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On qualitative behavior of multiple solutions of quasilinear parabolic functional equations

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Abstract. We shall consider weak solutions of initial-boundary value problems for semilinear and nonlinear parabolic differential equations for $t \in (0, \infty)$ with certain nonlocal terms. We shall prove theorems on the number of solutions and certain qualitative properties of the solutions. These statements are based on arguments for fixed points of some real functions and operators, respectively, and theorems on the existence, uniqueness and qualitative properties of the solutions of partial differential equations (without functional terms).

Keywords: partial functional differential equations, multiple solutions, qualitative properties.

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1 Introduction

It is well known that mathematical models in several applications are functional differential equations of one variable (e.g. delay equations). In the monograph by Jianhong Wu [7] semilinear evolutionary partial functional differential equations and applications are considered, where the book is based on the theory of semigroups and generators. In the monograph by A. L. Skubachevskii [6] linear elliptic functional differential equations (equations with nonlocal terms and nonlocal boundary conditions) and applications are considered. A nonlocal boundary value problem, arising in plasma theory, was considered by A. V. Bitsadze and A. A. Samarskii in [1].

It turned out that the theory of pseudomonotone operators is useful to study nonlinear (quasilinear) partial functional differential equations (both stationary and evolutionary equations) and to prove existence of weak solutions (see [2,4]).

In [5] we considered some nonlinear parabolic functional differential equations for $t \in (0, T)$ $(T < \infty)$ and proved existence of several weak solutions of initial-boundary boundary value problems.

In the present work we shall prove existence of weak solutions of some parabolic functional equations for $t \in (0, \infty)$ and show certain qualitative properties of the solutions (boundedness and stabilization as $t \to \infty$).

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First we remind the reader of the definition of weak solutions of initial-boundary value problems of nonlinear parabolic (functional) differential equation for $t \in (0,T)$ and $t \in (0,\infty)$ with zero initial and boundary conditions.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, $1 < p < \infty$. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space of real valued functions with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega (|Du|^p + |u|^p)\right]^{1/p}.$$

Further, let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace containing $C_0^\infty(\Omega)$, $V^\star$ the dual space of V, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$.

Denote by $L^p(0,T;V)$ the Banach space of functions $u : (0,T) \to V$ ($V \subset W^{1,p}(\Omega)$ is a closed linear subspace) with the norm

$$\|u\|_{L^p(0,T;V)} = \left[\int_0^T \|u(t)\|_V^p dt\right]^{1/p} \quad (1 < p < \infty).$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p + 1/q = 1$. (See, e.g. [8].) Let $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ be a given (nonlinear) operator and $F \in L^q(0,T;V^\star)$.

Weak solutions of

$$D_t u + A(u) = F \tag{1.1}$$

for $t \in (0,T)$ with zero initial and boundary condition is a function $u \in L^p(0,T;V)$ satisfying $D_t u \in L^q(0,T;V^\star)$, (1.1) and $u(0) = 0$. (For $p \geq 2$, $u \in L^p(0,T;V)$ and $D_t u \in L^q(0,T;V^\star)$ imply $u \in C([0,T];L^2(\Omega))$ thus the initial condition makes sense.)

Consider first the particular case (without functional terms) $A = \tilde{A}$ where

$$\langle [\tilde{A}(u)](t), v \rangle = \int_\Omega \left[\sum_{j=1}^n a_j(t,x,u,Du) D_j v + a_0(t,x,u,Du) v\right] dx \tag{1.2}$$

for all $v \in V$, almost all $t \in [0,T]$. By using the theory of monotone operators the following existence and uniqueness theorem is proved. (See, e.g., [3,4,8].)

(C1) The functions $a_j : (0,T) \times \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ ($j = 0,1,\dots,n$) satisfy the Carathéodory conditions, i.e. $(t,x) \mapsto a_j(t,x,\xi)$ is measurable for all $\xi \in \mathbb{R}^{n+1}$ and $\xi \mapsto a_j(t,x,\xi)$ is continuous for a.a. (t,x).

