Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **91**, 1–32; https://doi.org/10.14232/ejqtde.2019.1.91 www.math.u-szeged.hu/ejqtde/

C^1-smooth dependence on initial conditions and delay: spaces of initial histories of Sobolev type, and differentiability of translation in L^p

Junya Nishiguchi ✉

Mathematical Science Group, Advanced Institute for Materials Research (AIMR), Tohoku University, 2-1-1 Katahira, Aoba-ku, Sendai, 980-8577, Japan

Received 14 July 2019, appeared 11 December 2019
Communicated by Hans-Otto Walther

Abstract. The objective of this paper is to clarify the relationship between the C^1-smooth dependence of solutions to delay differential equations (DDEs) on initial histories (i.e., initial conditions) and delay parameters. For this purpose, we consider a class of DDEs which include a constant discrete delay. The problem of C^1-smooth dependence is fundamental from the viewpoint of the theory of differential equations. However, the above mentioned relationship is not obvious because the corresponding functional differential equations have the less regularity with respect to the delay parameter. In this paper, we prove that the C^1-smooth dependence on initial histories and delay holds by adopting spaces of initial histories of Sobolev type, where the differentiability of translation in L^p plays an important role.

Keywords: delay differential equations, constant discrete delay, smooth dependence on delay, history spaces of Sobolev type, differentiability of translation in L^p.

2010 Mathematics Subject Classification: 34K05, 46E35.

1 Introduction

Differential equations with constant discrete delays are used for mathematical models of various dynamic phenomena (e.g., see [8, Section 21], [18, Chapter 2], and [9]). In many cases, the precise values of delays are unknown. Therefore, it is important to study how the solutions behave as functions of delay parameters in order to investigate the validity of such mathematical models. This is known as the delay parameter identification problem (e.g., see [13] and [2]), where it is necessary to differentiate solutions to delay differential equations (DDEs) with respect to delay parameters. Indeed, the above mentioned differentiability problem is fundamental from the viewpoint of the theory of differential equations. However, the smoothness of the corresponding retarded functional differential equations (RFDEs) is closely related to the regularity of initial histories. Therefore, it is not obvious which spaces of initial histories

✉ Email: junya.nishiguchi.b1@tohoku.ac.jp

(called *history spaces* in this paper) should be chosen in order to obtain such differentiability or, in other words, the C^1-smooth dependence on delay.

The objective of this paper is to clarify the connection between the C^1-smooth dependence on initial histories and delay and the regularity of initial histories. For this purpose, we consider a DDE

$$\dot{x}(t) = f(x(t), x(t-r)) \tag{1.1}$$

and its initial value problem (IVP)

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-r)), & t \geq 0, \\ x(t) = \phi(t), & t \in [-R, 0] \end{cases} \tag{1.2}$$

for each $(\phi, r) \in C([-R, 0], \mathbb{R}^N) \times [0, R]$. Here $R > 0$ is the maximal delay which is constant, $r \in [0, R]$ is the delay parameter, $N \geq 1$ is an integer, and $f \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a function. $C([-R, 0], \mathbb{R}^N)$ denotes the Banach space of continuous functions from $[-R, 0]$ to $\mathbb{R}^N$ with the supremum norm

$$\|\phi\|_{C[-R,0]} := \sup_{\theta \in [-R,0]} |\phi(\theta)|,$$

where $|\cdot|$ is a norm on $\mathbb{R}^N$. Under the local Lipschitz continuity of f, (1.2) has the unique maximal solution

$$x(\cdot; \phi, r) \colon [-R, T_{\phi,r}) \to \mathbb{R}^N$$

for $0 < T_{\phi,r} \leq \infty$. We refer the reader to [12] as a general reference of the theory of RFDEs. Then the problem of the C^1-smooth dependence on initial histories and delay which will be studied in this paper is the continuous differentiability of

$$(\phi, r) \mapsto x(\cdot; \phi, r)$$

in an appropriate sense.

The difficulty about the C^1-smooth dependence on delay is the less smoothness of the corresponding functional F (called *history functional* in this paper) given by

$$F(\phi, r) := f(\phi(0), \phi(-r)) \tag{1.3}$$

with respect to the delay parameter r. In fact, the function $r \mapsto F(\phi, r)$ is not differentiable for general $\phi \in C([-R, 0], \mathbb{R}^N)$ even if the function $f \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is smooth. This phenomenon is similar to the lack of smoothness for history functionals corresponding to state-dependent DDEs (see [25]). We refer the reader to [16] as a reference of the theory of state-dependent DDEs.

It is natural to consider initial histories with better regularity in order to obtain the smooth dependence on initial histories and delay. The method of consideration in [25] is to adopt the Banach space $C^1([-R, 0], \mathbb{R}^N)$ of continuously differentiable functions from $[-R, 0]$ to $\mathbb{R}^N$ with the C^1-norm

$$\|\phi\|_{C^1[-R,0]} := \|\phi\|_{C[-R,0]} + \|\phi'\|_{C[-R,0]}$$

as a history space. Then the compatibility condition given by

$$\phi'(0) = f(\phi(0), \phi(-r))$$

for every initial history ϕ is necessary to keep the histories of solution of class C^1, and therefore, the solution manifold defined by

$$X_{f,r} = \{\, \phi \in C^1([-R, 0], \mathbb{R}^N) : \phi'(0) = f(\phi(0), \phi(-r)) \,\}$$

arises as the set of initial histories. However, the framework of the solution manifold is not suitable for the C^1-smooth dependence on delay because $X_{f,r}$ depends on r.

The first study of the C^1-smooth dependence on initial histories and delay seems to be done by Hale & Ladeira [11]. Their idea is to use the history space $C^{0,1}([-R,0],\mathbb{R}^N)$ endowed with the $\mathcal{W}^{1,1}$-norm. Here $C^{0,1}([-R,0],\mathbb{R}^N)$ denotes the set of Lipschitz continuous functions from $[-R,0]$ to $\mathbb{R}^N$, and $\mathcal{W}^{1,p}$-norm for $1 \le p < \infty$ is defined as follows for absolutely continuous functions:

$$\|\phi\|_{\mathcal{W}^{1,p}[-R,0]} := \left(|\phi(-R)|^p + \int_{-R}^{0} |\phi'(\theta)|^p \, \mathrm{d}\theta \right)^{\frac{1}{p}}$$

with the almost everywhere derivative ϕ' of ϕ. The contribution in [11] is the adoption of the Lipschitz continuous regularity for the C^1-smooth dependence on delay. In this case, the C^1-smooth dependence on delay is not trivial because the history functional given in (1.3) is not differentiable with respect to r for general $\phi \in C^{0,1}([-R,0],\mathbb{R}^N)$. It should be noticed that the differentiability of $r \mapsto x(\cdot;\phi,r)$ at $r = 0$ is not discussed in [11]. The continuous differentiability of

$$r \mapsto x(t;\phi,r) \in \mathbb{R}^N$$

for the time-dependent delay function $r = r(\cdot)$ is studied by Hartung [15] by assuming $\phi \in C^{0,1}([-R,0],\mathbb{R}^N)$, where the positivity $r(t) > 0$ is also assumed.

The method of the proof of the C^1-smooth dependence on initial histories and delay given in [11] is the fixed point argument, which is standard in the literature (ref. [12]). That is, IVP (1.2) is converted to the fixed point problem through the integral equation. Then the C^1-smooth dependence on initial histories and delay is obtained from the C^1-uniform contraction theorem (e.g., see [6, Theorem 2.2 in Chapter 2]), where history and delay are parameters. However, the history space

$$\left(C^{0,1}([-R,0],\mathbb{R}^N), \|\cdot\|_{\mathcal{W}^{1,1}[-R,0]} \right)$$

is not a Banach space but a *quasi-Banach space* in their terminology. Therefore, the usual C^1-uniform contraction theorem cannot be applicable, and it is necessary to invent the C^1-uniform contraction theorem for such quasi-Banach spaces ([11, Theorem 2.7]). It should be noticed that the Banach space $C^{0,1}([-R,0],\mathbb{R}^N)$ endowed with the $C^{0,1}$-norm

$$\|\phi\|_{C^{0,1}[-R,0]} := \max\{\|\phi\|_{C[-R,0]}, \mathrm{lip}(\phi)\},$$

where $\mathrm{lip}(\phi)$ is the Lipschitz constant of ϕ, is not suitable for a history space (see [20]).

Hale & Ladeira [11] gives an insight into the C^1-smooth dependence problem as mentioned above. However, the following questions which are related each other should arise:

- What is the essentiality of the Lipschitz continuous regularity for the C^1-smooth dependence on initial histories and delay?
- What happens if $\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ ($1 \le p < \infty$) is chosen as a history space?

Here $\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$, which will be called a *history space of Sobolev type* in this paper, is the linear space of absolutely continuous functions from $[-R,0]$ to $\mathbb{R}^N$ whose almost everywhere derivatives belong to $L^p([-R,0],\mathbb{R}^N)$ endowed with the $\mathcal{W}^{1,p}$-norm. When $p = 2$ and the norm $|\cdot|$ on $\mathbb{R}^N$ is the Euclidean norm, $\mathcal{W}^{1,2}([-R,0],\mathbb{R}^N)$ becomes a Hilbert space. This is an advantageous fact for numerical analysis.

In this paper, we show that $\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ can be chosen as a history space for the C^1-smooth dependence on initial histories and delay. It becomes clear that the *differentiability of translation in* L^p plays an important role in the proof. The method of the proof is standard but does not require the Lipschitz continuity of initial histories, which in fact make the proof simple. This is the reason why $\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ is appropriate and gives answer to the above questions. We also prove that the solution semiflow with a delay parameter, which is the solution semiflow generated by the IVPs of the extended system

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - r(t))), \\ \dot{r}(t) = 0, \end{cases} \tag{1.4}$$

is a C^1-maximal semiflow. We note that the extended system (1.4) is a special case of the following coupled system of DDE and ODE (see [1] and [4])

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - r(t))), \\ \dot{r}(t) = g(x(t), r(t)), \end{cases}$$

where $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a function. The extended system (1.4) also appears in bifurcation problems (ref. [19]).

Finally, we give another several comments about previous studies. (i) In [11], the function f is required to be of class C^2 for the C^1-smooth dependence on initial histories and delay. The results which will be given in this paper require that f is of class C^1 for such C^1-smooth dependence, which is same as [15]. (ii) It is mentioned in [11, Section 4] that similar results hold with the same proofs when the delay is time-dependent. However, this is incorrect because a simple counter example can be given as follows: We consider the function $f(x,y) = y$. Let $0 < T < R$. For each $c \in [0, R-T]$, we define $r_c \in C(\mathbb{R}, [0,R])$ by

$$r_c(t) = \begin{cases} c & (t \le 0), \\ t + c & (0 \le t \le T), \\ T + c & (t \ge T). \end{cases}$$

Then it can be shown that $[0, R-T] \ni c \mapsto r_c \in C(\mathbb{R}, [0,R])$ is differentiable but the solution

$$x(t; \phi, r_c) = \phi(0) + \int_0^t \phi(s - r_c(s))\, \mathrm{d}s = \phi(0) + t\phi(-c) \quad (\forall t \in [0,T])$$

is not differentiable with respect to c for general $\phi \in C^{0,1}([-R,0],\mathbb{R}^N)$. This example can be considered to be a critical case in the sense that the *delayed argument function*

$$t \mapsto t - r_c(t)$$

is constant. In [15], the C^1-smooth dependence on time-dependent delay with the Lipschitz continuous regularity of initial histories is studied under some strict monotonicity condition of the delayed argument function. See also [17] for state-dependent DDEs. (iii) In [2], the authors study the C^1-smoothness of the function

$$(0,R] \ni r \mapsto x(t; \phi, r) \in \mathbb{R}^N$$

without citing the previous studies. It seems that the argument relies on the differentiability of translation in L^2 assuming initial histories belong to $H^{1,\infty}$, however, the proof of the differentiability and the definition of $H^{1,\infty}$ are not given. The assumption given in [2] is also more stronger, namely, the boundedness of the norm of the Fréchet derivative of f is assumed.

