A model for spatial spreading and dynamics of fox rabies on a growing domain

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Abstract. In order to explore the impact of the growth rate of the habitat on the transmission of rabies, we consider a SEI model for fox rabies on a growing spatial domain. The basic reproduction number is introduced using the next infection operator, spectral analysis and the corresponding eigenvalue problem. The stability of equilibria is also established using the upper and lower solutions method in terms of this number. Our results show that a large growth rate of the domain has a negative impact on the prevention and control of rabies. Numerical simulations are presented to verify our theoretical results.

Keywords: SEI model, fox rabies, growing domain, basic reproduction number, stability.

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1 Introduction

Rabies, an acute infectious disease caused by virus infecting the central nervous system, is mainly transmitted by direct contact such as biting [3]. Most mammals are susceptible to the disease, and although only very few human fatalities occur every year, rabies is still a considerable threat to human beings on account of inefficient treatment and a nearly 100% mortality rate once it reaches the clinical stage [11]. In order to develop public policies for prevention and control of rabies, various mathematical models have been established to study the transmission mechanism of rabies.

The red fox is the main carrier of rabies in Europe [2]. The following SEI model for fox rabies was proposed and studied by Murray et al. in [17]:

$$\begin{cases} E_t = \beta I S - \sigma E - \left[b + (a - b)\frac{N}{K}\right] E, \\ I_t = D\Delta I + \sigma E - \alpha I - \left[b + (a - b)\frac{N}{K}\right] I, \\ S_t = (a - b)S\left(1 - \frac{N}{K}\right) - \beta I S, \end{cases} \quad (1.1)$$

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where $S(x,t)$, $E(x,t)$ and $I(x,t)$ are the densities of susceptible foxes, infected but non-infectious foxes and rabid foxes at location x and time t, respectively. $N = E + I + S$ is the total fox population. On account of the random wandering of the rabid foxes, the diffusion coefficient D is introduced in the equation for I. α represents the mortality rate of the rabid foxes and β is the disease transmission coefficient. We assume that infected foxes become infectious at the per capita rate σ. a is the birth rate, b is the intrinsic death rate and K is the environmental carrying capacity. The term $(a-b)\frac{N}{K}$ denotes the depletion of the food supply by all foxes, where $a > b$ ensures a sustainable population size. All coefficients in the model (1.1) are nonnegative constants.

Letting $W = K - S$, model (1.1) becomes

$$\begin{cases} E_t = \beta I(K-W) - \sigma E - \left[b + (a-b)\frac{N}{K}\right]E, \\ I_t = D\Delta I + \sigma E - \alpha I - \left[b + (a-b)\frac{N}{K}\right]I, \\ W_t = -(a-b)(K-W)\left(1 - \frac{N}{K}\right) + \beta I(K-W), \end{cases} \quad (1.2)$$

where $N = E + I + K - W$ is the total fox population.

Problems describing ecological models on fixed spatial domains have been extensively investigated in the literature. However, the habitats of species in nature are not invariable. Some habitats are affected by climate, temperature and rainfall, and the shifting boundaries are known, for example the area of Dongting Lake in China changes by season, that is, Dongting lake covers an average area of 1814 square kilometres in summer while it covers only 568 square kilometres in winter in the period 1996 to 2016, see [12,15,16,18,22,26] and references therein. Some habitats are influenced by the species itself and the boundaries are moving and unknown. Such boundaries have recently been described by free boundaries, which have been studied in [9,13,23] and [24] for invasive species and in [14] for the transmission of disease. Domain growth, as one possibility for domain evolution, plays an important role in the formation of living patterns.

Inspired by the aforementioned works, we consider a SEI model (1.2) on a growing domain as in [7] and [8]. Let $\Omega_t \subset \mathbb{R}^2$ be a bounded growing domain at time t, and its growing boundary is denoted $\partial\Omega_t$. Also we assume that $E(x(t),t)$, $I(x(t),t)$ and $W(x(t),t)$ are the densities of the three kinds of fox population at location $x(t) \in \Omega_t$ and time t. Additionally, the growth of the domain Ω_t generates a flow velocity $\mathbf{a} = \dot{x}(t)$, that is, the flow velocity is identical to the domain velocity. According to the principle of mass conservation and the Reynolds transport theorem [1], we can formulate the problem on a growing domain related to (1.2) as

$$\begin{cases} E_t + \mathbf{a}\cdot\nabla E + E(\nabla\cdot\mathbf{a}) = \beta I(K-W) - \sigma E - \left[b + (a-b)\frac{N}{K}\right]E & \text{in } \Omega_t, \\ I_t - D\Delta I + \mathbf{a}\cdot\nabla I + I(\nabla\cdot\mathbf{a}) = \sigma E - \alpha I - \left[b + (a-b)\frac{N}{K}\right]I & \text{in } \Omega_t, \\ W_t + \mathbf{a}\cdot\nabla W + W(\nabla\cdot\mathbf{a}) = -(a-b)(K-W)\left(1-\frac{N}{K}\right) + \beta I(K-W) & \text{in } \Omega_t, \\ E(x(t),t) = I(x(t),t) = W(x(t),t) = 0 & \text{on } \partial\Omega_t, \\ E(x(0),0) = E_0(x), I(x(0),0) = I_0(x), W(x(0),0) = W_0(x) & \text{in } \overline{\Omega}_0. \end{cases} \quad (1.3)$$

Here $\mathbf{a}\cdot\nabla E$, $\mathbf{a}\cdot\nabla I$ and $\mathbf{a}\cdot\nabla W$ are called advection terms related to the transport of material across $\partial\Omega_t$ with the flow $\mathbf{a}$, and other extra terms introduced by the growth of the domain Ω_t are the dilution terms $E(\nabla\cdot\mathbf{a})$, $I(\nabla\cdot\mathbf{a})$ and $W(\nabla\cdot\mathbf{a})$ due to the local volume expansion [5]. The null Dirichlet boundary conditions mean that there is no infection outside the growing domain and on the boundary.