(C2) There exist a constant c_1 and a function $k_1 \in L^q((0,T) \times \Omega)$ ($1/p + 1/q = 1$, $p \geq 2$) such that

$$|a_j(t,x,\xi)| \leq c_1[1 + |\xi|^{p-1}] + k_1(t,x),$$

$j = 0,1,\dots,n$, for a.a. $(t,x) \in (0,T) \times \Omega$, each $\xi \in \mathbb{R}^{n+1}$.

(C3) The inequality

$$\sum_{j=0}^n [a_j(t,x,\xi) - a_j(t,x,\xi^\star)](\xi_j - \xi_j^\star) \geq c_2 |\xi - \xi^\star|^p$$

holds with come constant $c_2 > 0$.

Theorem 1.1. *Assume (C1)–(C3). Then for any $F \in L^q(0,T;V^\star)$ there exists a unique $u \in L^p(0,T;V)$ weak solution of (1.1) with $A = \tilde{A}$ which depends on F continuously.*

A more general case is when $[A(u)](t)$ is depending not only on $u(t)$ and $(Du)(t)$, then (1.1) is a functional equation. By using the theory of pseudomonotone operators, one can prove existence of solutions for $t \in [0, T]$ in this more general case. (See, e.g., [4].)

Now we formulate a theorem on weak solutions of (1.1) for $t \in (0, \infty)$. The set $L^p_{\mathrm{loc}}(0, \infty; V)$ consists of all functions $f : (0, \infty) \to V$ for which the restriction $f|_{(0,T)}$ belongs to $L^p(0, T; V)$ for each finite $T > 0$. Furthermore, by using the notations $Q_T = (0, T) \times \Omega$, $Q_\infty = (0, \infty) \times \Omega$ denote by $L^p_{\mathrm{loc}}(Q_\infty)$ the set of functions $f : Q_\infty \to \mathbb{R}$ for which $f|_{Q_T} \in L^p(Q_T)$ with arbitrary $T > 0$. Assume that

(C_∞1) Functions $a_j : Q_\infty \times \mathbb{R}^{n+1}$ satisfy the Carathéodory conditions.

(C_∞2) There exist a constant c_1 and a function $k_1 \in L^q(\Omega)$ such that

$$|a_j(t, x, \xi)| \le c_1|\xi|^{p-1} + k_1(x).$$

(C_∞3) For a.a. $(t, x) \in Q_\infty$, all $\xi, \xi^\star \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n}[a_j(t, x, \xi) - a_j(t, x, \xi^\star)](\xi_j - \xi^\star) \ge c_2|\xi - \xi^\star|^p$$

with some constant $c_2 > 0$.

Theorem 1.2. *Assume (C_∞1)–(C_∞3). Then for arbitrary $F \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ there is a unique $u \in L^p_{\mathrm{loc}}(0, \infty; V)$ such that $u' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ and*

$$D_t u(t) + [\tilde{A}(u)](t) = F(t) \quad \text{for a.a. } t \in (0, \infty), \quad u(0) = 0$$

with the operator $\tilde{A}$ defined in (1.2).

If $\|F(t)\|_{V^\star}$ is bounded for a.a. $t \in (0, \infty)$ then for a solution u, $\|u(t)\|_{L^2(\Omega)}$ is bounded and

$$\int_{T_1}^{T_2} \|u(t)\|_V^p dt \le c_3(T_2 - T_1) \quad \text{with some constant } c_3. \tag{1.3}$$

Now we formulate a theorem on the stabilization of $u(t)$ as $t \to \infty$.

Theorem 1.3. *Assume that the assumptions of the above theorem are satisfied. Further, there exist Carathéodory functions $a_{j,\infty} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$, a continuous function $\Phi : (0, \infty) \to (0, \infty)$ and $F_\infty \in V^\star$ such that*

$$|a_j(t, x, \xi) - a_{j,\infty}(x, \xi)| \le \Phi(t)(|\xi|^{p-1} + 1), \quad \text{where } \lim_{\infty} \Phi = 0, \tag{1.4}$$

$$\|F(t) - F_\infty\|_{V^\star} \le \Phi(t) \quad \text{for a.a. } t > 0. \tag{1.5}$$

Then

$$\lim_{t\to 0} \|u(t) - u_\infty\|_{L^2(\Omega)} = 0, \qquad \lim_{T\to\infty} \int_{T-a}^{T+a} \|u(t) - u_\infty\|_V^p dt = 0 \tag{1.6}$$

for arbitrary fixed $a > 0$ where $u_\infty \in V$ is the unique solution $z \in V$ to

$$\sum_{j=1}^{n} \int_\Omega a_{j,\infty}(x, z, Dz) D_j v dx + \int_\Omega a_{0,\infty}(x, z, Dz) v dx = \langle F_\infty, v\rangle, \qquad v \in V.$$

(For the proofs, see, e.g., [4].)