This paper is organized as follows. In Section 2, we define history spaces of Sobolev type and investigate their fundamental properties. Section 3 are divided into two parts: a simple case (Subsection 3.1) and a general case (Subsection 3.2). In Subsection 3.1, we concentrate our consideration on a DDE

$$\dot{x}(t) = f(x(t-r)) \tag{1.5}$$

and its IVP

$$\begin{cases} \dot{x}(t) = f(x(t-r)), & t \geq 0, \\ x(t) = \phi(t), & t \in [-R, 0] \end{cases} \tag{1.6}$$

for each $(\phi, r) \in C([-R, 0], \mathbb{R}^N) \times (0, R]$. Then the problem of the C^1-smooth dependence on delay is very simplified, and the result directly follows by the continuity and differentiability of translation in L^p. In Subsection 3.2, we consider a general class of DDEs of the form given in (1.1). Here we prove the main results of this paper, which consist of the C^1-smooth dependence on initial histories and delay (Theorem 3.15) and the C^1-smoothness of the solution semiflow with a delay parameter (Theorem 3.18). As mentioned above, the differentiability of translation in L^p plays an important role in the proof.

We have two appendices. In Appendix A, we give a proof of this differentiability result (Corollary A.4) together with the discussion about the estimate of the double integral for the translation of L^p-functions (Corollary A.2). The latter is also used in the proof of the C^1-smooth dependence result. We give the proof and some fundamental properties about Fréchet differentiability to keep this paper self-contained. In Appendix B, we give definitions about maximal semiflows and prove the theorem (Theorem B.13) which ensures that a maximal semiflow is of class C^1.

2 Preliminary: History spaces of Sobolev type

Let $R > 0$ be a constant and $N \geq 1$ be an integer. The linear space of all functions from $[-R, 0]$ to $\mathbb{R}^N$ is denoted by $\mathrm{Map}([-R, 0], \mathbb{R}^N)$. Let $\mathbb{R}_+$ denote the set of all nonnegative real numbers.

Definition 2.1 (History)**.** Let $R > 0$ be a given constant. Let $a < b$ be real numbers so that $a + R < b$ and $\gamma \colon [a, b] \to \mathbb{R}^N$ be a function. For every $t \in [a + R, b]$, the function $R_t\gamma \in \mathrm{Map}([-R, 0], \mathbb{R}^N)$ defined by

$$R_t\gamma \colon [-R, 0] \ni \theta \mapsto \gamma(t + \theta) \in \mathbb{R}^N$$

is called the *history* of γ at t.

Definition 2.2 (History space)**.** A linear subspace $H \subset \mathrm{Map}([-R, 0], \mathbb{R}^N)$ is called a *history space* with the past interval $[-R, 0]$ if the topology of H is given so that the linear operations on H are continuous.

Definition 2.3 (Static prolongation)**.** For each $\phi \in \mathrm{Map}([-R, 0], \mathbb{R}^N)$, the function $\bar{\phi} \colon [-R, +\infty) \to \mathbb{R}^N$ defined by

$$\bar{\phi}(t) = \begin{cases} \phi(t) & (t \in [-R, 0]), \\ \phi(0) & (t \in \mathbb{R}_+) \end{cases}$$

is called the *static prolongation* of ϕ.

Definition 2.4 (History space of Sobolev type)**.** Let $1 \le p < \infty$ and $a < b$ be real numbers. For each absolutely continuous function $x\colon [a,b] \to \mathbb{R}^N$, let

$$\|x\|_{\mathcal{W}^{1,p}[a,b]} := \left(|x(a)|^p + \|x'\|_{L^p[a,b]}^p\right)^{\frac{1}{p}},$$

where x' denotes the almost everywhere derivative of x. Let $\mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$ denote the normed space

$$\left\{ x \in \mathrm{AC}([a,b],\mathbb{R}^N) : x' \in L^p([a,b],\mathbb{R}^N) \right\}$$

endowed with the norm $\|\cdot\|_{\mathcal{W}^{1,p}[a,b]}$. The history space $\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ is called the *history space of Sobolev type*.

Remark 2.5. History spaces of Sobolev type appear for the investigation of neutral delay differential equations. See [7] for $p = 1$ and [22] for $1 \le p < \infty$ for examples.

Lemma 2.6. *Let $1 \le p < \infty$ and $a < b$ be real numbers. We define a norm $\|\cdot\|$ on $\mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$ by*

$$\|x\| := \|x\|_{C[a,b]} + \|x'\|_{L^p[a,b]}.$$

Then $\|\cdot\|$ is equivalent to $\|\cdot\|_{\mathcal{W}^{1,p}[a,b]}$.

Proof. Let $x \in \mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$.

Step 1. By the relationships between ℓ^p-norms, we have

$$\|x\|_{\mathcal{W}^{1,p}[a,b]} = \left(|x(a)|^p + \|x'\|_{L^p[a,b]}^p\right)^{\frac{1}{p}} \le |x(a)| + \|x'\|_{L^p[a,b]} \le \|x\|.$$

Step 2. By the fundamental theorem of calculus for absolutely continuous functions, we have

$$|x(t)| \le |x(a)| + \int_a^t |x'(s)|\,\mathrm{d}s \le |x(a)| + \|x'\|_{L^1[a,b]}$$

for all $t \in [a,b]$. This shows

$$\|x\|_{C[a,b]} \le |x(a)| + (b-a)^{\frac{1}{q}} \|x'\|_{L^p[a,b]},$$

where q is the Hölder conjugate of p. Therefore,

$$\begin{aligned}
\|x\| &= \|x\|_{C[a,b]} + \|x'\|_{L^p[a,b]} \\
&\le \left[(b-a)^{\frac{1}{q}} + 1\right]\left(|x(a)| + \|x'\|_{L^p[a,b]}\right) \\
&\le 2^{\frac{1}{q}}\left[(b-a)^{\frac{1}{q}} + 1\right]\|x\|_{\mathcal{W}^{1,p}[a,b]},
\end{aligned}$$

where the relation between ℓ^p-norms is used.

By the above steps, the conclusion holds. □

Remark 2.7. Lemma 2.6 means that a sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$ converges to x if and only if $x_n \to x$ uniformly and $x_n' \to x'$ in L^p.

Lemma 2.8. *Let $1 \le p < \infty$ and $a < b$ be real numbers. Then $\mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$ is a Banach space.*

Proof. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$. Then $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $C([a,b],\mathbb{R}^N)$, and $(x_n')_{n=1}^\infty$ is a Cauchy sequence in $L^p([a,b],\mathbb{R}^N)$. Since these spaces are complete, there are $x \in C([a,b],\mathbb{R}^N)$ and $y \in L^p([a,b],\mathbb{R}^N)$ such that

$$\|x - x_n\|_{C[a,b]} \to 0 \text{ and } \|y - x_n'\|_{L^p[a,b]} \to 0$$

as $n \to \infty$. By the fundamental theorem of calculus for absolutely continuous functions, we have

$$x_n(t) = x_n(a) + \int_a^t x_n'(s)\,\mathrm{d}s \quad (t \in [a,b]).$$

Then by taking the limit as $n \to \infty$, we obtain

$$x(t) = x(a) + \int_a^t y(s)\,\mathrm{d}s \quad (x \in [a,b])$$

because

$$\begin{aligned}\left|\int_a^t (y(s) - x_n'(s))\,\mathrm{d}s\right| &\le \|y - x_n'\|_{L^1[a,b]} \\ &\le (b-a)^{\frac{1}{q}}\|y - x_n'\|_{L^p[a,b]}.\end{aligned}$$

Here q is the Hölder conjugate of p. This shows $x \in \mathrm{AC}([a,b],\mathbb{R}^N)$ and $x' = y \in L^p([a,b],\mathbb{R}^N)$. Therefore, $(x_n)_{n=1}^\infty$ converges to x in $\mathcal{W}^{1,p}([a,b],\mathbb{R}^N)$. □

Lemma 2.9. *Let $1 \le p < \infty$ and $R, T > 0$ be given. Then for all $x \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$, the orbit*

$$[0,T] \ni t \mapsto R_t x \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$$

is continuous.

Proof. Let $t_0 \in [0,T]$ be fixed. For all $t \in [0,T]$, we have

$$\|R_t x - R_{t_0} x\|_{\mathcal{W}^{1,p}[-R,0]}^p = |x(t-R) - x(t_0 - R)|^p + \int_{-R}^0 |x'(t+\theta) - x'(t_0+\theta)|^p\,\mathrm{d}\theta,$$

where the right-hand side converges to 0 as $t \to t_0$ by the continuity of x and by the continuity of translation in L^p. □

Lemma 2.10. *Let $1 \le p < \infty$ and $R, T > 0$ be given. Then the family of history operators given by*

$$\mathcal{W}^{1,p}([-R,T],\mathbb{R}^N) \ni x \mapsto R_t x \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N),$$

where $t \in [0,T]$, is pointwise equicontinuous.

Proof. It is sufficient to show the equicontinuity at 0 because the maps are linear. Let $t \in [0,T]$. Then for all $x \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$,

$$\begin{aligned}\|R_t x\|_{C[-R,0]} + \|(R_t x)'\|_{L^p[-R,0]} &= \sup_{\theta\in[-R,0]} |x(t+\theta)| + \left(\int_{-R}^0 |x'(t+\theta)|^p\,\mathrm{d}\theta\right)^{\frac{1}{p}} \\ &\le \|x\|_{C[-R,T]} + \|x'\|_{L^p[-R,T]}.\end{aligned}$$

This shows the conclusion. □

Remark 2.11. By the preceding two lemmas,

$$[0,T] \times \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N) \ni (t,x) \mapsto R_t x \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$$

is continuous.

Lemma 2.12 (Continuity of the prolongation operator)**.** *Let $1 \le p < \infty$ and $R, T > 0$ be given. Then the prolongation operator given by*

$$\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N) \ni \phi \mapsto \bar{\phi}|_{[-R,T]} \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$$

is a continuous linear map. In particular,

$$\|\bar{\phi}|_{[-R,T]}\|_{\mathcal{W}^{1,p}[-R,T]} = \|\phi\|_{\mathcal{W}^{1,p}[-R,0]}$$

holds for all $\phi \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$.

Proof. For every $\phi \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$, we have

$$\begin{aligned}\|\bar{\phi}|_{[-R,T]}\|_{\mathcal{W}^{1,p}[-R,T]} &= \left(|\bar{\phi}(-R)|^p + \int_{-R}^{T} |\bar{\phi}'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\ &= \left(|\phi(-R)|^p + \int_{-R}^{0} |\phi'(\theta)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\ &= \|\phi\|_{\mathcal{W}^{1,p}[-R,0]}.\end{aligned}$$

Therefore, the conclusion holds. □

3 Main results

In the proofs, the function space $\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ is abbreviated as $\mathcal{W}^{1,p}[-R,0]$. This is similar to other function spaces.

3.1 A special case

Let $N \ge 1$ be an integer, $f\colon \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function, and $R > 0$ be a constant. We consider a DDE (1.5)

$$\dot{x}(t) = f(x(t-r))$$

and its IVP (1.6)

$$\begin{cases} \dot{x}(t) = f(x(t-r)), & t \ge 0, \\ x(t) = \phi(t), & t \in [-R,0] \end{cases}$$

for each $(\phi, r) \in C([-R,0],\mathbb{R}^N) \times (0,R]$. The solution $x(\cdot;\phi,r)$ of (1.6) is expressed by

$$x(t;\phi,r) = \phi(0) + \int_0^t f(\phi(s-r))\,\mathrm{d}s$$

on the interval $[0,r]$, which is continued to $[-R,+\infty)$ by the method of steps. Let $|\cdot|$ be a norm on $\mathbb{R}^N$. The operator norm of a linear map $L\colon \mathbb{R}^N \to \mathbb{R}^N$ with respect to the above norm $|\cdot|$ will be denoted by $\|L\|$.

Proposition 3.1. *Let $1 \le p < \infty$ and $0 < T < R$ be given. Let $\phi \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$. If f is of class C^1, then*

$$[T,R] \ni r \mapsto x(\cdot;\phi,r) \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$$

is a continuously differentiable function whose derivative is given by

$$\left(\frac{\partial}{\partial r}x(\cdot;\phi,r)\right)(t) = B_{\phi,r}(t) := \begin{cases} 0 & (t \in [-R,0]), \\ -\int_0^t (f\circ\phi)'(s-r)\,\mathrm{d}s & (t \in [0,T]) \end{cases}$$

in $\mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$.

Proof. Since f is locally Lipschitz continuous, $f\circ\phi\colon [-R,0] \to \mathbb{R}^N$ is also absolutely continuous. Then $f\circ\phi$ is differentiable almost everywhere, and

$$(f\circ\phi)'(\theta) = Df(\phi(\theta))\phi'(\theta)$$

holds for almost all $\theta \in [-R,0]$. Therefore,

$$\begin{aligned}\int_{-R}^0 |(f\circ\phi)'(\theta)|^p\,\mathrm{d}\theta &\le \int_{-R}^0 \|Df(\phi(\theta))\|^p|\phi'(\theta)|^p\,\mathrm{d}\theta \\ &\le \sup_{\theta\in[-R,0]} \|Df(\phi(\theta))\|^p \cdot \|\phi'\|^p_{L^p[-R,0]},\end{aligned}$$

which shows $f\circ\phi \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$.