In order to simplify problem (1.3), we assume that the growth of the domain Ω_t is uniform and isotropic. Biologically, the infected domain Ω_t is supposed to grow at the same rate $\rho(t)$ in all directions as time t increases. Mathematically, we can formulate this as

$$x(t) = \rho(t)y \quad \text{for all } x(t) \in \Omega_t \text{ and } (y,t) \in \Omega_0 \times [0, +\infty),$$

where $\rho(t) \in C^1[0, +\infty)$ is called the growth function and satisfies

$$\rho(0) = 1, \qquad \dot{\rho}(t) > 0, \qquad \lim_{t \to \infty} \rho(t) = \rho_\infty > 1 \quad \text{and} \quad \lim_{t \to \infty} \dot{\rho}(t) = 0.$$

By Lagrangian transformations (see e.g. [4]), we define $E(x(t), t) = u_1(y, t)$, $I(x(t), t) = u_2(y, t)$ and $W(x(t), t) = u_3(y, t)$. Then we have

$$u_{1t} = E_t + \mathbf{a} \cdot \nabla E, \; u_{2t} = I_t + \mathbf{a} \cdot \nabla I, \; u_{3t} = W_t + \mathbf{a} \cdot \nabla W,$$

$$\mathbf{a} = \dot{x}(t) = \dot{\rho}(t)y = \frac{\dot{\rho}(t)}{\rho(t)}x(t),$$

$$\nabla \cdot \mathbf{a} = \frac{n\dot{\rho}(t)}{\rho(t)}, \; \Delta I = \frac{1}{\rho^2(t)}\Delta u_2$$

and problem (1.3) can be transformed into the following reaction-diffusion model on the fixed domain Ω_0

$$\begin{cases} u_{1t} = \beta u_2(K - u_3) - \sigma u_1 - \left[b + (a - b)\frac{N}{K}\right]u_1 - \frac{n\dot{\rho}(t)}{\rho(t)}u_1, & y \in \Omega_0, \; t > 0, \\ u_{2t} - \frac{D}{\rho^2(t)}\Delta u_2 = \sigma u_1 - \alpha u_2 - \left[b + (a - b)\frac{N}{K}\right]u_2 - \frac{n\dot{\rho}(t)}{\rho(t)}u_2, & y \in \Omega_0, \; t > 0, \\ u_{3t} = -(a - b)(K - u_3)\left(1 - \frac{N}{K}\right) + \beta u_2(K - u_3) - \frac{n\dot{\rho}(t)}{\rho(t)}u_3, & y \in \Omega_0, \; t > 0, \\ u_1(y, t) = u_2(y, t) = u_3(y, t) = 0, & y \in \partial\Omega_0, \; t > 0, \\ u_1(y, 0) := \eta_1(y), u_2(y, 0) := \eta_2(y), u_3(y, 0) := \eta_3(y), & y \in \overline{\Omega}_0, \end{cases} \quad (1.4)$$

where $N = u_1 + u_2 + K - u_3$ is the total fox population.

The rest of the paper is organized as follows: Section 2 is devoted to the basic reproduction number of problem (1.4) as well as its analytic properties. In Section 3, we investigate the stability of the disease-free steady state. Numerical simulations and the discussion are finally presented in Sections 4 and 5, respectively.

2 The basic reproduction number

In this section, we first present the principal eigenvalue R_0^* of the linearized system of problem (1.4) at $(0, 0, 0)$, then define the basic reproduction number R_0 and analyze its properties. Epidemiologically, the basic reproduction number is a critical threshold that reflects whether the disease will be spread or disappear.

Problem (1.4) admits a disease-free steady state $(0, 0, 0)$. Linearizing system (1.4) at $(0, 0, 0)$ and recalling that $\dot{\rho}(t) \to 0$ as $t \to \infty$, we are led to consider the system

$$\begin{cases} u_t = \beta K v - (\sigma + a)u, & y \in \Omega_0, \; t > 0, \\ v_t - \frac{D\Delta v}{\rho_\infty^2} = \sigma u - (\alpha + a)v, & y \in \Omega_0, \; t > 0, \\ w_t = (a - b)(u + v - w) + \beta K v, & y \in \Omega_0, \; t > 0. \end{cases} \quad (2.1)$$

Since the first two equations of (2.1) are decoupled from the last equation, we consider the following eigenvalue problem

$$
\begin{cases}
0 = \frac{\beta K \psi}{R_0^*} - (\sigma + a)\phi, & y \in \Omega_0, \\
-\frac{D \Delta \psi}{\rho_\infty^2} = \frac{\sigma \phi}{R_0^*} - (\alpha + a)\psi, & y \in \Omega_0, \\
\phi(y) = \psi(y) = 0, & y \in \partial\Omega_0,
\end{cases}
\tag{2.2}
$$

which is equivalent to the eigenvalue problem

$$
\begin{cases}
-\frac{D \Delta \psi}{\rho_\infty^2} = \frac{\sigma \beta K \psi}{(\sigma+a)(R_0^*)^2} - (\alpha + a)\psi, & y \in \Omega_0, \\
\phi(y) = 0, & y \in \partial\Omega_0.
\end{cases}
\tag{2.3}
$$

Direct calculation shows that the principal eigenvalue of problem (1.4)

$$
R_0^* = \sqrt{\frac{\sigma \beta K}{(\sigma + a)(\frac{D}{\rho_\infty^2}\lambda_1 + \alpha + a)}},
\tag{2.4}
$$

where $(\lambda_1, \zeta(y))$ is the principal eigen-pair of the eigenvalue problem

$$
\begin{cases}
-\Delta \zeta = \lambda_1 \zeta, & y \in \Omega_0, \\
\zeta(y) = 0, & y \in \partial\Omega_0.
\end{cases}
\tag{2.5}
$$