By using the above results, we shall consider parabolic functional equations (equations containing some nonlocal terms) of certain particular type. In Section 2 equations with real valued functionals and in Section 3 equations with certain operators will be studied.

2 Parabolic equations with real valued functionals, applied to the solution

Case 1. First consider a semilinear parabolic functional equation for $t \in (0,\infty)$

$$D_t u + \tilde{B}u = D_t u - \sum_{j,k=1}^{n} D_j[a_{jk}(t,x)D_k u] + a_0(t,x)u = k(M(u))F_1 + F_2 \tag{2.1}$$

(i.e. the elliptic operator $\tilde{A}$ in (1.2) is linear), where $M : L^2(0,T_0;V) \to \mathbb{R}$ is a given linear continuous functional ($T_0 < \infty$), $V \subset W^{1,2}(\Omega)$, $k : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $F_1, F_2 \in L^2_{\text{loc}}(0,\infty;V^\star)$. Further, $a_{jk}, a_0 \in L^2_{\text{loc}}((0,\infty)\times\Omega)$, $a_{jk} = a_{kj}$ and the functions a_{jk} satisfy the uniform ellipticity condition

$$c_1|\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k + a_0(t,x)\xi_0^2 \le c_2|\xi|^2$$

for all $\xi = (\xi_0,\xi_1,\dots,\xi_n) \in \mathbb{R}^{n+1}$, $x \in \Omega$, $t \in (0,\infty)$ with some positive constants c_1, c_2.

Remark 2.1. The linear continuous functional $M : L^2(0,T_0;V) \to \mathbb{R}$ may have the form

$$M(u) = \int_0^{T_0}\int_\Omega \left[K_0(t,x)u(t,x) + \sum_{j=1}^{n} K_j(t,x)D_j u(t,x)\right] dxdt \tag{2.2}$$

where $K_0, K_1 \in L^2((0,T_0)\times\Omega)$.

According to Theorem 1.2, for arbitrary $F \in L^2_{\text{loc}}(0,\infty;V^\star)$ there is a unique solution $u \in L^2_{\text{loc}}(0,\infty;V)$ of

$$D_t u + \tilde{B}u = F,$$

denoted by $u = (D_t + \tilde{B})^{-1}F$.

Theorem 2.2. *A function $u \in L^2_{\text{loc}}(0,\infty;V)$ is a weak solution of (2.1) if and only if $\lambda = Mu$ satisfies the equation*

$$\lambda = k(\lambda)M[(D_t+\tilde{B})^{-1}F_1] + M[(D_t+\tilde{B})^{-1}F_2]. \tag{2.3}$$

and

$$u = k(\lambda)(D_t+\tilde{B})^{-1}F_1 + (D_t+\tilde{B})^{-1}F_2. \tag{2.4}$$

Proof. By Theorem 1.2 function $u \in L^2_{\text{loc}}(0,\infty;V)$ is a weak solution of (2.1) if and only if

$$u = k(M(u))(D_t+\tilde{B})^{-1}F_1 + (D_t+\tilde{B})^{-1}F_2,$$

thus

$$M(u) = k(M(u))M[(D_t+\tilde{B})^{-1}F_1 + (D_t+\tilde{B})^{-1}F_2]$$

which implies the theorem. □

Corollary 2.3. *The number of weak solutions of (2.1) (with zero initial-boundary conditions) equals the number of solutions λ of equation (2.3). Consequently, it is easy to show that for any natural number N or for $N = \infty$ one can choose functions k such that (2.1) has exactly N solutions.*

Remark 2.4. If we know the values of $M[(D_t + B)^{-1}F_1]$ and $M[(D_t + B)^{-1}F_2]$ then by using some numerical procedure one can calculate the λ roots of (2.3). Further, it is easy to show simple sufficient conditions on $M[(D_t + B)^{-1}F_1]$, $M[(D_t + B)^{-1}F_2]$ and the function k which imply that (2.3) has zero, exactly one (two or three) roots.