Let $r_0 \in [T,R]$ be fixed. Then for all $r \in [R,T]$ and all $t \in [-R,T]$,

$$\begin{aligned}&\frac{1}{r-r_0}(x(t;\phi,r) - x(t;\phi,r_0)) - B_{\phi,r_0}(t) \\ &\quad= \begin{cases} 0 & (t \in [-R,0]), \\ \frac{1}{r-r_0}\int_0^t \big(f(\phi(s-r)) - f(\phi(s-r_0)) + (r-r_0)(f\circ\phi)'(s-r_0)\big)\,\mathrm{d}s & (t \in [0,T]). \end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|\frac{1}{r-r_0}(x(\cdot;\phi,r) - x(\cdot;\phi,r_0)) - B_{\phi,r_0}\right\|_{\mathcal{W}^{1,p}[-R,T]} \\ &\quad= \frac{1}{|r-r_0|}\left(\int_0^T |(f\circ\phi)(t-r) - (f\circ\phi)(t-r_0) + (r-r_0)(f\circ\phi)'(t-r_0)|\,\mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\to 0\end{aligned}$$

as $r \to r_0$ by the differentiability of translation in L^p (Corollary A.4). The continuity of the derivative also holds because

$$\begin{aligned}\|B_{\phi,r} - B_{\phi,r_0}\|_{\mathcal{W}^{1,p}[-R,T]} &= \left(\int_0^T |(f\circ\phi)'(t-r) - (f\circ\phi)'(t-r_0)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \\ &\to 0\end{aligned}$$

as $r \to r_0$ by the continuity of translation in L^p. □

Proposition 3.2. *Let $1 \le p < \infty$ and $0 < T < R$ be given. Suppose that f is of class C^1. Then the family of functions*

$$\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N) \ni \phi \mapsto B_{\phi,r} \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N),$$

where $r \in [T,R]$, is pointwise equicontinuous.

Proof. Let $\phi_0 \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ be fixed and $r \in [T,R]$ be a parameter. Then we have

$$\begin{aligned}
&|(f\circ\phi)'(t-r) - (f\circ\phi_0)'(t-r)| \\
&\quad \le \|Df(\phi(t-r)) - Df(\phi_0(t-r))\||\phi'(t-r)| \\
&\qquad + \|Df(\phi_0(t-r))\||\phi'(t-r) - \phi_0'(t-r)|
\end{aligned}$$

for all $\phi \in \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ and all $t \in [0,T]$. Therefore, by the Minkowski inequality,

$$\begin{aligned}
&\|B_{\phi,r} - B_{\phi_0,r}\|_{\mathcal{W}^{1,p}[-R,T]} \\
&\quad = \left(\int_0^T |(f\circ\phi)'(t-r) - (f\circ\phi_0)'(t-r)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad \le \left(\int_0^T \|Df(\phi(t-r)) - Df(\phi_0(t-r))\|^p |\phi'(t-r)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\qquad + \left(\int_0^T \|Df(\phi_0(t-r))\|^p |\phi'(t-r) - \phi_0'(t-r)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad \le \sup_{\theta\in[-R,0]} \|Df(\phi(\theta)) - Df(\phi_0(\theta))\| \cdot \|\phi\|_{\mathcal{W}^{1,p}[-R,0]} \\
&\qquad + \sup_{\theta\in[-R,0]} \|Df(\phi_0(\theta))\| \cdot \|\phi - \phi_0\|_{\mathcal{W}^{1,p}[-R,0]}.
\end{aligned}$$

The right-hand side converges to 0 as $\|\phi - \phi_0\|_{\mathcal{W}^{1,p}[-R,0]} \to 0$ uniformly in r because Df is uniformly continuous on any closed and bounded set. □

Corollary 3.3. *Let $1 \le p < \infty$ and $0 < T < R$ be given. Suppose that f is of class C^1. Then*

$$\mathcal{W}^{1,p}([-R,0],\mathbb{R}^N) \times [T,R] \ni (\phi,r) \mapsto B_{\phi,r} \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$$

is continuous.

Proof. Let $(\phi_0, r_0) \in \mathcal{W}^{1,p}[-R,0] \times [T,R]$ be fixed. Then for all $(\phi,r) \in \mathcal{W}^{1,p}[-R,0] \times [T,R]$, we have

$$\|B_{\phi,r} - B_{\phi_0,r_0}\|_{\mathcal{W}^{1,p}[-R,T]} \le \|B_{\phi,r} - B_{\phi_0,r}\|_{\mathcal{W}^{1,p}[-R,T]} + \|B_{\phi_0,r} - B_{\phi_0,r_0}\|_{\mathcal{W}^{1,p}[-R,T]},$$

where the right-hand side converges to 0 as $(\phi,r) \to (\phi_0,r_0)$ from the above propositions. □

3.2 A general case

Let $N \ge 1$ be an integer, $f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function, and $R > 0$ be a constant. We consider a DDE (1.1)

$$\dot{x}(t) = f(x(t), x(t-r))$$

and its IVP (1.2)

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-r)), & t \ge 0, \\ x(t) = \phi(t), & t \in [-R,0] \end{cases}$$

for each $(\phi,r) \in C([-R,0],\mathbb{R}^N) \times [0,R]$. We note that the case $r = 0$ is permitted. Let $|\cdot|$ be a norm on $\mathbb{R}^N$. The following product norm on $\mathbb{R}^N \times \mathbb{R}^N$

$$\|(x_1,x_2)\| := |x_1| + |x_2|$$

will be used. The operator norms of linear maps $L_1 \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $L_2 \colon \mathbb{R}^N \to \mathbb{R}^N$ with respect to the corresponding norms are denoted by $\|L_1\|$ and $\|L_2\|$, respectively.

Let

$$y(t) := x(t) - \bar{\phi}(t) \quad (t \in [0, T])$$

for some $T > 0$. Then x is a solution of (1.2) on $[0, T]$ if and only if y satisfies

$$\begin{aligned} y(t) &= \mathcal{T}(y, \phi, r)(t) \\ &:= \begin{cases} 0 & (t \in [-R, 0]), \\ \int_0^t f\big((y+\bar{\phi})(s), (y+\bar{\phi})(s-r)\big)\,\mathrm{d}s & (t \in [0, T]). \end{cases} \end{aligned}$$

The above argument means that y is a fixed point of $\mathcal{T}(\cdot, \phi, r)$ if and only if $x := y + \bar{\phi}$ is a solution of (1.2).

For any continuous y, $\mathcal{T}(y, \phi, r)$ is absolutely continuous and

$$\|\mathcal{T}(y, \phi, r)\|_{\mathcal{W}^{1,p}[-R,T]} = \left(\int_0^T \left| f\big((y+\bar{\phi})(t), (y+\bar{\phi})(t-r)\big) \right|^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty$$

because the integrand is continuous.

3.2.1 Uniform contraction

Notation 1. Let $T > 0$ be given. For each $\delta > 0$, let

$$\begin{aligned} \Gamma(\delta) &:= \left\{ \gamma \in C([-R, T], \mathbb{R}^N) : R_0\gamma = 0, \ \|\gamma\|_{C[-R,T]} < \delta \right\}, \\ \bar{\Gamma}(\delta) &:= \left\{ \gamma \in C([-R, T], \mathbb{R}^N) : R_0\gamma = 0, \ \|\gamma\|_{C[-R,T]} \le \delta \right\}, \end{aligned}$$

which are considered to be metric spaces with the metric induced by supremum norm.

Notation 2. Let $T > 0$ be given. For each $1 \le p < \infty$ and $\delta > 0$, let

$$\begin{aligned} \Gamma_{1,p}(\delta) &:= \left\{ \gamma \in \mathcal{W}^{1,p}([-R, T], \mathbb{R}^N) : R_0\gamma = 0, \ \|\gamma\|_{\mathcal{W}^{1,p}[-R,T]} < \delta \right\}, \\ \bar{\Gamma}_{1,p}(\delta) &:= \left\{ \gamma \in \mathcal{W}^{1,p}([-R, T], \mathbb{R}^N) : R_0\gamma = 0, \ \|\gamma\|_{\mathcal{W}^{1,p}[-R,T]} \le \delta \right\}, \end{aligned}$$

which are considered to be metric spaces with the metric induced by $\mathcal{W}^{1,p}$-norm.

Lemma 3.4. *Let $1 \le p < \infty$, $B \subset C([-R, 0], \mathbb{R}^N)$ be a bounded set, and $\delta > 0$. Then for all sufficiently small $T > 0$, the family of maps*

$$\mathcal{T}(\cdot, \phi, r) \colon \bar{\Gamma}(\delta) \to \Gamma_{1,p}(\delta),$$

where $(\phi, r) \in B \times [0, R]$, is well-defined.

Proof. Let $y \in \bar{\Gamma}(\delta)$. Then for all $(\phi, r) \in B \times [0, R]$,

$$\begin{aligned} \sup_{t \in [0,T]} \left\| \big((y+\bar{\phi})(t), (y+\bar{\phi})(t-r)\big) \right\| &= \sup_{t \in [0,T]} \big(|(y+\bar{\phi})(t)| + |(y+\bar{\phi})(t-r)|\big) \\ &\le 2(\|y\|_{C[-R,T]} + \|\phi\|_{C[-R,0]}). \end{aligned}$$

Since f is bounded on any bounded set of $\mathbb{R}^N \times \mathbb{R}^N$, there is $M > 0$ such that

$$\sup_{t\in[0,T]} \left| f\big((y+\bar{\phi})(t), (y+\bar{\phi})(t-r)\big) \right| \le M$$

for all $(\phi, r) \in B \times [0, R]$. By choosing $0 < T < (\delta/M)^p$,

$$\|\mathcal{T}(y,\phi,r)\|_{\mathcal{W}^{1,p}[-R,T]} \le \left(\int_0^T M^p \, \mathrm{d}t \right)^{\frac{1}{p}} = MT^{\frac{1}{p}} < \delta$$

holds for all such (ϕ, r). This shows the conclusion. □

Lemma 3.5. *Let $1 \le p < \infty$, $B \subset C([-R,0], \mathbb{R}^N)$ be a bounded set, and $\delta > 0$. If f is locally Lipschitz continuous, then for all sufficiently small $T > 0$, the family of maps*

$$\mathcal{T}(\cdot,\phi,r) \colon \bar{\Gamma}(\delta) \to \Gamma_{1,p}(\delta),$$

where $(\phi, r) \in B \times [0, R]$, is a well-defined uniform contraction.

Proof. The well-definedness follows by the preceding lemma. Since f is Lipschitz continuous on any bounded set of $\mathbb{R}^N \times \mathbb{R}^N$, there is $L > 0$ such that

$$\begin{aligned} &\left| f\big((y_1+\bar{\phi})(t), (y_1+\bar{\phi})(t-r)\big) - f\big((y_2+\bar{\phi})(t), (y_2+\bar{\phi})(t-r)\big) \right| \\ &\quad \le L(|(y_1-y_2)(t)| + |(y_1-y_2)(t-r)|) \\ &\quad \le 2L\|y_1-y_2\|_{C[-R,T]} \end{aligned}$$

for all $y_1, y_2 \in \bar{\Gamma}(\delta)$, $\phi \in B$, and $r \in [0, R]$. This implies that we have

$$\begin{aligned} \|\mathcal{T}(y_1,\phi,r) - \mathcal{T}(y_2,\phi,r)\|_{\mathcal{W}^{1,p}[-R,T]} &\le \left(\int_0^T (2L\|y_1-y_2\|_{C[-R,T]})^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\ &\le 2LT^{\frac{1}{p}} \cdot \|y_1-y_2\|_{C[-R,T]} \end{aligned}$$

for all such (y_1, ϕ, r) and (y_2, ϕ, r). Therefore, the family of maps becomes a well-defined uniform contraction by choosing sufficiently small $0 < T < 1/(2L)^p$. □

Remark 3.6. The uniform contraction means that there is $0 < c < 1$ such that for all $(\phi, r) \in B \times [0, R]$ and $y_1, y_2 \in \bar{\Gamma}(\delta)$,

$$\|\mathcal{T}(y_1,\phi,r) - \mathcal{T}(y_2,\phi,r)\|_{\mathcal{W}^{1,p}[-R,T]} \le c \cdot \|y_1-y_2\|_{C[-R,T]}$$

holds. Therefore, the families of maps

$$\begin{aligned} &\mathcal{T}(\cdot,\phi,r) \colon \bar{\Gamma}(\delta) \to \Gamma(\delta), \\ &\mathcal{T}(\cdot,\phi,r) \colon \bar{\Gamma}_{1,p}(\delta) \to \Gamma_{1,p}(\delta), \end{aligned}$$

where $(\phi, r) \in B \times [0, R]$, are also uniform contractions. We note that the domains of the two operators correspond to different T.