Now we define the basic reproduction number R_0. Similarly as in [25] and [27], we write the first two equations of (2.1) as the following equivalent single equation:

$$
\begin{cases}
U_t = d\Delta U + FU - VU, & y \in \Omega_0, \ t > 0, \\
u = v = 0, & y \in \partial\Omega_0, \ t > 0,
\end{cases}
$$

where $U = (u, v)^T$, $d = (0, D)^T$,

$$
F = \begin{pmatrix} 0 & \beta K \\ 0 & 0 \end{pmatrix},
$$

$$
V = \begin{pmatrix} \sigma + a & 0 \\ -\sigma & \alpha + a \end{pmatrix}.
$$

Let $X_1 = C(\overline{\Omega}_0, \mathbb{R}^2)$ and $X_1^+ := C(\overline{\Omega}_0, \mathbb{R}_+^2)$, and let $T(t)$ be the solution semigroup of the following system on X_1

$$
\begin{cases}
U_t = d\Delta U - VU, & y \in \Omega_0, \ t > 0, \\
u = v = 0, & y \in \partial\Omega_0, \ t > 0,
\end{cases}
$$

and let $\phi(y)$ be the density of the initial infectious fox population. Define the next infection operator L by

$$
L(\phi)(y) := \int_0^\infty F(y)[T(t)\phi](y)dt = F(y)\int_0^\infty [T(t)\phi](y)dt.
$$

Then $R_0 = r(L)$, where $r(L)$ is the spectral radius of L. We have the following result, we refer to Theorem 11.3.3 in [27] for more details:

Lemma 2.1. $R_0 = R_0^*$ *and* $\text{sign}\,(1 - R_0) = \text{sign}\,\lambda^*$, *where* λ^* *is the principal eigenvalue of the following eigenvalue problem*

$$\begin{cases} 0 = \beta K \psi - (\sigma + a)\phi + \lambda \phi, & y \in \Omega_0, \\ -\frac{D\Delta \psi}{\rho_\infty^2} = \sigma \phi - (\alpha + a)\psi + \lambda \psi, & y \in \Omega_0, \\ \phi(y) = \psi(y) = 0, & y \in \partial\Omega_0. \end{cases} \tag{2.6}$$

According to the explicit expression of R_0, we can list some properties of R_0.

Theorem 2.2. *The following assertions hold.*

(i) $R_0(\rho_\infty, \Omega)$ *is a positive and strictly increasing function with respect to* Ω, *that is*, $R_0(\rho_\infty, \Omega_1) \leq R_0(\rho_\infty, \Omega_2)$ *provided that* $\Omega_1 \subseteq \Omega_2$, *with strict inequality if* $\Omega_2 \backslash \overline{\Omega}_1$ *is a non-empty open set*;

(ii) $R_0(\rho_\infty, \Omega)$ *is a monotonically increasing function with respect to* ρ_∞, *in the sense that* $R_0(\rho_\infty, \Omega) < R_0(\rho_\infty^*, \Omega)$ *provided that* $\rho_\infty < \rho_\infty^*$.

Proof. The proof of the monotonicity in (i) is similar to Corollary 2.3 in [6]. The proof of (ii) follows directly from (2.4). $\qquad \square$

Remark 2.3. The basic reproduction number is used as a threshold parameter for the transmission mechanism of the disease and plays a central role in mathematical epidemiology. Biologically, R_0 is the average number of new infections produced by a typical infective individual over its infection period. R_0 can be obtained by the second generation matrix method [10] for epidemic models described by spatially-independent systems, and it can be calculated as the spectral radius of the next-generation operator for models in a constant environment [25] or in a periodic environment [27].

3   The stability of the disease-free equilibrium

In this section we will investigate the stability of the disease-free equilibrium $(0, 0, 0)$ in terms of the threshold R_0. First we introduce the definition of the pair of coupled upper and lower solutions.

Definition 3.1. Let $(\tilde{u}_1(y, t), \tilde{u}_2(y, t), \tilde{u}_3(y, t)), (\hat{u}_1(y, t), \hat{u}_2(y, t), \hat{u}_3(y, t))$ be a pair of (triplets of) functions in $C^{2,1}(\Omega_0 \times (0, +\infty)) \bigcap C(\bar{\Omega}_0 \times [0, +\infty))$, satisfying $(0, 0, 0) \leq (\hat{u}_1, \hat{u}_2, \hat{u}_3) \leq (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \leq (K, K, K)$. The pair (of triplets) is called coupled upper and lower solutions of (1.4), if the following relations are satisfied:

$$\begin{cases} \hat{u}_{1t} \leq \beta \hat{u}_2 (K - \tilde{u}_3) - \sigma \hat{u}_1 - \left[b + (a - b)\frac{\hat{u}_1 + \tilde{u}_2 + K - \hat{u}_3}{K} \right] \hat{u}_1 - \frac{n\dot{\rho}(t)}{\rho(t)} \hat{u}_1, \\ \hat{u}_{2t} - \frac{D}{\rho^2(t)} \Delta \hat{u}_2 \leq \sigma \hat{u}_1 - \alpha \hat{u}_2 - \left[b + (a - b)\frac{\tilde{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K} \right] \hat{u}_2 - \frac{n\dot{\rho}(t)}{\rho(t)} \hat{u}_2, \\ \hat{u}_{3t} \leq -(a - b)(K - \hat{u}_3)\left(1 - \frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K} \right) + \beta \hat{u}_2 (K - \hat{u}_3) - \frac{n\dot{\rho}(t)}{\rho(t)} \hat{u}_3, \\ \tilde{u}_{1t} \geq \beta \tilde{u}_2 (K - \hat{u}_3) - \sigma \tilde{u}_1 - \left[b + (a - b)\frac{\tilde{u}_1 + \hat{u}_2 + K - \tilde{u}_3}{K} \right] \tilde{u}_1 - \frac{n\dot{\rho}(t)}{\rho(t)} \tilde{u}_1, \\ \tilde{u}_{2t} - \frac{D}{\rho^2(t)} \Delta \tilde{u}_2 \geq \sigma \tilde{u}_1 - \alpha \tilde{u}_2 - \left[b + (a - b)\frac{\hat{u}_1 + \tilde{u}_2 + K - \tilde{u}_3}{K} \right] \tilde{u}_2 - \frac{n\dot{\rho}(t)}{\rho(t)} \tilde{u}_2, \\ \tilde{u}_{3t} \geq -(a - b)(K - \tilde{u}_3)\left(1 - \frac{\tilde{u}_1 + \tilde{u}_2 + K - \tilde{u}_3}{K} \right) + \beta \tilde{u}_2 (K - \tilde{u}_3) - \frac{n\dot{\rho}(t)}{\rho(t)} \tilde{u}_3, & y \in \Omega_0,\ t > 0, \\ \hat{u}_1(y, t) = 0 \leq \tilde{u}_1(y, t), \hat{u}_2(y, t) = 0 \leq \tilde{u}_2(y, t), \hat{u}_3(y, t) = 0 \leq \tilde{u}_3(y, t), & y \in \partial\Omega_0,\ t > 0, \\ \hat{u}_1(y, 0) \leq \eta_1(y), \hat{u}_2(y, 0) \leq \eta_2(y), \hat{u}_3(y, 0) \leq \eta_3(y), & y \in \overline{\Omega}_0, \\ \tilde{u}_1(y, 0) \geq \eta_1(y), \tilde{u}_2(y, 0) \geq \eta_2(y), \tilde{u}_3(y, 0) \geq \eta_3(y), & y \in \overline{\Omega}_0. \end{cases} \tag{3.1}$$

R_0 is a threshold value for the local stability of the disease-free equilibrium [25]. In the following we investigate the local stability of the disease-free equilibrium $(0,0,0)$ in the two cases $R_0 < 1$ and $R_0 > 1$.

Theorem 3.2. *If $R_0 < 1$, then the disease-free steady state $(0,0,0)$ is a locally asymptotically stable equilibrium for problem* (1.4).

Proof. The upper and lower solutions method is used to prove this theorem. Let

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3)(y,t) = (0,0,0), \qquad (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)(y,t) = (\varepsilon\phi(y), \varepsilon\psi(y), \varepsilon\xi(y)), \tag{3.2}$$

where ε is sufficiently small, $\phi(y)$ and $\psi(y)$ are the normalized positive eigenfunctions in problem (2.2), and $\xi(y)$ satisfies

$$0 = \frac{(a-b)\phi + (a-b+\beta K)\psi}{R_0} - (a-b)\xi. \tag{3.3}$$

Plugging (3.2) back into (3.1), it is easy to verify that the first three inequalities in (3.1) hold. The fourth inequality becomes

$$0 \geq \beta K\psi - \sigma\phi - \left[b + (a-b)\frac{\varepsilon\phi + K - \varepsilon\xi}{K} \right]\phi - \frac{n\dot{\rho}(t)}{\rho(t)}\phi.$$

According to the first equation in (2.2), we only need to prove that

$$b + (a-b)\frac{\varepsilon\phi + K - \varepsilon\xi}{K} + \sigma + \frac{n\dot{\rho}(t)}{\rho(t)} \geq R_0(\sigma + a). \tag{3.4}$$

Since $R_0 < 1$ and ε is sufficiently small, (3.4) holds and the fourth inequality in (3.1) holds. The fifth inequality becomes

$$-\frac{D\Delta\psi}{\rho(t)^2} \geq \sigma\phi - \alpha\psi - \left[b + (a-b)\frac{\varepsilon\psi + K - \varepsilon\xi}{K} \right]\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi. \tag{3.5}$$

It is easy to check that $\psi(y) = \zeta(y)$, where $\zeta(y)$ satisfies (2.5). We have $-\frac{D\Delta\psi}{\rho^2(t)} \geq -\frac{D\Delta\psi}{\rho_\infty^2}$ due to $\Delta\psi = \Delta\zeta = -\lambda_1\zeta \leq 0$. We have that (3.5) is satisfied if

$$-\frac{D\Delta\psi}{\rho_\infty^2} \geq \sigma\phi - \alpha\psi - \left[b + (a-b)\frac{\varepsilon\psi + K - \varepsilon\xi}{K} \right]\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi \tag{3.6}$$

holds. From the second equation in (2.2), (3.5) becomes

$$\left(\frac{1}{R_0} - 1 \right)\sigma\phi \geq \left\{ a - \left[b + (a-b)\frac{\varepsilon\psi + K - \varepsilon\xi}{K} \right] \right\}\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi. \tag{3.7}$$

Since $R_0 < 1$ and that the right of (3.7) tends to 0 as $\varepsilon \to 0$, the fifth inequality in (3.1) holds for sufficiently small ε. The sixth inequality in (3.1) is equivalent to

$$0 \geq (a-b)(K - \varepsilon\xi)\frac{\phi + \psi - \xi}{K} + \beta\phi(K - \varepsilon\xi) - \frac{n\dot{\rho}(t)}{\rho(t)}. \tag{3.8}$$

Due to (3.3), (3.8) becomes

$$(a-b)(1 - R_0) + \frac{n\dot{\rho}(t)}{\rho(t)} \geq -\varepsilon\left[(a-b)\frac{\phi + \psi - \xi}{K} + \beta\psi \right]. \tag{3.9}$$

Since $R_0 < 1$ and $\dot{\rho}(t) > 0$, (3.8) is also true for sufficiently small ε.