From Theorem 1.3 it directly follows

Theorem 2.5. *If there exist measurable functions $a_{j,k,\infty}, a_{0,\infty} \in L^\infty(\Omega)$ and $F_{1,\infty}, F_{2,\infty} \in V^\star$ such that*

$$|a_0(t,x) - a_{0,\infty}(x)| \le \Phi(t), \qquad |a_{j,k}(t,x) - a_{j,k,\infty}(x)| \le \Phi(t), \quad \text{where } \lim_{\infty} \Phi = 0,$$

$$\|F_1(t) - F_{1,\infty}\|_{V^\star} \le \Phi(t), \qquad \|F_2(t) - F_{2,\infty}\|_{V^\star} \le \Phi(t) \quad \text{for a.a. } t > 0$$

then we have (1.6) where $u_\infty \in V$ is the unique solution $z \in V$ to

$$\sum_{j,k=1}^{n} \int_\Omega a_{j,k,\infty}(x)(D_j z)(D_k v)dx + \int_\Omega a_{0,\infty}(x) z v dx = \langle k(M(u))F_{1,\infty}, v\rangle + \langle F_{2,\infty}, v\rangle, \qquad v \in V.$$

Case 2. Now consider *nonlinear* parabolic functional equations of the form

$$D_t u + [lM(u))]^\gamma \tilde{A}(u) = [lM(u))]^\beta F, \qquad t \in (0,\infty), \quad u(0) = 0 \tag{2.5}$$

where the nonlinear operator $\tilde{A}$ has the form (1.2) and has the property

$$\tilde{A}(\mu u) = \mu^{p-1}\tilde{A}(u), \quad \text{for all } \mu > 0 \text{ with some } p \ge 2 \tag{2.6}$$

(e.g. $\tilde{A}(u) = -\triangle_p u + c_0 u|u|^{p-2}$ with $c_0 > 0$ has this property), further, $M : L^p(0,T_0;V) \to \mathbb{R}$ ($V \subset W^{1,p}(\Omega)$) is (homogeneous) functional with the property

$$M(\mu u) = \mu^\sigma M(u) \quad \text{for all } \mu > 0 \text{ with some } \sigma > 0; \tag{2.7}$$

l is a given positive continuous function and the numbers β, γ satisfy

$$\gamma = \beta(2-p), \qquad \beta > 0.$$

A simple calculation shows

Theorem 2.6. *A function $u \in L^p_{\text{loc}}(0,\infty;V)$ satisfies (2.5) in weak sense if and only if*

$$\tilde{u} = [l(M(u))]^{-\beta} u \quad \textit{satisfies} \quad D_t\tilde{u} + \tilde{A}(\tilde{u}) = F.$$

This theorem implies

Theorem 2.7. *A function $u \in L^p_{\text{loc}}(0,\infty;V)$ is a weak solution of (2.5) with zero initial and boundary condition if and only if $\lambda = M(u)$ satisfies the equation*

$$\lambda = [l(\lambda)]^{\beta\sigma} M[B_0^{-1}(F)] \quad \textit{and} \quad u = [l(\lambda)]^\beta B_0^{-1}(F) \tag{2.8}$$

where B_0 is defined by $B_0(u) = D_t u + \tilde{A}(u)$, i.e. $B_0^{-1}(F)$ is the unique weak solution of (1.1) (with $A = \tilde{A}$ and zero initial and boundary condition). If $F \in L^\infty(0,\infty;V^\star)$ then $\|u(t)\|_{L^2(\Omega)}$ is bounded and (1.3) holds.