Proposition 3.7. *Let $B \subset C([-R,0], \mathbb{R}^N)$ be a bounded set. If f is locally Lipschitz continuous, then there exists $T > 0$ such that for every $(\phi, r) \in B \times [0, R]$, IVP (1.2) has the unique solution $x \colon [-R, T] \to \mathbb{R}^N$.*

Proof. Let $\delta > 0$ be given.

Step 1. We choose $M, T > 0$ so that $MT \le \delta$ and for all $(y, \phi) \in \bar{\Gamma}(\delta) \times B$ and $r \in [0, R]$,

$$\sup_{t \in [0,T]} \left| f\big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r)\big) \right| \le M.$$

Let $(\phi, r) \in B \times [0, R]$ be given. Then for every solution $x \colon [-R, T] \to \mathbb{R}^N$ of IVP (1.2), the function $y \colon [-R, T] \to \mathbb{R}^N$ defined by $y = x - \bar{\phi}$ necessarily belongs to $\bar{\Gamma}(\delta)$.

Step 2. From the preceding lemma, there is sufficiently small $T > 0$ such that the family of maps

$$\mathcal{T}(\cdot, \phi, r) \colon \bar{\Gamma}(\delta) \to \bar{\Gamma}(\delta),$$

where $(\phi, r) \in B \times [0, R]$, is a uniform contraction. Then the Banach fixed point theorem implies that for each $(\phi, r) \in B \times [0, R]$, $\mathcal{T}(\cdot, \phi, r)$ has the unique fixed point $y(\cdot, \phi, r) \in \bar{\Gamma}(\delta)$ because $\bar{\Gamma}(\delta)$ is a complete metric space. Then $x \colon [-R, T] \to \mathbb{R}^N$ defined by

$$x := y(\cdot, \phi, r) + \bar{\phi}$$

is a solution of IVP (1.2). The uniqueness follows by Step 1. □

Remark 3.8. Under the assumption of the local Lipschitz continuity of f, IVP (1.2) has the unique maximal solution

$$x(\cdot; \phi, r) \colon [-R, T_{\phi,r}) \to \mathbb{R}^N, \quad \text{where } 0 < T_{\phi,r} \le \infty,$$

for every $(\phi, r) \in C([-R, 0], \mathbb{R}^N) \times [0, R]$.

3.2.2 C^1-smoothness with respect to delay

Let $1 \le p < \infty$ and $T > 0$ be given. The following notation will be used.

Notation 3. For each $(y, \phi) \in C[-R, T] \times C[-R, 0]$, $r \in [0, R]$, and $t \in [0, T]$, let

$$\rho(y, \phi, r, t) := \big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r)\big).$$

Then

$$\rho(y_1, \phi_1, r, t) - \rho(y_2, \phi_2, r, t) = \rho(y_1 - y_2, \phi_1 - \phi_2, r, t)$$

holds.

Lemma 3.9. *Let $(y, \phi) \in C([-R, T], \mathbb{R}^N) \times C([-R, 0], \mathbb{R}^N)$ and $r_0 \in [0, R]$ be fixed. If f is of class C^1, then for $r \in [0, R]$,*

$$\sup_{t \in [0,T]} \| Df(\rho(y, \phi, r, t)) - Df(\rho(y, \phi, r_0, t)) \| \to 0$$

as $r \to r_0$.

Proof. Since

$$\|\rho(y, \phi, r, t)\| \le 2(\|y\|_{C[-R,T]} + \|\phi\|_{C[-R,0]})$$

holds for all $r \in [0, R]$ and $t \in [0, T]$, $\rho(y, \phi, r, t)$ is contained in some bounded set B for all such r, t.

Let $\varepsilon > 0$. The uniform continuity of Df on B implies that there is $\delta_1 > 0$ such that for all $(x_1, y_1), (x_2, y_2) \in B$,

$$|x_1 - x_2| + |y_1 - y_2| < \delta_1 \implies \|Df(x_1, y_1) - Df(x_2, y_2)\| < \varepsilon.$$

By the uniform continuity of $y + \bar{\phi} \colon [-R, T] \to \mathbb{R}^N$, there is $\delta_2 > 0$ such that $|r - r_0| < \delta_2$ implies

$$\sup_{t \in [0,T]} |(y + \bar{\phi})(t - r) - (y + \bar{\phi})(t - r_0)| < \delta_1.$$

In view of

$$\|\rho(y, \phi, r, t) - \rho(y, \phi, r_0, t)\| = |(y + \bar{\phi})(t - r) - (y + \bar{\phi})(t - r_0)|,$$

the above argument shows that $|r - r_0| < \delta_2$ implies

$$\|Df(\rho(y, \phi, r, t)) - Df(\rho(y, \phi, r_0, t))\| < \varepsilon$$

for all $t \in [0, T]$. □

Theorem 3.10. *Let $y \in \mathcal{W}^{1,p}([-R, T], \mathbb{R}^N)$ and $\phi \in \mathcal{W}^{1,p}([-R, 0], \mathbb{R}^N)$ be fixed. If f is of class C^1, then*

$$\mathcal{T}(y, \phi, \cdot) \colon [0, R] \to \mathcal{W}^{1,p}([-R, T], \mathbb{R}^N)$$

is a continuously differentiable function whose derivative is given by

$$\left(\frac{\partial}{\partial r}\mathcal{T}(y, \phi, r)\right)(t) = B_{y,\phi,r}(t) := \begin{cases} 0 & (t \in [-R, 0]), \\ -\int_0^t D_2 f\big((y + \bar{\phi})(s), (y + \bar{\phi})(s - r)\big)(y + \bar{\phi})'(s - r)\,\mathrm{d}s & (t \in [0, T]) \end{cases}$$

in $\mathcal{W}^{1,p}([-R, T], \mathbb{R}^N)$.

Proof.

Step 1. Let $r_0 \in [0, R]$ be fixed. For $y \in \mathcal{W}^{1,p}[-R, T]$ and $\phi \in \mathcal{W}^{1,p}[-R, 0]$, let

$$L(u, t, r) := D_2 f\Big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r_0) + u\big((y + \bar{\phi})(t - r) - (y + \bar{\phi})(t - r_0)\big)\Big)$$

for each $(u, t, r) \in [0, 1] \times [0, T] \times [0, R]$. We note

$$\begin{aligned} L(0, t, r) &:= D_2 f\big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r_0)\big) = D_2 f(\rho(y, \phi, r_0, t)), \\ L(1, t, r) &:= D_2 f\big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r)\big) = D_2 f(\rho(y, \phi, r, t)). \end{aligned}$$

Then

$$\begin{aligned} &f\big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r)\big) - f\big((y + \bar{\phi})(t), (y + \bar{\phi})(t - r_0)\big) \\ &\qquad = \int_0^1 L(u, t, r)\,\mathrm{d}u \cdot \big((y + \bar{\phi})(t - r) - (y + \bar{\phi})(t - r_0)\big) \end{aligned}$$

holds for all $(t,r) \in [0,T] \times [0,R]$. Therefore, we have

$$\begin{aligned}
&\left\| \frac{1}{r-r_0}(\mathcal{T}(y,\phi,r) - \mathcal{T}(y,\phi,r_0)) - B_{y,\phi,r_0} \right\|_{\mathcal{W}^{1,p}[-R,T]} \\
&= \frac{1}{|r-r_0|} \left(\int_0^T \left| \int_0^1 L(u,t,r)\,\mathrm{d}u \cdot \big((y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0)\big) \right.\right. \\
&\qquad \left.\left. + (r-r_0) L(0,t,r)(y+\bar{\phi})'(t-r_0) \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&=: \frac{1}{|r-r_0|} \left(\int_0^T g(t,r)^p \,\mathrm{d}t \right)^{\frac{1}{p}}
\end{aligned}$$

for all $r \in [0,R]$.

Step 2. For all $(t,r) \in [0,T] \times [0,R]$,

$$\begin{aligned}
g(t,r) &\le \int_0^1 \|L(u,t,r) - L(0,t,r)\|\,\mathrm{d}u \cdot |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0)| \\
&\quad + \|L(0,t,r)\| \cdot |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0) + (r-r_0)(y+\bar{\phi})'(t-r_0)| \\
&\le \sup_{(u,t)\in[0,1]\times[0,T]} \|L(u,t,r) - L(0,t,r)\| \cdot |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0)| \\
&\quad + \sup_{t\in[0,T]} \|L(0,t,r)\| \cdot |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0) + (r-r_0)(y+\bar{\phi})'(t-r_0)| \\
&=: g_1(t,r) + g_2(t,r).
\end{aligned}$$

Therefore,

$$\frac{1}{|r-r_0|} \left(\int_0^T g(t,r)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \le \frac{1}{|r-r_0|} \left(\int_0^T g_1(t,r)^p \,\mathrm{d}t \right)^{\frac{1}{p}} + \frac{1}{|r-r_0|} \left(\int_0^T g_2(t,r)^p \,\mathrm{d}t \right)^{\frac{1}{p}}$$

by the Minkowski inequality.

Step 3. Let $\varepsilon > 0$. In the same way as the preceding lemma, there is $\delta > 0$ such that for all $r \in [0,R]$, $|r-r_0| < \delta$ implies

$$\sup_{(u,t)\in[0,1]\times[0,T]} \|L(u,t,r) - L(0,t,r)\| \le \varepsilon$$

because

$$\left\| \Big(0, u\big((y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0)\big)\Big) \right\| \le |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0)|.$$

Therefore, for such r,

$$g_1(t,r) \le \varepsilon |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0)| = \varepsilon \cdot \left| \int_{-r_0}^{-r} (y+\bar{\phi})'(t+\theta)\,\mathrm{d}\theta \right|,$$

and we have

$$\begin{aligned}
\frac{1}{|r-r_0|} \left(\int_0^T g_1(t,r)^p \,\mathrm{d}t \right)^{\frac{1}{p}} &\le \frac{\varepsilon}{|r-r_0|} \left(\int_0^T \left| \int_{-r_0}^{-r} |(y+\bar{\phi})'(t+\theta)|\,\mathrm{d}\theta \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\le \varepsilon \cdot \|y+\bar{\phi}\|_{\mathcal{W}^{1,p}[-R,T]},
\end{aligned}$$

where the last inequality follows from Corollary A.2.

Step 4. For all $r \in [0, R]$, we have

$$\begin{aligned} &\frac{1}{|r-r_0|}\left(\int_0^T g_2(t,r)^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad \le \sup_{t\in[0,T]} \|L(0,t,r)\| \\ &\qquad \cdot \frac{1}{|r-r_0|}\left(\int_0^T |(y+\bar{\phi})(t-r) - (y+\bar{\phi})(t-r_0) + (r-r_0)(y+\bar{\phi})'(t-r_0)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}, \end{aligned}$$

where the last term converges to 0 by the differentiability of translation in L^p (Corollary A.4).

Step 5. By the above steps, we have

$$\left\| \frac{1}{r-r_0}(\mathcal{T}(y,\phi,r) - \mathcal{T}(y,\phi,r_0)) - B_{y,\phi,r_0} \right\|_{\mathcal{W}^{1,p}[-R,T]} \to 0$$

as $r \to r_0$, which shows the Fréchet differentiability. The continuity of the derivative also holds because

$$\begin{aligned} &\|B_{y,\phi,r} - B_{y,\phi,r_0}\|_{\mathcal{W}^{1,p}[-R,T]} \\ &\quad = \left(\int_0^T |L(1,t,r)(y+\bar{\phi})'(t-r) - L(0,t,r)(y+\bar{\phi})'(t-r_0)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad \le \left(\int_0^T \|L(1,t,r) - L(0,t,r)\|^p |(y+\bar{\phi})'(t-r)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\qquad + \left(\int_0^T \|L(0,t,r)\|^p |(y+\bar{\phi})'(t-r) - (y+\bar{\phi})'(t-r_0)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad \le \sup_{t\in[0,T]} \|L(1,t,r) - L(0,t,r)\| \cdot \|y+\bar{\phi}\|_{\mathcal{W}^{1,p}[-R,T]} \\ &\qquad + \sup_{t\in[0,T]} \|L(0,t,r)\| \left(\int_0^T |(y+\bar{\phi})'(t-r) - (y+\bar{\phi})'(t-r_0)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}$$

This shows that $\|B_{y,\phi,r} - B_{y,\phi,r_0}\|_{\mathcal{W}^{1,p}[-R,T]}$ converges to 0 as $r \to r_0$ by the preceding lemma and by the continuity of translation in L^p. □

3.2.3 C^1-smoothness with respect to prolongation and history

Let $1 \le p < \infty$ and $T > 0$ be given.