Therefore, the function-pairs

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3)(y, t) = (0, 0, 0), \qquad (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)(y, t) = (\varepsilon\phi(y), \varepsilon\psi(y), \varepsilon\xi(y))$$

are the upper and lower solutions of problem (1.4). This implies that the solutions of problem (1.4) lies between the lower solutions and the upper solutions as long as the initial values belong to the prescribed intervals. Therefore, given the condition $R_0 < 1$, we can conclude local stability of the disease-free equilibrium $(0, 0, 0)$. □

The next result shows that the disease-free equilibrium $(0, 0, 0)$ is unstable if $R_0 > 1$.

Theorem 3.3. *If $R_0 > 1$, then there exists a $\delta_0 > 0$ such that any positive solution of problem (1.4) satisfies* $\limsup_{t \to \infty} \| (u_1(\cdot, t), u_2(\cdot, t), u_3(\cdot, t)) - (0, 0, 0) \| \geq \delta_0$.

Proof. We argue by contradiction and assume that for any $\delta \in (0, K)$, there exists a $T_\delta > 0$ such that

$$0 < u_1(y, t), u_2(y, t), u_3(y, t) < \delta \quad \text{for all } y \in \Omega_0, t \geq T_\delta. \tag{3.10}$$

We consider the following eigenvalue problem:

$$\begin{cases} 0 = \beta u_2(K - \delta) - \sigma u_1 - \left[b + (a - b)\frac{K+3\delta}{K}\right] u_1 - \delta u_1 + \lambda u_1, & y \in \Omega_0, \\ -\frac{D\Delta u_2}{(\rho_\infty - \delta)^2} = \sigma u_1 - \alpha u_2 - \left[b + (a - b)\frac{K+3\delta}{K}\right] u_2 - \delta u_2 + \lambda u_2, & y \in \Omega_0, \\ u_1 = u_2 = 0, & y \in \partial\Omega_0. \end{cases} \tag{3.11}$$

Problem (3.11) has a principal eigenvalue λ_δ^* and a pair of positive corresponding eigenfunctions $(\phi_\delta^*(y), \psi_\delta^*(y))$. It is easy to check that $\psi_\delta^*(y) = \zeta(y)$, where $\zeta(y)$ satisfies (2.5). By Lemma 2.1, $R_0 > 1$ implies that $\lambda^* < 0$. Therefore, $\lim_{\delta \to 0} \lambda_\delta^* = \lambda^* < 0$. We can fix a small $\delta_0 \in (0, K)$ such that $\lambda_{\delta_0}^* < 0$. Then there exists a $T_1 > 0$ such that

$$0 < u_1(y, t), u_2(y, t), u_3(y, t) < \delta_0 \quad \text{for all } y \in \Omega_0, t \geq T_1.$$

Since $\lim_{t \to \infty} \rho(t) = \rho_\infty$, there exists a $T_2 > 0$ such that

$$\rho_\infty - \delta_0 < \rho(t) \leq \rho_\infty \quad \text{for } t \geq T_2.$$

Similarly, the limit $\lim_{t \to \infty} \frac{n\dot{\rho}(t)}{\rho(t)} = 0$ implies that there exists a $T_3 > 0$ such that

$$\frac{n\dot{\rho}(t)}{\rho(t)} < \delta_0 \quad \text{for } t \geq T_3.$$

Now choose a large $T^* = \max\{T_1, T_2, T_3\}$. Note that any positive solution (u_1, u_2, u_3) of the problem (1.4) satisfies

$$\begin{cases} u_{1t} \geq \beta u_2(K - \delta_0) - \sigma u_1 - \left[b + (a - b)\frac{K+3\delta_0}{K}\right] u_1 - \delta_0 u_1, \\ u_{2t} - \frac{D\Delta u_2}{\rho(t)^2} \geq \sigma u_1 - \alpha u_2 - \left[b + (a - b)\frac{K+3\delta_0}{K}\right] u_2 - \delta_0 u_2, \end{cases}$$

for all $y \in \Omega_0, t \geq T^*$. Define $(\underline{u}_1(y, t), \underline{u}_2(y, t))$ to be a positive solution of the problem

$$\begin{cases} \underline{u}_{1t} = \beta\underline{u}_2(K - \delta_0) - \sigma\underline{u}_1 - \left[b + (a - b)\frac{K+3\delta_0}{K}\right] \underline{u}_1 - \delta_0\underline{u}_1, & y \in \Omega_0, t \geq T^*, \\ \underline{u}_{2t} - \frac{D\Delta\underline{u}_2}{\rho^2(t)} = \sigma\underline{u}_1 - \alpha\underline{u}_2 - \left[b + (a - b)\frac{K+3\delta_0}{K}\right] \underline{u}_2 - \delta_0\underline{u}_2, & y \in \Omega_0, t \geq T^*, \\ \underline{u}_1 = \underline{u}_2 = 0, & y \in \partial\Omega_0, t \geq T^*, \\ \underline{u}_1(y, T^*) = u_1(y, T^*), \underline{u}_2(y, T^*) = u_2(y, T^*), & y \in \overline{\Omega}_0. \end{cases} \tag{3.12}$$

It then follows from the comparison principle that

$$(u_1(y,t), u_2(y,t)) \geq (\underline{u}_1(y,t), \underline{u}_2(y,t)) > (0,0) \quad \text{for all } y \in \Omega_0, t \geq T^*. \tag{3.13}$$