Corollary 2.8. *The number of weak solutions of* (2.5) *equals the number of roots of* (2.8)*. Further, assuming* $M[B_0^{-1}(F)] > 0$*, for arbitrary* $N = 1, 2, \ldots, \infty$ *one can construct a continuous positive function* l *such that* (2.5) *has exactly* N *solutions, in the following way. Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $g(\lambda) + \lambda > 0$ *for all* $\lambda \in \mathbb{R}$ *and* g *has* N *real roots. Then for*

$$l(\lambda) = \left[\frac{g(\lambda) + \lambda}{M(B_0^{-1}(F))} \right]^{1/(\beta\sigma)}$$

(2.5) *has* N *weak solutions.*

Remark 2.9. An example for functional M with property (2.7) is integral operator

$$M(u) = \int_0^T \int_\Omega K(t,x)|u(t,x)|^\sigma dtdx.$$

By Theorems 1.3 and 2.6 one obtains

Theorem 2.10. *If the assumptions* (1.4), (1.5) *are satisfied then we have* (1.6) *where* $u_\infty \in V$ *is the unique solution* $z \in V$ *to*

$$\sum_{j=1}^n \int_\Omega a_{j,\infty}(x,z,Dz) D_j v dx + \int_\Omega a_{0,\infty}(x,z,Dz) v dx$$
$$= (l(\lambda))^\beta \langle F_\infty, v \rangle = [l(M(u))]^\beta \langle F_\infty, v \rangle, \qquad v \in V.$$

3 Parabolic equations with nonlocal operators

Now consider partial functional equations of the form

$$D_t u + \tilde{A}(u) = C(u) \tag{3.1}$$

where $\tilde{A}$ is nonlinear differential operator (1.2) satisfying (C_∞1)–(C_∞3) (or $\tilde{A} = \tilde{B}$ is a uniformly elliptic linear differential operator (see (2.1)) and $C : L^p_{\text{loc}}(0,\infty;V) \to L^p_{\text{loc}}(0,\infty;V^\star)$ is a given (possibly nonlinear) operator. Clearly, $u \in L^p_{\text{loc}}(0,\infty;V)$ satisfies (3.1) if and only if

$$u = (D_t + \tilde{A})^{-1}[C(u)] =: G(u) \tag{3.2}$$

where $G : L^p_{\text{loc}}(0,\infty;V) \to L^p_{\text{loc}}(0,\infty;V)$ is a given (possibly nonlinear) operator, i.e. u is a fixed point of G. Then

$$C(u) = (D_t + \tilde{A})[G(u)]. \tag{3.3}$$

Now we consider three particular cases for G.

Case 1. The operator G is defined by

$$[G(u)](t,x) = (Lu)(t,x) + F(t,x) = \int_0^\infty \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy + F(t,x) \tag{3.4}$$

where $K \in L^2((0,\infty) \times (0,\infty) \times \Omega \times \Omega)$; $u, F \in L^2((0,\infty) \times \Omega)$.

By using (3.1) and (3.3) we find

Theorem 3.1. *If K and F are sufficiently smooth and "good" then the solution $u \in L^2(0,\infty) \times \Omega)$ of (3.2) with the operator (3.4) belongs to $L^p_{\text{loc}}(0,\infty;V)$, $D_t u$ belongs to $L^q_{\text{loc}}(0,\infty;V^\star)$ (in the linear case $\tilde{A} = \tilde{B}$, $p = q = 2$), $u(0) = 0$ and the equation (3.1) has the form*

$$\begin{aligned} D_t u + (\tilde{A}(u))(t,x) = & \int_0^\infty \int_\Omega D_t K(t,\tau,x,y) u(\tau,y) dx dy + D_t F(t,x) \\ & + \tilde{A}_x \left[\int_0^\infty \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy + F(t,x) \right]. \end{aligned} \tag{3.5}$$

In the linear case $\tilde{A} = \tilde{B}$

$$\begin{aligned} D_t u + (\tilde{B}u)(t,x) = & \int_0^\infty \int_\Omega D_t K(t,\tau,x,y) u(\tau,y) dx dy + D_t F(t,x) \\ & + \int_0^\infty \int_\Omega \tilde{B}_x K(t,\tau,x,y) u(\tau,y) d\tau dy + \tilde{B}_x F(t,x). \end{aligned} \tag{3.6}$$

($\tilde{A}_x K(t,\tau,x,y)$ denotes the differential operator $\tilde{A}$ applied to $x \mapsto K(t,\tau,x,y)$ and $\tilde{B}_x F(t,x)$ denotes the differential operator $\tilde{B}$ applied to $x \mapsto F(t,x)$.)