Lemma 3.11. *Let $(y_0, \phi_0) \in C([-R,T], \mathbb{R}^N) \times C([-R,0], \mathbb{R}^N)$ be fixed. If f is of class C^1, then for $(y, \phi) \in C([-R,T], \mathbb{R}^N) \times C([-R,0], \mathbb{R}^N)$,*

$$\sup_{t\in[0,T]} \|Df(\rho(y,\phi,r,t)) - Df(\rho(y_0,\phi_0,r,t))\| \to 0$$

as $(y, \phi) \to (y_0, \phi_0)$ uniformly in $r \in [0, R]$.

Proof. We may assume that there is a bounded set $B \subset \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\rho(y, \phi, r, t) \in B$$

holds for all $(y, \phi) \in C[-R, T] \times C[-R, 0]$, $r \in [0, R]$, and $t \in [0, T]$ because

$$\begin{aligned}\|\rho(y, \phi, r, t)\| &\leq \|\rho(y, \phi, r, t) - \rho(y_0, \phi_0, r, t)\| + \|\rho(y_0, \phi_0, r, t)\| \\ &= \|\rho(y - y_0, \phi - \phi_0, r, t)\| + \|\rho(y_0, \phi_0, r, t)\| \\ &\leq 2(\|y - y_0\|_{C[-R,T]} + \|\phi - \phi_0\|_{C[-R,0]} + \|y_0\|_{C[-R,T]} + \|\phi_0\|_{C[-R,0]}).\end{aligned}$$

Let $\varepsilon > 0$. The uniform continuity of Df on B implies that there is $\delta > 0$ such that for all $(x_1, y_1), (x_2, y_2) \in B$,

$$|x_1 - x_2| + |y_1 - y_2| < \delta \implies \|Df(x_1, y_1) - Df(x_2, y_2)\| < \varepsilon.$$

Therefore,

$$\|y - y_0\|_{C[-R,T]} + \|\phi - \phi_0\|_{C[-R,0]} < \frac{\delta}{2}$$

implies

$$\|Df(\rho(y, \phi, r, t)) - Df(\rho(y_0, \phi_0, r, t))\| < \varepsilon$$

for all $t \in [0, T]$ uniformly in r. □

Theorem 3.12. *Let $r \in [0, R]$ be fixed. If f is of class C^1, then*

$$\mathcal{T}(\cdot, \cdot, r)\colon C([-R, T], \mathbb{R}^N) \times C([-R, 0], \mathbb{R}^N) \to \mathcal{W}^{1,p}([-R, T], \mathbb{R}^N)$$

is continuously Fréchet differentiable. The Fréchet derivative is given by

$$D_{y,\phi}\mathcal{T}(y, \phi, r) = A_{y,\phi,r},$$

where

$$[A_{y,\phi,r}(\eta, \chi)](t) = \begin{cases} 0 & (t \in [-R, 0]), \\ \int_0^t Df\big((y + \bar{\phi})(s), (y + \bar{\phi})(s - r)\big)\big((\eta + \bar{\chi})(s), (\eta + \bar{\chi})(s - r)\big)\,\mathrm{d}s & (t \in [0, T]) \end{cases}$$

for all $(\eta, \chi) \in C([-R, T], \mathbb{R}^N) \times C([-R, 0], \mathbb{R}^N)$. In particular,

$$\|A_{y,\phi,r} - A_{y_0,\phi_0,r}\| \leq 2T^{\frac{1}{p}} \sup_{t \in [0,T]} \big\|Df\big((y + \bar{\phi})(t), (y + \bar{\phi})(y - r)\big) - Df\big((y_0 + \bar{\phi}_0)(t), (y_0 + \bar{\phi}_0)(t - r)\big)\big\|$$

holds, where $\|\cdot\|$ denotes the corresponding operator norm.

Proof. Let

$$\|(\eta, \chi)\| := \|\eta\|_{C[-R,T]} + \|\chi\|_{C[-R,0]}$$

for each $(\eta, \chi) \in C[-R, T] \times C[-R, 0]$.

Step 1. Let $(y, \phi) \in C[-R, T] \times C[-R, 0]$ be fixed. Then for all $(\eta, \chi) \in C[-R, T] \times C[-R, 0]$,

$$\begin{aligned} \|A_{y,\phi,r}(\eta,\chi)\|_{\mathcal{W}^{1,p}[-R,T]} &= \left(\int_0^T |Df(\rho(y,\phi,r,t))\rho(\eta,\chi,r,t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\ &\le \left(2T^{\frac{1}{p}} \sup_{t\in[0,T]} \|Df(\rho(y,\phi,r,t))\| \right) \|(\eta,\chi)\|. \end{aligned}$$

This shows that

$$A_{y,\phi,r} \colon C[-R,T] \times C[-R,0] \to \mathcal{W}^{1,p}[-R,T]$$

is a bounded linear operator.

Step 2. Let $(y_0, \phi_0) \in C[-R, T] \times C[-R, 0]$ be fixed. For $(\eta, \chi) \in C[-R, T] \times C[-R, 0]$, let

$$y := y_0 + \eta \text{ and } \phi := \phi_0 + \chi.$$

Then

$$f(\rho(y,\phi,r,t)) - f(\rho(y_0,\phi_0,r,t)) = \int_0^1 Df\big(\rho(y_0,\phi_0,r,t) + u\rho(\eta,\chi,r,t)\big) \, \mathrm{d}u \cdot \rho(\eta,\chi,r,t)$$

holds for all $r \in [0, R]$ and $t \in [0, T]$.

Let $\varepsilon > 0$. In the same way as the preceding lemma, there is $\delta > 0$ such that $\|(\eta, \chi)\| \le \delta$ implies

$$\sup_{t\in[0,T]} \big\|Df\big(\rho(y_0,\phi_0,r,t) + u\rho(\eta,\chi,r,t)\big) - Df(\rho(y_0,\phi_0,r,t))\big\| \le \varepsilon$$

because

$$\|u\rho(\eta,\chi,r,t)\| \le 2\|(\eta,\chi)\|.$$

Therefore, for such (η, χ), we have

$$\begin{aligned} &\|\mathcal{T}(y,\phi,r) - \mathcal{T}(y_0,\phi_0,r) - A_{y_0,\phi_0,r}(\eta,\chi)\|_{\mathcal{W}^{1,p}[-R,T]} \\ &\quad = \left(\int_0^T |f(\rho(y,\phi,r,t)) - f(\rho(y_0,\phi_0,r,t)) - Df(\rho(y_0,\phi_0,r,t))\rho(\eta,\chi,r,t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\ &\quad \le \left(\int_0^T \left(\int_0^1 \big\|Df\big(\rho(y_0,\phi_0,r,t) + u\rho(\eta,\chi,r,t)\big) - Df(\rho(y_0,\phi_0,r,t))\big\| \, \mathrm{d}u \right)^p \right. \\ &\qquad\quad \left. \cdot \|\rho(\eta,\chi,r,t)\|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\ &\quad \le 2\varepsilon T^{\frac{1}{p}} \|(\eta,\chi)\|. \end{aligned}$$

This shows the Fréchet differentiability of $\mathcal{T}(\cdot, \cdot, r)$ at (y_0, ϕ_0).

Step 3. Let $(y_0, \phi_0) \in C[-R, T] \times C[-R, 0]$ be fixed. For all $(y, \phi), (\eta, \chi) \in C[-R, T] \times C[-R, 0]$,

$$\begin{aligned} &\|(A_{y,\phi,r} - A_{y_0,\phi_0,r})(\eta,\chi)\|_{\mathcal{W}^{1,p}[-R,T]} \\ &\quad = \left(\int_0^T |[Df(\rho(y,\phi,r,t)) - Df(\rho(y_0,\phi_0,r,t))]\rho(\eta,\chi,r,t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\ &\quad \le \left(2T^{\frac{1}{p}} \sup_{t\in[0,T]} \|Df(\rho(y,\phi,r,t)) - Df(\rho(y_0,\phi_0,r,t))\| \right) \|(\eta,\chi)\|. \end{aligned}$$

This shows

$$\|A_{y,\phi,r} - A_{y_0,\phi_0,r}\| \le 2T^{\frac{1}{p}} \sup_{t\in[0,T]} \|Df(\rho(y,\phi,r,t)) - Df(\rho(y_0,\phi_0,r,t))\|,$$

which converges to 0 uniformly in r as $(y,\phi) \to (y_0,\phi_0)$ by the preceding lemma. Therefore, $(y,\phi) \mapsto A_{y,\phi,r}$ is continuous at (y_0,ϕ_0).

This completes the proof. □

Remark 3.13. The function

$$\mathcal{T}(\cdot,\cdot,r)\colon \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N) \times \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N) \to \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$$

is also continuously Fréchet differentiable because the inclusion

$$\mathcal{W}^{1,p}([a,b],\mathbb{R}^N) \subset C([a,b],\mathbb{R}^N), \quad \text{where } a < b,$$

is continuous (see Lemma 2.6).

3.2.4 C^1-smoothness with respect to prolongation, history, and delay

Let $1 \le p < \infty$ and $T > 0$ be given. We continue to use the following notations used in Theorems 3.10 and 3.12.

Notation 4. Let $(y,\phi) \in \mathcal{W}^{1,p}[-R,T] \times \mathcal{W}^{1,p}[-R,0]$ and $r \in [0,R]$.

$$A_{y,\phi,r}\colon \mathcal{W}^{1,p}[-R,T] \times \mathcal{W}^{1,p}[-R,0] \to \mathcal{W}^{1,p}[-R,T]$$

is the bounded linear operator defined by

$$[A_{y,\phi,r}(\eta,\chi)](t) = \begin{cases} 0 & (t \in [-R,0]), \\ \int_0^t Df(\rho(y,\phi,r,s))\rho(\eta,\chi,r,s)\,\mathrm{d}s & (t \in [0,T]). \end{cases}$$

Notation 5. Let $(y,\phi) \in \mathcal{W}^{1,p}[-R,T] \times \mathcal{W}^{1,p}[-R,0]$ and $r \in [0,R]$. $B_{y,\phi,r} \in \mathcal{W}^{1,p}[-R,T]$ is defined by

$$B_{y,\phi,r}(t) = \begin{cases} 0 & (t \in [-R,0]), \\ -\int_0^t D_2 f(\rho(y,\phi,r,s))(y+\bar{\phi})'(s-r)\,\mathrm{d}s & (t \in [0,T]). \end{cases}$$

Theorem 3.14. *Suppose that f is of class C^1. Then*

$$\mathcal{T}\colon \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N) \times \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N) \times [0,R] \to \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$$

is continuously Fréchet differentiable whose Fréchet derivative at (y,ϕ,r) is given by

$$[D\mathcal{T}(y,\phi,r)](\eta,\chi,\xi) = A_{y,\phi,r}(\eta,\chi) + \xi B_{y,\phi,r}$$

for all $(\eta,\chi,\xi) \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N) \times \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N) \times \mathbb{R}$.

Proof. It is sufficient to show the continuity of

$$(y, \phi, r) \mapsto A_{y,\phi,r} \text{ and } (y, \phi, r) \mapsto B_{y,\phi,r}$$

with respect to the corresponding operator norms. Let

$$\|(\eta, \chi)\| := \|\eta\|_{\mathcal{W}^{1,p}[-R,T]} + \|\chi\|_{\mathcal{W}^{1,p}[-R,0]}$$

for each $(\eta, \chi) \in \mathcal{W}^{1,p}[-R, T] \times \mathcal{W}^{1,p}[-R, 0]$.

Step 1. The family of functions

$$(y, \phi) \mapsto A_{y,\phi,r},$$

where $r \in [0, R]$, is pointwise equicontinuous from Theorem 3.12. Therefore, we only have to show the continuity of

$$r \mapsto A_{y,\phi,r}$$

for each fixed $(y, \phi) \in \mathcal{W}^{1,p}[-R, T] \times \mathcal{W}^{1,p}[-R, 0]$.