Now we conclude that $(\underline{u}_1(y,T^*), \underline{u}_2(y,T^*)) \geq (\mu\phi_{\delta_0}^*(y), \mu\psi_{\delta_0}^*(y))$ in $\overline{\Omega}_0$ for sufficiently small μ. In fact, since $\underline{u}_1(y,T^*)$, $\underline{u}_2(y,T^*)$, $\phi_{\delta_0}^*(y)$ and $\psi_{\delta_0}^*(y)$ are all > 0 for $y \in \Omega_0$, we have $\frac{\partial \underline{u}_1(y,T^*)}{\partial \eta}\big|_{\partial\Omega_0}$, $\frac{\partial \underline{u}_2(y,T^*)}{\partial \eta}\big|_{\partial\Omega_0}$, $\frac{\partial \phi_{\delta_0}^*(y)}{\partial \eta}\big|_{\partial\Omega_0}$ and $\frac{\partial \psi_{\delta_0}^*(y)}{\partial \eta}\big|_{\partial\Omega_0} < 0$ by the strong maximum principle [19], where η is the outer unit normal on $\partial\Omega_0$. For $y_0 \in \partial\Omega_0$, there exists a small $\varepsilon(y_0) > 0$ such that

$$\frac{\partial \underline{u}_1(y,T^*)}{\partial \eta} < \frac{1}{2}\frac{\partial \underline{u}_1(y,T^*)}{\partial \eta}\bigg|_{\partial\Omega_0} < 0, \qquad \frac{\partial \underline{u}_2(y,T^*)}{\partial \eta} < \frac{1}{2}\frac{\partial \underline{u}_2(y,T^*)}{\partial \eta}\bigg|_{\partial\Omega_0} < 0,$$

$$\frac{\partial \phi_{\delta_0}^*(y)}{\partial \eta} < \frac{1}{2}\frac{\partial \phi_{\delta_0}^*(y)}{\partial \eta}\bigg|_{\partial\Omega_0} < 0, \qquad \frac{\partial \psi_{\delta_0}^*(y)}{\partial \eta} < \frac{1}{2}\frac{\partial \psi_{\delta_0}^*(y)}{\partial \eta}\bigg|_{\partial\Omega_0} < 0$$

for $y \in \overline{B(y_0, \varepsilon(y_0)) \bigcap \Omega_0}$. Set $\mu_1 = \min\left\{\frac{\partial \underline{u}_1(y,T^*)}{\partial \eta} / \frac{\partial \phi_{\delta_0}^*(y)}{\partial \eta}, \frac{\partial \underline{u}_2(y,T^*)}{\partial \eta} / \frac{\partial \psi_{\delta_0}^*(y)}{\partial \eta}\right\}$, $y \in \overline{B(y_0, \varepsilon(y_0)) \bigcap \Omega_0}$, then

$$\frac{\partial \underline{u}_1(y,T^*)}{\partial \eta} \geq \mu_1 \frac{\partial \phi_{\delta_0}^*(y)}{\partial \eta}, \qquad \frac{\partial \underline{u}_2(y,T^*)}{\partial \eta} \geq \mu_1 \frac{\partial \psi_{\delta_0}^*(y)}{\partial \eta} \quad \text{for } y \in \overline{B(y_0, \varepsilon(y_0)) \bigcap \Omega_0}.$$

By the mean value theorem, we have

$$\underline{u}_1(y,T^*) \geq \mu_1 \phi_{\delta_0}^*(y), \qquad \underline{u}_2(y,T^*) \geq \mu_1 \psi_{\delta_0}^*(y) \quad \text{for } y \in \overline{B(y_0, \varepsilon(y_0)) \bigcap \Omega_0}.$$

Since $\partial\Omega_0$ is bounded, we can find finitely many points $y_0^i \in \partial\Omega_0$, radii $\varepsilon(y_0^i) > 0$ $(i = 1, \ldots, N)$ such that $\partial\Omega_0 \subset \bigcup_{i=1}^{N} B(y_0^i, \varepsilon(y_0^i))$, hence there exists a small $h = \min_i \varepsilon(y_0^i) > 0$ such that

$$\underline{u}_1(y,T^*) \geq \mu_1 \phi_{\delta_0}^*(y), \qquad \underline{u}_2(y,T^*) \geq \mu_1 \psi_{\delta_0}^*(y) \quad \text{for } y \in \{y \in \Omega_0 | \text{ dist}(y, \partial\Omega_0) \leq h\}.$$

Meanwhile, for any $y \in \{y \in \Omega_0 \mid \text{dist}(y, \partial\Omega_0) > h\}$, since $\underline{u}_1(y,T^*)$, $\underline{u}_2(y,T^*)$, $\phi_{\delta_0}^*(y)$ and $\psi_{\delta_0}^*(y)$ are all > 0, there exists a small $\mu_2 > 0$ such that $\frac{\underline{u}_1(y,T^*)}{\phi_{\delta_0}^*(y)}$ and $\frac{\underline{u}_2(y,T^*)}{\psi_{\delta_0}^*(y)} \geq \mu_2$ for $y \in \{y \in \Omega_0 \mid \text{dist}(y, \partial\Omega_0) > h\}$. Therefore, a sufficiently small $\mu > 0$ satisfying $\mu \leq \min\{\mu_1, \mu_2\}$ can be chosen to make sure $(\underline{u}_1(y,T^*), \underline{u}_2(y,T^*)) \geq (\mu\phi_{\delta_0}^*(y), \mu\psi_{\delta_0}^*(y))$ in $\overline{\Omega}_0$.