Further, if 1 is an eigenvalue of the linear integral operator $L : L^2((0,\infty)\Omega) \to L^2((0,\infty)\Omega)$ with multiplicity N then (for certain functions F) (3.6) may have N "linearly independent" solutions.

The proof is similar to the previous ones.

Remark 3.2. The value of solution u at some time t is connected with the values of u for all $t \in (0,\infty)$ (and for all $t \in [0,T_0]$ if $K(t,\tau,x,y) = 0$ for $\tau > T_0$).

By using (3.2), (3.4) and the Cauchy–Schwarz inequality, one obtains

Theorem 3.3. *Assume that there exist sufficiently smooth $K_\infty \in L^2((0,\infty) \times \Omega \times \Omega) = L^2(Q)$ and $F_\infty \in L^2(\Omega)$ such that*

$$\lim_{t\to\infty} \|K(t,\tau,x,y) - K_\infty(\tau,x,y)\|_{L^2(Q)} = 0,$$
$$\lim_{t\to\infty} \|F(t,x) - F_\infty(x)\|_{L^2(\Omega)} = 0.$$

Then

$$\lim_{t\to\infty} \|u(t,x) - u_\infty(x)\|_{L^2(\Omega)} = 0,$$

where

$$u_\infty(x) = \int_0^\infty \int_\Omega K_\infty(\tau,x,y) u(\tau,y) d\tau dy + F_\infty(x)$$

and u_∞ satisfies

$$[\tilde{A}(u_\infty)](x) = \tilde{A}_x \left[\int_0^\infty \int_\Omega K_\infty(\tau,x,y) u(\tau,y) d\tau dy + F_\infty(x) \right].$$

Case 2. Now consider operators G of the form

$$G(u) = Lu + h(Pu)F + H, \qquad t \in (0,\infty) \tag{3.7}$$

where operator L is defined by

$$(Lu)(t,x) = \int_0^t \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy,$$

$K \in L^2((0,\infty) \times (0,\infty) \times \Omega \times \Omega)$, $u \in L^2((0,\infty) \times \Omega)$ and the kernel K has the same smoothness property as in Theorem 3.1, $P : L^2(0, T_0; V) \to \mathbb{R}$ is a linear continuous functional ($T_0 < \infty$), $h : \mathbb{R} \to \mathbb{R}$ is a given continuous function and $F, H \in L^2((0,\infty) \times \Omega)$, $D_t F, D_t H \in L^2((0,\infty) \times \Omega)$. In this case the integral operator L is of Volterra type and so $(I - L)^{-1} : L^2((0,\infty) \times \Omega) \to L^2((0,\infty) \times \Omega)$ exists.

Theorem 3.4. *If $\tilde{A} = \tilde{B}$ (i.e. $\tilde{A}$ is linear) then equation (3.1) has the form*

$$\begin{aligned} D_t u + \tilde{B} u = & \int_0^t \int_\Omega [D_t K(t,\tau,x,y) + \tilde{B}_x K(t,\tau,x,y)] u(\tau,y) d\tau dy \\ & + \int_\Omega K(t,t,x,y) u(t,y) dy + h(Pu)(D_t + \tilde{B})F + (D_t + \tilde{B})H, \qquad u(0,x) = 0. \end{aligned} \tag{3.8}$$

Further, $u \in L^2((0,\infty) \times \Omega)$ is a weak solution of (3.8) if and only if $u = h(\lambda)[(I-L)^{-1}F] + (I-L)^{-1}H$ where λ is a root of the equation

$$\lambda = h(\lambda) P[(I-L)^{-1}F] + P[(I-L)^{-1}H]. \tag{3.9}$$

Thus the number of solutions of (3.8) equals the number of the roots of (3.9).

