand side of identity (2.19) is equal to $\tilde{x}(t)\tilde{L}[\tilde{x}](t))$ and by the asymptotic formulas for equation (1.6) derived in [11].

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