Let $r_0 \in [0, R]$ be fixed. Since

$$\begin{aligned}
&|Df(\rho(y, \phi, r, t))\rho(\eta, \chi, r, t) - Df(\rho(y, \phi, r_0, t))\rho(\eta, \chi, r_0, t)| \\
&\quad \le \|Df(\rho(y, \phi, r, t)) - Df(\rho(y, \phi, r_0, t))\| \|\rho(\eta, \chi, r, t)\| \\
&\qquad + \|Df(\rho(y, \phi, r_0, t))\| \|\rho(\eta, \chi, r, t) - \rho(\eta, \chi, r_0, t)\| \\
&\quad \le 2 \sup_{t\in[0,T]} \|Df(\rho(y, \phi, r, t)) - Df(\rho(y, \phi, r_0, t))\| \cdot \|(\eta, \chi)\| \\
&\qquad + \sup_{t\in[0,T]} \|Df(\rho(y, \phi, r_0, t))\| \cdot |(\eta + \bar{\chi})(t - r) - (\eta + \bar{\chi})(t - r_0)|
\end{aligned}$$

for all $t \in [0, T]$, $r \in [0, R]$, and $(\eta, \chi) \in \mathcal{W}^{1,p}[-R, T] \times \mathcal{W}^{1,p}[-R, 0]$, we have

$$\begin{aligned}
&\|(A_{y,\phi,r} - A_{y,\phi,r_0})(\eta, \chi)\|_{\mathcal{W}^{1,p}[-R,T]} \\
&\quad = \left(\int_0^T |Df(\rho(y, \phi, r, t))\rho(\eta, \chi, r, t) - Df(\rho(y, \phi, r_0, t))\rho(\eta, \chi, r_0, t)|^p \, dt \right)^{\frac{1}{p}} \\
&\quad \le \left(2T^{\frac{1}{p}} \sup_{t\in[0,T]} \|Df(\rho(y, \phi, r, t)) - Df(\rho(y, \phi, r_0, t))\| \right) \|(\eta, \chi)\| \\
&\qquad + \sup_{t\in[0,T]} \|Df(\rho(y, \phi, r_0, t))\| \left(\int_0^T |(\eta + \bar{\chi})(t - r) - (\eta + \bar{\chi})(t - r_0)|^p \, dt \right)^{\frac{1}{p}}
\end{aligned}$$

by the Minkowski inequality. Here the estimate

$$\begin{aligned}
\left(\int_0^T |(\eta + \bar{\chi})(t - r) - (\eta + \bar{\chi})(t - r_0)|^p \, dt \right)^{\frac{1}{p}} &= \left(\int_0^T \left| \int_{-r_0}^{-r} (\eta + \bar{\chi})'(t + \theta) \, d\theta \right|^p dt \right)^{\frac{1}{p}} \\
&\le \|\eta + \bar{\chi}\|_{\mathcal{W}^{1,p}[-R,T]} |r - r_0| \\
&\le |r - r_0| \cdot \|(\eta, \chi)\|
\end{aligned}$$

follows from from Corollary A.2. As the conclusion, we obtain

$$\begin{aligned}
\|A_{y,\phi,r} - A_{y,\phi,r_0}\| \le 2T^{\frac{1}{p}} &\sup_{t\in[0,T]} \|Df(\rho(y, \phi, r, t)) - Df(\rho(y, \phi, r_0, t))\| \\
&+ \sup_{t\in[0,T]} \|Df(\rho(y, \phi, r_0, t))\| |r - r_0|,
\end{aligned}$$

which shows $\lim_{r\to r_0} \|A_{y,\phi,r} - A_{y,\phi,r_0}\| = 0$.

Step 2. From Theorem 3.10, the function

$$r \mapsto B_{y,\phi,r}$$

is continuous for each fixed $(y,\phi) \in \mathcal{W}^{1,p}[-R,T] \times \mathcal{W}^{1,p}[-R,0]$. Therefore, we only have to show the pointwise equicontinuity of the family of functions

$$(y,\phi) \mapsto B_{y,\phi,r},$$

where $r \in [0,R]$. This is indeed true in view of the following calculation:

$$\begin{aligned}
&\|B_{y,\phi,r} - B_{y_0,\phi_0,r}\|_{\mathcal{W}^{1,p}[-R,T]} \\
&\quad = \left(\int_0^T |D_2 f(\rho(y,\phi,r,t))(y+\bar{\phi})'(t-r) - D_2 f(\rho(y_0,\phi_0,r,t))(y_0+\bar{\phi}_0)'(t-r)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad \le \left(\sup_{t\in[0,T]} \|D_2 f(\rho(y,\phi,r,t)) - D_2 f(\rho(y_0,\phi_0,r,t))\|\right) \|y+\bar{\phi}\|_{\mathcal{W}^{1,p}[-R,T]} \\
&\qquad + \left(\sup_{t\in[0,T]} \|D_2 f(\rho(y_0,\phi_0,r,t))\|\right) \|(y-y_0,\phi-\phi_0)\|.
\end{aligned}$$

The detail has been omitted because this is similar to the case of the special case discussed in the previous subsection.

This completes the proof. □

See Definition 3.17 for the Fréchet differentiability of functions defined on sets which are not necessarily open.

3.2.5 C^1-smooth dependence of solutions on initial histories and delay

Let $1 \le p < \infty$ be given.

Theorem 3.15. *Let $B \subset \mathcal{W}^{1,p}([-R,0],\mathbb{R}^N)$ be an open subset which is bounded with respect to the supremum norm. Suppose that f is of class C^1. Then there exists $T > 0$ such that the function*

$$B \times [0,R] \ni (\phi,r) \mapsto x(\cdot;\phi,r)|_{[-R,T]} \in \mathcal{W}^{1,p}([-R,T],\mathbb{R}^N)$$

is well-defined and continuously Fréchet differentiable.

Proof.
Step 1. From the unique existence theorem (Proposition 3.7), there is $T > 0$ such that the family of functions

$$x(\cdot;\phi,r)|_{[-R,T]} \colon [-R,T] \to \mathbb{R}^N,$$

where $(\phi,r) \in B \times [0,R]$, is well-defined, i.e.,

$$T_{\phi,r} > T \quad (\forall (\phi,r) \in B \times [0,R]).$$

Step 2. By choosing small $T > 0$, we may assume that the family of maps

$$\mathcal{T}(\cdot;\phi,r) \colon \bar{\Gamma}_{1,p}(\delta) \to \Gamma_{1,p}(\delta),$$

where $(\phi, r) \in B \times [0, R]$, is a uniform contraction by the uniform contraction lemma (Lemma 3.5). Then the Banach fixed point theorem implies that $\mathcal{T}(\cdot;\phi, r)$ has the unique fixed point $y(\cdot;\phi, r) \in \Gamma_{1,p}(\delta)$ for each $(\phi, r) \in B \times [0, R]$ because $\bar{\Gamma}_{1,p}(\delta)$ is a complete metric space. By the uniqueness,

$$x(\cdot;\phi, r)|_{[-R,T]} = y(\cdot;\phi, r) + \bar{\phi}_{[-R,T]}$$

holds.

Step 3. By the C^1-smoothness theorem (Theorem 3.14), the function

$$\mathcal{T}\colon \Gamma_{1,p}(\delta) \times B \times [0, R] \to \Gamma_{1,p}(\delta)$$

is continuously Fréchet differentiable. Therefore, C^1-uniform contraction theorem implies that

$$B \times [0, R] \ni (\phi, r) \mapsto y(\cdot;\phi, r) \in \Gamma_{1,p}(\delta)$$

is continuously Fréchet differentiable. This shows that

$$B \times [0, R] \ni (\phi, r) \mapsto x(\cdot;\phi, r)|_{[-R,T]} \in \mathcal{W}^{1,p}[-R, T]$$

is also continuously Fréchet differentiable because

$$\mathcal{W}^{1,p}[-R, 0] \ni \phi \mapsto \bar{\phi}|_{[-R,T]} \in \mathcal{W}^{1,p}[-R, T]$$

is a continuous linear map. □

Remark 3.16. In the above theorem, the parameter set is not an open set. However, the C^1-uniform contraction theorem holds in this case by adopting the following Fréchet differentiability.

Definition 3.17 (Fréchet differentiability)**.** Let X, Y be normed spaces, $U \subset X$ be a subset, $x_0 \in U$ be a limit point of U, and $f\colon U \to Y$ be a function. f is said to be *Fréchet differentiable* at x_0 if there exists a unique linear approximation $A \in \mathcal{L}(X, Y)$ such that

$$\lim_{\|x - x_0\| \to 0 \text{ in } U} \frac{\|f(x) - f(x_0) - A(x - x_0)\|}{\|x - x_0\|} = 0.$$

Here $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X to Y. The above A is called the *Fréchet derivative* of f at x_0 and is denoted by $Df(x_0)$. f is said to be *Fréchet differentiable* when U is contained in the set of all limit points of U and f is Fréchet differentiable at every $x_0 \in U$.

3.2.6 C^1-smoothness of solution semiflow with a delay parameter

Let $1 \le p < \infty$ be given.

Theorem 3.18. *We define a map*

$$\Phi\colon \mathbb{R}_+ \times \mathcal{W}^{1,p}([-R, 0], \mathbb{R}^N) \times [0, R] \supset \mathrm{dom}(\Phi) \to \mathcal{W}^{1,p}([-R, 0], \mathbb{R}^N) \times [0, R]$$

by

$$\mathrm{dom}(\Phi) = \bigcup_{(\phi,r) \in \mathcal{W}^{1,p}[-R,0] \times [0,R]} [0, T_{\phi,r}) \times \{(\phi, r)\}, \quad \Phi(t, \phi, r) = (R_t x(\cdot;\phi, r), r).$$

Suppose that f is of class C^1. Then Φ is a C^1-maximal semiflow.

Proof.

Step 1. The unique existence theorem (Proposition 3.7) implies that Φ is a maximal semiflow with the escape time function $(\phi, r) \mapsto T_{\phi,r}$.

Step 2. By the continuity of orbit (Lemma 2.9),

$$[0, T_{\phi,r}) \ni t \mapsto \Phi(t, \phi, r) \in \mathcal{W}^{1,p}[-R, 0] \times [0, R]$$

is continuous for every $(\phi, r) \in \mathcal{W}^{1,p}[-R, 0] \times [0, R]$.

Step 3. Let $B \subset \mathcal{W}^{1,p}[-R, 0]$ be an open subset which is bounded with respect to the supremum norm. By the C^1-smooth dependence theorem (Theorem 3.15), there is $T > 0$ such that

$$B \times [0, R] \ni (\phi, r) \mapsto x(\cdot; \phi, r)|_{[-R,T]} \in \mathcal{W}^{1,p}[-R, T]$$

is a well-defined continuously Fréchet differentiable function, which implies

$$[0, T] \times B \times [0, R] \subset \operatorname{dom}(\Phi).$$

By combining the above Fréchet differentiability and the continuity of

$$[0, T] \times \mathcal{W}^{1,p}[-R, T] \ni (t, x) \mapsto R_t x \in \mathcal{W}^{1,p}[-R, 0]$$

(see Lemmas 2.9 and 2.10), we obtain the following properties:

- the continuity of $\Phi|_{[0,T]\times B\times[0,R]}$, i.e,
$$[0, T] \times B \times [0, R] \ni (t, \phi, r) \mapsto (R_t x(\cdot; \phi, r), r).$$
- the continuous Fréchet differentiability of $\Phi(t, \cdot, \cdot)|_{B\times[0,R]}$, i.e.,
$$B \times [0, R] \ni (\phi, r) \mapsto (R_t x(\cdot; \phi, r), r)$$
for each $t \in [0, T]$.

The above steps imply that Φ is a C^1-maximal semiflow from Theorems B.9 and B.13. □

4 Comments and discussion

This paper reveals that the history spaces of Sobolev type $\mathcal{W}^{1,p}([-R, 0], \mathbb{R}^N)$ $(1 \le p < \infty)$ arise as the history spaces for the C^1-smooth dependence on initial histories and delay, whose adoption is natural from the viewpoint of the differentiability of translation in L^p. This paper also extends the regularity of initial histories from the Lipschitz continuity and show that the topology induced by $\mathcal{W}^{1,p}$-norm is adapted, where the history space of the Lipschitz continuous functions with the topology induced by $\mathcal{W}^{1,1}$-norm is used in the previous studies (see [11] and [15]). Another feature of this paper is to prove the differentiability of solutions with respect to r at $r = 0$. It seems that there is some relationship with the C^1-smoothness of *special flow* for the small delay studied by Chicone [5].

The extension of this work to the time- and state- dependent delay case will be a next task. By a preparatory study, it is expected that this extension explains a meaning of the strict monotonicity of the delayed argument function, which is called the *temporal order of reactions* by Walther [26]. The study of the higher-order smoothness of solutions with respect to delay will also be a next task. The results in Subsection 3.1 suggest that it is appropriate to choose history spaces of higher-order Sobolev type, where other spaces based on $W^{k,\infty}$ are used in previous studies (see [4] and [14]).