Proof. Equation (3.8) is fulfilled if and only if

$$u(t,x) = \int_0^t \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy + h(Pu)F(t,x) + H(t,x), \tag{3.10}$$

i.e.

$$(I-L)u = h(Pu)F + H, \qquad u = h(Pu)[(I-L)^{-1}F] + (I-L)^{-1}H. \tag{3.11}$$

Let $u_\lambda = h(\lambda)(I-L)^{-1}F + (I-L)^{-1}H$ then

$$P(u_\lambda) = h(\lambda) P[(I-L)^{-1}F] + P[(I-L)^{-1}H].$$

Consequently, (3.11) (and so (3.8)) is satisfied if and only if $\lambda = Pu$ satisfies (3.9). □

Corollary 3.5. *If $P[(I-L)^{-1}F] \neq 0$ then for arbitrary N $(= 0, 1, \ldots, \infty)$ we can construct h such that (3.8) has N solutions, in the following way. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous functions having N zeros. Then (3.8) has N solutions if*

$$h(\lambda) = \frac{g(\lambda) + \lambda - P[(I-L)^{-1}H]}{P[(I-L)^{-1}F]}.$$

Remark 3.6. The linear functional $P : L^2(0, T_0; V) \to \mathbb{R}$ may have the form (2.2).

By (3.10) and the Cauchy–Schwarz inequality we obtain

Theorem 3.7. *Assume that there exist sufficiently smooth $F_\infty, H_\infty \in L^2(\Omega)$ and $K_\infty \in L^2((0,\infty) \times \Omega \times \Omega)$ such that*

$$\lim_{t\to\infty} \|F(t,x) - F_\infty(x)\|_{L^2(\Omega)} = 0, \qquad \lim_{t\to\infty} \|H(t,x) - H_\infty(x)\|_{L^2(\Omega)} = 0,$$

$$\lim_{t\to\infty} \int_\Omega \left[\int_0^t \int_\Omega [K(t,\tau,x,y) - K_\infty(\tau,x,y)]^2 d\tau dy \right] dx = 0.$$

Then

$$\lim_{t\to\infty} \|u(t,x) - u_\infty(x)\|_{L^2(\Omega)} = 0,$$

where

$$u_\infty(x) = \int_0^\infty \int_\Omega K_\infty(\tau,x,y)u(\tau,y)d\tau dy + h(\lambda)F_\infty(x) + H_\infty(x),$$

$\lambda = P(u)$ *and* u_∞ *satisfies*

$$(\tilde{B}u_\infty)(x) = \int_0^\infty \int_\Omega \tilde{B}_x[K_\infty(\tau,x,y)]u(\tau,y)d\tau dy + h(\lambda)(\tilde{B}F_\infty)(x) + (\tilde{B}H_\infty)(x).$$

Case 3. Finally, consider the case

$$[G(u)](t,x) = \hat{P}(\hat{M}u(t))F(t,x), \qquad (t,x) \in (0,\infty)\times\Omega$$

where

$$(\hat{M}u))(t) = \int_0^t \int_\Omega \tilde{M}(\tau,y)u(\tau,y)d\tau dy, \qquad \tilde{M} \in C([0,\infty]\times\overline{\Omega}),$$

$\hat{P}: \mathbb{R} \to \mathbb{R}$ is a given continuously differentiable function, $\hat{P}(0) = 0$, F is sufficiently smooth, $F(0,x) = 0$, $F(t,x) = 0$ for $x \in \partial\Omega$.

Theorem 3.8. *In this case the partial functional equation (with possibly nonlinear operator $\tilde{A}$)* (1.2) *has the form*

$$D_t u + \tilde{A}(u) = \hat{P}'(\hat{M}u(t))F\int_\Omega \tilde{M}(t,y)u(t,y)dy + \hat{P}(\hat{M}u(t))D_tF \\ + \tilde{A}_x[\hat{P}(\hat{M}u(t))F], \quad u(0,x) = 0, \quad u(t,x) = 0 \quad \text{for } x \in \partial\Omega \tag{3.12}$$

which is satisfied if and only if

$$u(t,x) = \hat{P}(\hat{M}u(t))F(t,x). \tag{3.13}$$

Then $v(t) = \hat{M}u(t)$ satisfies the separable differential equation

$$v'(t) = \int_\Omega \tilde{M}(t,y)u(t,y)dy = \hat{P}(v(t))\int_\Omega \tilde{M}(t,y)F(t,y)dy \quad \text{and} \quad v(0) = 0. \tag{3.14}$$

Conversely, if v satisfies (3.14) *then $u(t,x) = \hat{P}(v(t))F(t,x)$ satisfies* (3.13).