Set

$$U_1 = \mu e^{-\lambda_{\delta_0}^*(t-T^*)}\phi_{\delta_0}^*(y) \quad \text{and} \quad U_2 = \mu e^{-\lambda_{\delta_0}^*(t-T^*)}\psi_{\delta_0}^*(y).$$

It is easy to verify that $(U_1(y,t), U_2(y,t))$ is a positive solution of the problem

$$\begin{cases} U_{1t} = \beta U_2(K - \delta_0) - \sigma U_1 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right]U_1 - \delta_0 U_1, & y \in \Omega_0, \ t \geq T^*, \\ U_{2t} = \frac{D\Delta U_2}{(\rho_\infty - \delta_0)^2} + \sigma u_1 - \alpha U_2 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right]U_2 - \delta_0 U_2, & y \in \Omega_0, \ t \geq T^*, \\ U_1 = U_2 = 0, & y \in \partial\Omega_0, \ t \geq T^*, \\ U_1(y,T^*) = \mu\phi_{\delta_0}^*(y), U_2(y,T^*) = \mu\psi_{\delta_0}^*(y), & y \in \overline{\Omega}_0. \end{cases}$$

Recalling that $\Delta\psi_{\delta_0}^*(y) = \Delta\zeta(y) = -\lambda_1\zeta(y) \leq 0$ yields

$$U_{2t} \leq \frac{D\Delta U_2}{\rho^2(t)} + \sigma u_1 - \alpha U_2 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right]U_2 - \delta_0 U_2 \quad \text{for all } y \in \Omega_0, t \geq T^*,$$

which means that $(U_1(y,t), U_2(y,t))$ is a lower solution of problem (3.12), so by the comparison principle we have that

$$(\underline{u}_1(y,t), \underline{u}_2(y,t)) \geq (U_1(y,t), U_2(y,t)) \quad \text{for all } y \in \Omega_0, t \geq T^*,$$

which together with (3.13) gives

$$(u_1(y,t), u_2(y,t)) \geq (U_1(y,t), U_2(y,t)) = (\mu e^{-\lambda^*_{\delta_0}(t-T^*)} \phi^*_{\delta_0}(y), \mu e^{-\lambda^*_{\delta_0}(t-T^*)} \psi^*_{\delta_0}(y)),$$

for all $y \in \overline{\Omega}_0, t \geq T^*$. But since $\lambda^*_{\delta_0} < 0$, $u_1(y,t)$ and $u_2(y,t)$ tends to ∞ as t goes to ∞, for any fixed $y \in \overline{\Omega}_0$ which contradicts (3.10). The proof is now completed. □

4 Numerical simulations

In this section we carry out some numerical simulations in one space dimension to illustrate our theoretical analysis.

Regarding the domain growth, we choose $\Omega(t) = (0, x(t)) = (0, \rho(t)y)$, where $\rho(t) = \frac{e^t}{1+\frac{1}{m}(e^t-1)}$ and $y \in \Omega_0 = (0,1)$. Then, the domain grows like $\rho(t)$ from initial rate $\rho(0) = 1$ to the final rate $\rho_\infty = m$ with $m > 1$. To highlight the impacts of the domain growth on the transmission of rabies, we first fix the following parameters

$$D = 1, \quad a = 1, \quad b = 0.2, \quad K = 1000, \quad \alpha = 0.01, \quad \beta = 0.08, \quad \sigma = 0.05$$

and subsequently obtain $\lambda_1 = \pi^2$. Next, we choose a different growth rate $\rho(t)$ for the domain and study the asymptotic behavior of the solution to the problem (1.4).

Example 4.1. Set $m = 1.2$ and we have

$$R_0 = \sqrt{\frac{\sigma \beta K}{(\sigma + a)(\frac{D}{\rho^2_{1\infty}}\lambda_1 + \alpha + a)}} = 0.64 < 1.$$

By Theorem 3.2, we know that the disease-free equilibrium of problem (1.4) is stable. One can see from Fig. 4.1 that the solution $(u_1(y,t), u_2(y,t), u_3(y,t))$ decays to zero, which consists with the result of Theorem 3.2.

Example 4.2. Set $m = 4$ and a direct calculation shows that

$$R_0 = \sqrt{\frac{\sigma \beta K}{(\sigma + a)(\frac{D}{\rho^2_{2\infty}}\lambda_1 + \alpha + a)}} = 1.05 > 1.$$

Theorem 3.3 shows that the disease-free equilibrium $(0,0,0)$ is now unstable. It is easily seen from Fig. 4.2 that (u_1, u_2, u_3) stabilizes to a positive steady state.