A Differentiability of translation in L^p

We refer the reader to [24] and [3] for general references of theories of Lebesgue integration and Sobolev spaces, respectively.

Lemma A.1. *Let $f \in L^1(\mathbb{R}, \mathbb{R})$ and $a < b$ be given real numbers. Then for all $s, t \in \mathbb{R}$,*

$$\int_a^b \left| \int_s^t |f(x+y)| \, \mathrm{d}y \right| \mathrm{d}x \le \|f\|_{L^1(\mathbb{R})} |t-s|$$

holds.

Proof. It is sufficient to consider the case $s < t$. Let $A(t,s) \subset \mathbb{R}^2$ be the closed subset given by

$$A(t,s) := \{ (x,y) : a \le x \le b, \, x+s \le y \le x+t \},$$

which is Lebesgue measurable. Then the function

$$\mathbb{R}^2 \ni (x,y) \mapsto |f(y)| 1_{A(t,s)}(x,y) \in \mathbb{R}$$

is Lebesgue measurable. Since for each fixed $x \in [a,b]$,

$$\{ y \in \mathbb{R} : (x,y) \in A(t,s) \} = [x+s, x+t],$$

we have

$$\begin{aligned} \int_a^b \left(\int_s^t |f(x+y)| \, \mathrm{d}y \right) \mathrm{d}x &= \int_a^b \left(\int_{x+s}^{x+t} |f(y)| \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int_{[a,b]} \left(\int_{\mathbb{R}} |f(y)| 1_{A(t,s)}(x,y) \, \mathrm{d}y \right) \mathrm{d}x. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_a^b \left(\int_s^t |f(x+y)| \, \mathrm{d}y \right) \mathrm{d}x &= \int_{\mathbb{R}} \left(\int_{[a,b]} |f(y)| 1_{A(t,s)}(x,y) \, \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)| \left(\int_{[a,b]} 1_{A(t,s)}(x,y) \, \mathrm{d}x \right) \mathrm{d}y \\ &\le (t-s) \|f\|_{L^1(\mathbb{R})} \\ &< \infty \end{aligned}$$

is valid by Tonelli's theorem. □

Corollary A.2. *Let $1 \le p < \infty$, $f \in L^p(\mathbb{R}, \mathbb{R}^N)$, and $a < b$ be given real numbers. Then for all $s, t \in \mathbb{R}$,*

$$\left(\int_a^b \left| \int_s^t |f(x+y)| \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \le \|f\|_{L^p(\mathbb{R})} |t-s|$$

holds.

Proof. It is sufficient to consider the case $s < t$. Let q be the Hölder conjugate of p. Then for each fixed x, we have

$$\int_s^t |f(x+y)| \, \mathrm{d}y \le \left(\int_s^t |f(x+y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \cdot (t-s)^{\frac{1}{q}}.$$

Since $|f|^p \in L^1(\mathbb{R}, \mathbb{R})$, we obtain

$$\begin{aligned}\int_a^b \left(\int_s^t |f(x+y)| \, \mathrm{d}y \right)^p \mathrm{d}x &\le (t-s)^{\frac{p}{q}} \int_a^b \left(\int_s^t |f(x+y)|^p \, \mathrm{d}y \right) \mathrm{d}x \\ &\le (t-s)^{\frac{p}{q}} \cdot \| |f|^p \|_{L^1(\mathbb{R})} (t-s) \\ &\le (t-s)^{\frac{p}{q}+1} \|f\|_{L^p(\mathbb{R})}^p\end{aligned}$$

by applying Lemma A.1. Then the inequality is obtained because $(1/p) + (1/q) = 1$. □

Theorem A.3. *Let $1 \le p < \infty$, $f \in L^p(\mathbb{R}, \mathbb{R}^N)$, and $a < b$ be real numbers. Then for all $s, t, u \in \mathbb{R}$,*

$$\left(\int_a^b \left| \int_s^t (f(x+y) - f(x+u)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} = o(|t-s|)$$

as $|t-s| \to 0$ uniformly in u between s and t.

Proof. Let

$$F(x; s, t, u) := \int_s^t (f(x+y) - f(x+u)) \, \mathrm{d}y.$$

Then for each fixed $x \in [a, b]$, we have

$$\begin{aligned}F(x; s, t, u) &= \int_s^t f(x+y) \, \mathrm{d}y - (t-s) f(x+u) \\ &= \int_{s+x}^{t+x} f(y) \, \mathrm{d}y - (t-s) f(x+u),\end{aligned}$$

which is Lebesgue measurable in x.

Let $\varepsilon > 0$ be given. We choose $g \in C_\mathrm{c}(\mathbb{R}, \mathbb{R}^N)$ so that

$$\|f - g\|_{L^p(\mathbb{R})} \le \frac{\varepsilon}{3}.$$

Here $C_\mathrm{c}(\mathbb{R}, \mathbb{R}^N)$ denotes the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^N$ with compact support. By the Minkowski inequality,

$$\begin{aligned}\|F(\cdot; s, t, u)\|_{L^p[a,b]} &\le \left(\int_a^b \left| \int_s^t (f(x+y) - g(x+y)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\quad + \left(\int_a^b \left| \int_s^t (g(x+y) - g(x+u)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\quad + \left(\int_a^b \left| \int_s^t (g(x+u) - f(x+u)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}}.\end{aligned}$$

First term. By applying Corollary A.2, we obtain

$$\begin{aligned}\left(\int_a^b \left| \int_s^t (f(x+y) - g(x+y)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} &\le \left(\int_a^b \left| \int_s^t |f(x+y) - g(x+y)| \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\le \|f - g\|_{L^p(\mathbb{R})} |t-s|.\end{aligned}$$

Second term. Since g is uniformly continuous, there is $\delta > 0$ such that for all x, y, u, $|y-u| \le \delta$ implies

$$|g(x+y) - g(x+u)| \le \frac{\varepsilon}{3(b-a)^{1/p}}.$$

Therefore, $|t-s| \le \delta$ implies

$$\left| \int_s^t |g(x+y) - g(x+u)| \, \mathrm{d}y \right|^p \le \left[\frac{\varepsilon}{3(b-a)^{1/p}} |t-s| \right]^p$$

uniformly in u between s and t. Thus,

$$\begin{aligned} \left(\int_a^b \left| \int_s^t (g(x+y) - g(x+u)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} &\le \left(\int_a^b \left| \int_s^t |g(x+y) - g(x+u)| \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\le \left\{ \left[\frac{\varepsilon}{3(b-a)^{1/p}} |t-s| \right]^p (b-a) \right\}^{\frac{1}{p}} \\ &\le \frac{\varepsilon}{3} |t-s|. \end{aligned}$$

Third term. We have

$$\begin{aligned} \left(\int_a^b \left| \int_s^t (g(x+u) - f(x+u)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} &\le \left(\int_a^b \left| \int_s^t |g(x+u) - f(x+u)| \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &= \left(|t-s|^p \int_a^b |g(x+u) - f(x+u)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\ &\le \|f - g\|_{L^p(\mathbb{R})} |t-s|. \end{aligned}$$

By combining the above estimates, we finally obtain

$$\left(\int_a^b \left| \int_s^t (f(x+y) - f(x+u)) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \le \varepsilon |t-s|$$

for all $|t-s| \le \delta$ uniformly in u between s and t. □

Corollary A.4. *Let $1 \le p < \infty$. Let $a < b$ and $c, d \ge 0$ be given real numbers. If $f \in \mathcal{W}^{1,p}([a-c, b+d], \mathbb{R}^N)$, then for all $s, t, u \in [-c, d]$,*

$$\left(\int_a^b |f(x+t) - f(x+s) - (t-s) f'(x+u)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} = o(|t-s|)$$

as $|t-s| \to 0$ uniformly in u between s and t.

Proof. Let $x \in [a, b]$ and $s, t \in [-c, d]$. By the fundamental theorem of calculus for absolutely continuous functions, we have

$$f(x+t) - f(x+s) = \int_s^t f'(x+y) \, \mathrm{d}y,$$

which implies that for all u between s and t,

$$f(x+t) - f(x+s) - (t-s) f'(x+u) = \int_s^t (f'(x+y) - f'(x+u)) \, \mathrm{d}y.$$

Therefore, the conclusion is obtained applying Theorem A.3 for the extension of $f' \in L^p([a-c, b+d], \mathbb{R}^N)$ by 0 outside $[a-c, b+d]$. □

Remark A.5. The similar statement is given in [3, Exercise 8.13 in Chapter 8].

B Continuity and smoothness of maximal semiflows

Definition B.1 (Maximal semiflows)**.** Let X be a set and $D \subset \mathbb{R}_+ \times X$ be a subset. A map $\Phi\colon D \to X$ is called a *maximal semiflow* in X if the following conditions are satisfied:

(i) There exists a function $T_\Phi\colon X \to (0, \infty]$ such that

$$D = \bigcup_{x \in X} \bigl([0, T_\Phi(x)) \times \{x\}\bigr).$$

(ii) For all $x \in X$, $\Phi(0, x) = x$.

(iii) For all $t, s \in \mathbb{R}_+$ and all $x \in X$, both of the conditions $(t, x) \in D$ and $(s, \Phi(t, x)) \in D$ imply

$$(t + s, x) \in D \quad \text{and} \quad \Phi(t + s, x) = \Phi(s, \Phi(t, x)).$$

The above function T_Φ is called the *escape time function*.

Remark B.2. The condition (iii) means the maximality of domain of definition of Φ. In terms of the escape time function T_Φ, (iii) is equivalent to the following: both of $t < T_\Phi(x)$ and $s < T_\Phi(\Phi(t, x))$ imply $t + s < T_\Phi(x)$. The terminology of maximal semiflows comes from [21].

Definition B.3 (Time-t map)**.** Let Φ be a maximal semiflow in a set X with the escape time function $T_\Phi\colon X \to (0, \infty]$. For each $t \in \mathbb{R}_+$, the map $\Phi^t\colon \operatorname{dom}(\Phi^t) \to X$ defined by

$$\operatorname{dom}(\Phi^t) = \{\, x \in X : T_\Phi(x) > t \,\} \quad \text{and} \quad \Phi^t(x) = \Phi(t, x)$$

is called the *time-t map* of Φ.

Definition B.4 (Lower semicontinuity)**.** Let X be a topological space, $x_0 \in X$, and $f\colon X \to (0, \infty]$ be a function. f is said to be *lower semicontinuous* at x_0 if for every $M < f(x_0)$, there exists a neighborhood N of x_0 such that for all $x \in N$, $f(x) > M$. f is said to be *lower semicontinuous* if f is lower semicontinuous at every $x_0 \in X$.

Definition B.5 (C^0-maximal semiflows)**.** Let X be a topological space and $\Phi\colon \operatorname{dom}(\Phi) \to X$ be a maximal semiflow in X. Φ is called a *C^0-maximal semiflow* if Φ is a continuous map and the escape time function $T_\Phi\colon X \to (0, \infty]$ is lower semicontinuous.

Remark B.6. In [10], a C^0-maximal semiflow is called a continuous local semi-dynamical system.

The proofs of the following two lemmas are straightforward and can be omitted.

Lemma B.7. *Let $\Phi\colon \operatorname{dom}(\Phi) \to X$ be a maximal semiflow in a topological space X with the escape time function $T_\Phi\colon X \to (0, \infty]$. Then the following properties are equivalent:*

(a) $T_\Phi\colon X \to (0, \infty]$ *is lower semicontinuous.*

(b) $\operatorname{dom}(\Phi)$ *is an open set of* $\mathbb{R}_+ \times X$.

Lemma B.8. *Let Φ be a C^0-maximal semiflow in a topological space X with the escape time function $T_\Phi\colon X \to (0, \infty]$. Then for each $t \in \mathbb{R}_+$,*

$$\{\, x \in X : T_\Phi(x) > t \,\}$$

is an open subset of X.

The following theorem states that the local continuity property of maximal semiflows can induce their global continuity property. We omit the proof because a similar statement is proved in [23, Theorem A.7].

Theorem B.9. *Let $\Phi\colon \mathrm{dom}(\Phi) \to X$ be a maximal semiflow in a topological space X with the escape time function $T_\Phi\colon X \to (0,\infty]$. Suppose that for every $x \in X$, the orbit $[0, T_\Phi(x)) \ni t \mapsto \Phi(t,x) \in X$ is continuous. If for every $x \in X$, there exist $T > 0$ and a neighborhood N of x in such that $[0,T] \times N \subset \mathrm{dom}(\Phi)$ and $\Phi|_{[0,T]\times N}$ is continuous, then Φ is a C^0-maximal semiflow.*

Remark B.10. In [10, Theorem 15], the conclusion is obtained under the weaker assumption that for every $(t,x) \in \mathrm{dom}(\Phi)$, $\Phi([0,t] \times \{x\})$ is compact. The proof is based on the notion of germs.