Proof. Clearly, (3.12) is equivalent with (3.13). If u satisfies (3.13) then for

$$v(t) = (\hat{M}u)(t) = \int_0^t \int_\Omega \tilde{M}(\tau,y)u(\tau,y)d\tau dy \tag{3.15}$$

we have by (3.13)

$$v'(t) = \int_\Omega \tilde{M}(t,y)u(t,y)dy = \hat{P}((\hat{M}u)(t))\int_\Omega \tilde{M}(t,y)F(t,y)dy \\ = \hat{P}(v(t))\int_\Omega \tilde{M}(t,y)F(t,y)dy \quad \text{and, clearly,} \quad v(0) = 0.$$

Conversely, if v satisfies (3.14) then for

$$u(t,x) = \hat{P}(v(t))F(t,x) \tag{3.16}$$

we have $u(x,0) = 0$, $u(t,x) = 0$ for $x \in \Omega$ and by $v(0) = 0$

$$(\hat{M}u)(t) = \int_0^t \int_\Omega \tilde{M}(\tau,y)u(\tau,y)d\tau dy \\ = \hat{P}(v(t))\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)d\tau dy = \int_0^t v'(\tau)d\tau = v(t),$$

thus by (3.16)

$$u(t,x) = \hat{P}((\hat{M}u)(t))F(t,x).$$

□

Theorem 3.9. *Assume that $\hat{P}(w) > 0$ for $w > 0$ and $\hat{P}(0) = 0$, further,*

$$\hat{Q}(v) = \int_0^v \frac{dw}{\hat{P}(w)} < \infty, \qquad \lim_{v\to\infty} \hat{Q}(v) = \infty;$$

$$F(0,y) = 0 \quad \text{for all } y \in \Omega, \qquad \int_\Omega \tilde{M}(t,y)F(t,y)dy > 0 \quad \text{for all } t > 0.$$

Then we obtain for the solution of (3.14) $v = 0$ (v identically 0) and

$$v(t) = \hat{Q}^{-1}\left[\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)dyd\tau\right]$$

and, consequently, we have solutions $u = 0$ and

$$u(t,x) = \hat{P}(v(t))F(t,x) = \hat{P}\left\{\hat{Q}^{-1}\left[\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)dyd\tau\right]\right\}F(t,x). \tag{3.17}$$

Proof. By the assumptions on $\hat{P}$, $\hat{Q}$ is strictly monotone increasing, $\hat{Q}$ maps from $\mathbb{R}$ to $\mathbb{R}$, $\hat{Q}(0) = 0$, $\lim_{v\to\infty} \hat{Q}(v) = \infty$, thus

$$v(t) = \hat{Q}^{-1}\left[\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)dyd\tau\right], \qquad t \geq 0$$

is a solution of (3.14). By the previous theorem, function u, defined by (3.17) and $u = 0$ are solutions of (3.13) and (3.12). □

By using the continuity of functions $\hat{P}$ and $\hat{Q}^{-1}$, we obtain

Theorem 3.10. *Assume that there exist $F_\infty \in L^2(\Omega)$ and $c_0 \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \int_\Omega |F(t,y) - F_\infty(y)|^2 dy = 0, \tag{3.18}$$

$$\lim_{t\to\infty} \int_0^t \int_\Omega |\tilde{M}(\tau,y)F(\tau,y)dyd\tau = c_0. \tag{3.19}$$

Then for the nonzero solution u we have

$$\lim_{t\to\infty} \|u(t,x) - u_\infty(x)\|_{L^2(\Omega)} = 0$$

where

$$u_\infty(x) = \hat{P}(\hat{Q}^{-1}(c_0))F_\infty(x).$$

Remark 3.11. If there exists $\tilde{M}_\infty \in L^2(\Omega)$ such that

$$\lim_{t\to\infty} \int_\Omega \left[\int_0^t \tilde{M}(\tau,y)d\tau - \tilde{M}_\infty(y)\right] dy = 0$$

then (3.18) implies (3.19) with $c_0 = \int_\Omega \tilde{M}_\infty(y)F_\infty(y)dy$.

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