Comparing the above two cases, it can be seen that the infected but non-infectious population u_1 and rabid population u_2 vanish at small growth rate, but spread at large growth rate.
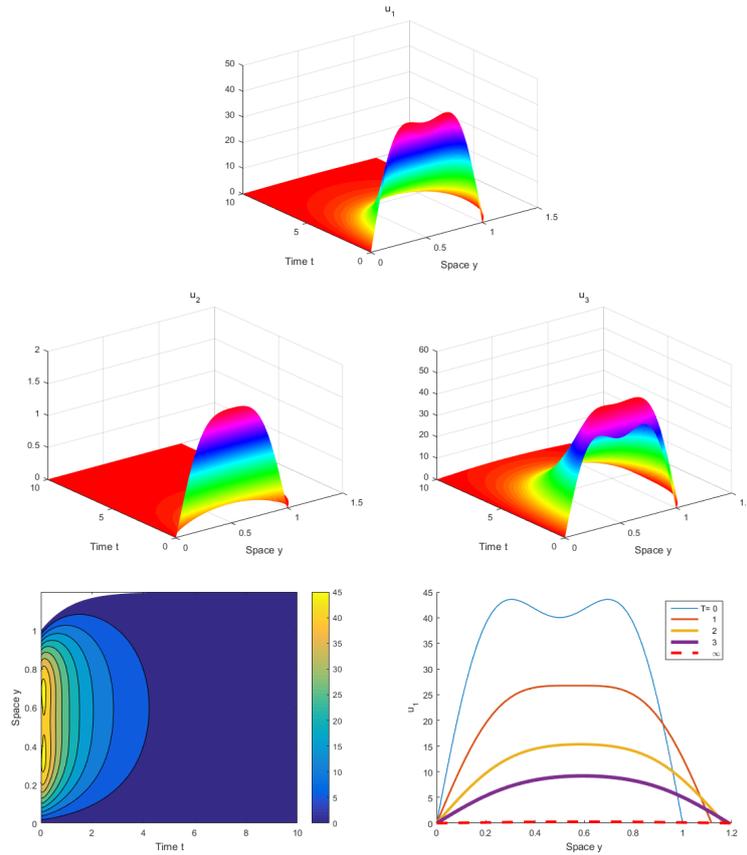
Figure 4.1: $\rho_1(t) = \frac{e^t}{1+\frac{1}{1.2}(e^t-1)}$. For small growth rate $\rho_1(t)$, we have $R_0 < 1$. The first three graphs show that (u_1, u_2, u_3) decays to zero quickly. The last two graphs in line 3 are the cross-sectional view (the left) and contour map (the right) of species u_1, respectively. The color bar in the graph of the cross-sectional view shows the density of the species u_1. The contour map shows the convergence of the temporal solution u_1 to the trivial solution (red dashed line).
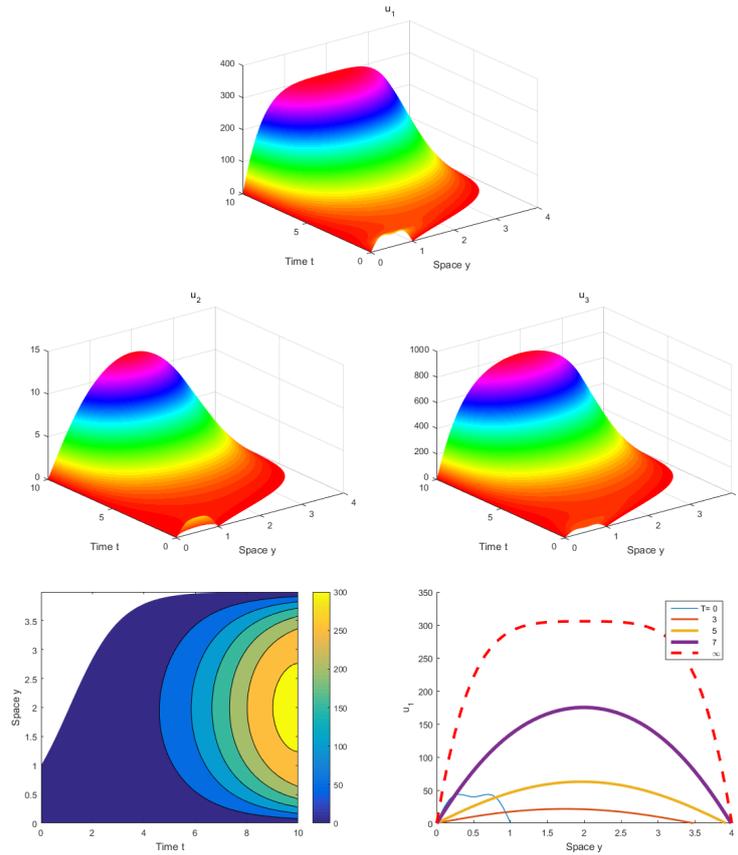
Figure 4.2: $\rho_2(t) = \frac{e^t}{1+\frac{1}{4}(e^t-1)}$. In this case, the growth rate $\rho_2(t)$ is now large enough to give that $R_0 > 1$. (u_1, u_2, u_3) tends to a positive steady state from the first three graphs. The last two graphs present the growth of the domain. The color bar in the graph of the cross-sectional view shows the density of the species u_1. The contour map shows the convergence of the temporal solution u_1 to the positive solution (red dashed line).

5   Discussion

Domain growth plays a significant role in the evolution of a biological population, and this has drawn much attention recently. In order to explore the impact of the domain growth on the transmission of fox rabies, we investigate a SEI model for fox rabies with uniform and isotropic domain growth.

We first transform the SEI model on the growing domain into a reaction-diffusion system on a fixed domain, and the basic reproduction number R_0 is introduced by spectral analysis and the so-called next infection operator. The relationship between R_0 and ρ_∞ directly follows by the explicit expression of R_0 which is determined by the variational method. Then, the stability of the disease-free equilibrium in terms of the threshold value R_0 is investigated by the upper and lower solutions method. It is proved in Theorem 3.2 that if $R_0 < 1$, the disease-free steady state $(0,0,0)$ for the problem (1.4) is locally asymptotically stable, while if $R_0 > 1$, the disease-free equilibrium $(0,0,0)$ is unstable according to Theorem 3.3. Finally our analytical results are clearly supported by numerical simulations. When $R_0 < 1$, the solution of (1.4) decays to zero when the domain growth is small (see Fig. 4.1) while when $R_0 > 1$, the disease-free equilibrium is unstable at a large domain growth (see Fig. 4.2). Our results show that a large growth of the domain has a negative effect on the stability of disease-free equilibrium, in the sense that it works against the prevention and control of rabies.

However, we can not derive the existence and uniqueness of the positive equilibrium. Moreover, all coefficients except $\rho(t)$ are constants in the problem (1.4), but in fact rabies is mainly affected by spatial heterogeneity and spatial distribution of habitats [20, 21], which implies that the diffusion coefficient D and the disease transmission coefficient β (and other constants) depend on the location x. We plan to investigate these problems in the future.

Acknowledgements

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