Definition B.11 (C^1-maximal semiflows)**.** Let X be a normed space and $\Omega \subset X$ be a subset contained in the set of all limit points of Ω. A C^0-maximal semiflow $\Phi\colon \mathrm{dom}(\Phi) \to \Omega$ is called a C^1-*maximal semiflow* if each time-t map Φ^t is continuously Fréchet differentiable.

Remark B.12. In the setting of Definition B.11, $\mathrm{dom}(\Phi^t)$ is open in Ω from Lemma B.8. Therefore, $\mathrm{dom}(\Phi^t) = U \cap \Omega$ holds for some open set U of X. This implies that $\mathrm{dom}(\Phi^t)$ is also contained in the set of all limit points of $\mathrm{dom}(\Phi^t)$, and it is meaningful to consider the continuous Fréchet differentiability of each Φ^t.

By definition, a C^1-maximal semiflow is not necessarily continuously Fréchet differentiable (see [21, p. 260]).

The following theorem ensures that a C^0-maximal semiflow is of class C^1 provided that the maximal semiflow has a local smoothness property. The proof is similar to that of [25, Theorem 1].

Theorem B.13. *Let X be a normed space, $\Omega \subset X$ be a subset contained in the set of all limit points of Ω, and $\Phi\colon \mathrm{dom}(\Phi) \to \Omega$ be a C^0-maximal semiflow with the escape time function $T_\Phi\colon \Omega \to (0,\infty]$. Suppose that for any function $f\colon \Omega \to X$, a linear approximation at every $x \in \Omega$ is unique if it exists. If for every $x \in \Omega$, there exist $T > 0$ and an open neighborhood N of x such that*

- *$[0,T] \times N \cap \Omega \subset \mathrm{dom}(\Phi)$ and*
- *$\Phi^t|_{N\cap\Omega}$ is continuously Fréchet differentiable for every $t \in [0,T]$,*

then Φ is a C^1-maximal semiflow.

Proof.

Step 1. For each $x \in \Omega$, we define a subset $S_x \subset (0, T_\Phi(x))$ by the following manner: $T \in S_x$ if there exists an open neighborhood N of x such that

- $[0,T] \times N \cap \Omega \subset \mathrm{dom}(\Phi)$ and
- $\Phi^t|_{N\cap\Omega}$ is continuously Fréchet differentiable for every $t \in [0,T]$.

By the assumptions, $S_x \neq \emptyset$, and therefore, $\sup(S_x) \in (0, T_\Phi(x)]$ exists. If $\sup(S_x) = T_\Phi(x)$ for all $x \in \Omega$, then every Φ^t is continuously Fréchet differentiable.

Let $x_0 \in \Omega$ be fixed.

Step 2. We suppose

$$t_* := \sup(S_{x_0}) < T_\Phi(x_0)$$

and derive a contradiction. We note that one cannot conclude $t_* \in S_{x_0}$ in general. Let

$$x_* := \Phi(t_*, x_0) \in \Omega.$$

By the assumptions, we can choose $T_* > 0$ and an open neighborhood N_* of x_* so that

- $[0, T_*] \times N_* \cap \Omega \subset \mathrm{dom}(\Phi)$ and
- $\Phi^t|_{N_* \cap \Omega}$ is continuously Fréchet differentiable for every $t \in [0, T_*]$.

Step 3. Since $[0, T_\Phi(x_0)) \ni t \mapsto \Phi(t, x_0)$ is continuous at t_*, we can choose t' so that

$$t_* - \frac{T_*}{2} < t' < t_* \text{ and } \Phi(t', x_0) \in N_* \cap \Omega.$$

We can also choose an open neighborhood N' of x_0 such that

- $[0, t'] \times N' \cap \Omega \subset \mathrm{dom}(\Phi)$,
- $\Phi^t|_{N' \cap \Omega}$ is continuously Fréchet differentiable for every $t \in [0, t']$, and
- $\Phi^{t'}(N' \cap \Omega) \subset N_* \cap \Omega$

because $t' < t_*$ and $\Phi^{t'}$ is continuous at x_0. Then for all $t \in [t', t' + T_*]$ and all $x \in N' \cap \Omega$,

$$(t', x) \in \mathrm{dom}(\Phi) \text{ and } (t - t', \Phi(t', x)) \in [0, T_*] \times N_* \cap \Omega \subset \mathrm{dom}(\Phi),$$

which implies

$$(t, x) = (t' + (t - t'), x) \in \mathrm{dom}(\Phi)$$

by the maximality. Therefore,

$$\begin{aligned}[0, t' + T_*] \times N' \cap \Omega &= ([0, t'] \times N' \cap \Omega) \cup ([t', t' + T_*] \times N' \cap \Omega) \\ &\subset \mathrm{dom}(\Phi).\end{aligned}$$

Step 4. For every $t \in [t', t' + T_*]$, we have

$$\Phi^t|_{N' \cap \Omega} = \Phi^{t-t'}|_{N* \cap \Omega} \circ \Phi^{t'}|_{N' \cap \Omega}.$$

Since the two maps in the right-hand side are continuously Fréchet differentiable, $\Phi^t|_{N' \cap \Omega}$ is also continuously Fréchet differentiable. Therefore,

$$t_* < t_* + \frac{T_*}{2} < t' + T_* \in S_{x_0},$$

which is a contradiction. Thus, $t_* = T_\Phi(x_0)$ follows.

By the above steps, the conclusion is obtained. □

Acknowledgements

This work was supported by the Research Alliance Center for Mathematical Sciences, Tohoku University, the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University, JSPS A3 Foresight Program, and JSPS KAKENHI Grant Number JP17H06460, JP19K14565.

References

[1] O. Arino, K. P. Hadeler, M. L. Hbid, Existence of periodic solutions for delay differential equations with state dependent delay, *J. Differential Equations* **144**(1998), No. 2, 263–301. https://doi.org/10.1006/jdeq.1997.3378; MR1616960; Zbl 0913.34057

[2] H. T. Banks, D. Robbins, K. L. Sutton, Theoretical foundations for traditional and generalized sensitivity functions for nonlinear delay differential equations, *Math. Biosci. Eng.* **10**(2013), No. 5–6, 1301–1333. https://doi.org/10.3934/mbe.2013.10.1301; MR3103332; Zbl 1273.34083

[3] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7; MR2759829; Zbl 1220.46002

[4] Y. Chen, Q. Hu, J. Wu, Second-order differentiability with respect to parameters for differential equations with adaptive delays, *Front. Math. China* **5**(2010), No. 2, 221–286. https://doi.org/10.1007/s11464-010-0005-9; MR2610723; Zbl 1201.34096

[5] C. Chicone, Inertial and slow manifolds for delay equations with small delays, *J. Differential Equations* **190**(2003), No. 2, 364–406. https://doi.org/10.1016/S0022-0396(02)00148-1; MR1970035; Zbl 1044.34026

[6] S. N. Chow, J. K. Hale, *Methods of bifurcation theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], Vol. 251, Springer-Verlag, New York–Berlin, 1982. https://doi.org/10.1007/978-1-4613-8159-4; MR0660633; Zbl 0487.47039

[7] R. D. Driver, Existence and continuous dependence of solutions of a neutral functional-differential equation, *Arch. Rational Mech. Anal.* **19**(1965), 149–166. https://doi.org/10.1007/BF00282279; MR0179406; Zbl 0148.05703

[8] R. D. Driver, *Ordinary and delay differential equations*, Applied Mathematical Sciences, Vol. 20. Springer-Verlag, New York–Heidelberg, 1977. https://doi.org/10.1007/978-1-4684-9467-9; MR0477368; Zbl 0374.34001

[9] T. Erneux, *Applied delay differential equations*, Surveys and Tutorials in the Applied Mathematical Sciences, Vol. 3, Springer, New York, 2009. https://doi.org/10.1007/978-0-387-74372-1; MR2498700; Zbl 1201.34002

[10] O. Hájek, Local characterisation of local semi-dynamical systems, *Math. Systems Theory* **2**(1968), 17–25. https://doi.org/10.1007/BF01691343; MR0239575; Zbl 0155.50703

[11] J. K. Hale, L. A. C. Ladeira, Differentiability with respect to delays, *J. Differential Equations* **92**(1991), No. 1, 14–26. https://doi.org/10.1016/0022-0396(91)90061-D; MR1113586; Zbl 0735.34045

[12] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7; MR1243878; Zbl 0787.34002

[13] F. Hartung, Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study, in: *Proceedings of the Third World Congress of Nonlinear Analysts, Part 7 (Catania, 2000)*, *Nonlinear Anal.* **47**(2001), No. 7, 4557–4566. https://doi.org/10.1016/S0362-546X(01)00569-7; MR1975850; Zbl 1042.34582

[14] F. Hartung, On second-order differentiability with respect to parameters for differential equations with state-dependent delays, *J. Dynam. Differential Equations* **25**(2013), No. 4, 1089–1138. https://doi.org/10.1007/s10884-013-9330-5; MR3138158; Zbl 1287.34052

[15] F. Hartung, Differentiability of solutions with respect to the delay function in functional differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 73, 1–16. https://doi.org/10.14232/ejqtde.2016.1.73; MR3547449; Zbl 1389.34190

[16] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: theory and applications, in: *Handbook of differential equations: ordinary differential equations*, *Vol. III*, pp. 435–545, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006. https://doi.org/10.1016/S1874-5725(06)80009-X; MR2457636

[17] F. Hartung, J. Turi, On differentiability of solutions with respect to parameters in state-dependent delay equations, *J. Differential Equations* **135** (1997), No. 2, 192–237. https://doi.org/10.1006/jdeq.1996.3238; MR1441270; Zbl 0877.34045

[18] V. Kolmanovskii, A. Myshkis, *Introduction to the theory and applications of functional-differential equations*, Mathematics and its Applications, Vol. 463, Kluwer Academic Publishers, Dordrecht, 1999. https://doi.org/10.1007/978-94-017-1965-0; MR1680144; Zbl 0917.34001

[19] Y. A. Kuznetsov, *Elements of applied bifurcation theory*, 3rd edition, Applied Mathematical Sciences, Vol. 112. Springer-Verlag, New York, 2004. https://doi.org/10.1007/978-1-4757-3978-7; MR2071006; Zbl 1082.37002

[20] M. Louihi, M. L. Hbid, O. Arino, Semigroup properties and the Crandall Liggett approximation for a class of differential equations with state-dependent delays, *J. Differential Equations* **181**(2002), No. 1, 1–30. https://doi.org/10.1006/jdeq.2001.4076; MR1900458; Zbl 1006.34061

[21] J. E. Marsden, M. McCracken, *The Hopf bifurcation and its applications*, With contributions by P. Chernoff, G. Childs, S. Chow, J. R. Dorroh, J. Guckenheimer, L. Howard, N. Kopell, O. Lanford, J. Mallet-Paret, G. Oster, O. Ruiz, S. Schecter, D. Schmidt and S. Smale. Applied Mathematical Sciences, Vol. 19, Springer-Verlag, New York, 1976. https://doi.org/10.1007/978-1-4612-6374-6; MR0494309; Zbl 0545.58002

[22] W. R. Melvin, A class of neutral functional differential equations, *J. Differential Equations* **12**(1972), 524–534. https://doi.org/10.1016/0022-0396(72)90023-X; MR0340761; Zbl 0234.34083

[23] J. Nishiguchi, A necessary and sufficient condition for well-posedness of initial value problems of retarded functional differential equations, *J. Differential Equations* **263**(2017), No. 6, 3491–3532. https://doi.org/10.1016/j.jde.2017.04.038; MR3659369; Zbl 1370.34121

[24] T. Tao, *An introduction to measure theory*, Graduate Studies in Mathematics, Vol. 126, American Mathematical Society, Providence, RI, 2011. https://doi.org/10.1090/gsm/126; MR2827917; Zbl 1231.28001

[25] H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, *J. Differential Equations* **195**(2003), No. 1, 46–65. https://doi.org/10.1016/j.jde.2003.07.001; MR2019242; Zbl 1045.34048

[26] H.-O. Walther, Algebraic-delay differential systems, state-dependent delay, and temporal order of reactions, *J. Dynam. Differential Equations* **21**(2009), No. 1, 195–232. https://doi.org/10.1007/s10884-009-9129-6; MR2482014; Zbl 1167.34026