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Delayed linear difference equations: the method of $\mathcal{Z}$-transform

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Abstract. A system of nonhomogeneous linear difference equations with linear parts given by non-commutative matrices is studied. Representation of its solution is derived by means of newly defined delayed perturbation of discrete matrix exponential using the $\mathcal{Z}$-transform. We discard the invertibility condition of matrix of non delayed term used in recent works related to the representation of solutions for delayed linear difference systems.

Keywords: $\mathcal{Z}$-transform, difference equations.

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1 Introduction

Throughout the paper we denote:

- Θ and I the $d \times d$ zero and identity matrix, respectively;
- $\mathbb{Z}_a^b := \{a, a+1, \dots, b\}$ for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a \leq b$;
- An empty sum $\sum_{i=a}^{b} z(i) = 0$ and an empty product $\prod_{i=a}^{b} z(i) = 1$ for integers $a < b$, where $z(i)$ is a given function which does not have to be defined for each $i \in \mathbb{Z}_b^a$ in this case;
- $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator;

In the present paper we consider the following discrete systems with delay,

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \qquad k \geq 0, \tag{1.1}$$

where $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}_0^\infty$, A, B are constant $d \times d$ matrices, $x : \mathbb{Z}_{-m}^\infty \to \mathbb{R}^d$ is an unknown solution, C is a constant $d \times d$ matrix and $f : \mathbb{Z}_0^\infty \to \mathbb{R}^d$ is a function.

Let $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^d$ be a function. We consider an initial value problem

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^0. \tag{1.2}$$

We recall that the initial problem (1.1), (1.2) has a unique solution in $\mathbb{Z}_{-m}^{\infty}$.

In 2006, J. Diblik and D. Ya. Khusainov published two papers [2,3] on a matrix representation of solutions of linear discrete systems with a single delay using so called delayed discrete matrix exponential. In [8,9] the concept of discrete matrix delayed exponential is extended to two matrices with a representation derived of solutions to systems with two delayed linear terms. Along these lines, [21] presents rather general results giving a representation of solutions to discrete systems with multiple delayed terms assuming that matrices of these terms pairwise permute, while the paper by the author [15] treats the case of non-permutable matrices. The results of these papers are widely used. These basic results of these papers are widely used to deal with control theory, iterative learning control and stability analysis for time-delay equations; see for example, [1,4,5,7,11–14,16,18–20,22,23] and references therein.

In the paper [6] is an open problem formulated - to prove that the case of non-permutable matrice can be treated with the method of $\mathcal{Z}$-transform. This paper gives positive answer to this problem in the case of two matrices. Representation of solutions is derived by means of newly defined delayed perturbation of matrix exponential using the $\mathcal{Z}$-transform where the existence of inverse of the matrix A is not assumed (the assumption of regularity of matrix A plays important role in [15]).

The $\mathcal{Z}$-transform is a mathematical device similar to a generating function which provides an alternate method for solving linear difference equations as well as certain summation equations. The $\mathcal{Z}$-transform is important in the analysis and design of digital control systems. Note that in [21] the $\mathcal{Z}$-transform is applied to the following multiple delayed linear discrete systems with permutable matrices:

$$x(k+1) = x(k) + \sum_{j=1}^{m} B_j x(k - m_j) + f(k), \qquad k \geq 0$$
$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^{0},$$

where $B_1, \ldots, B_m$ are pairwise permutable matrices.

Motivated by [21] we apply the $\mathcal{Z}$-transform to study the problem (1.1), (1.2) assuming that the linear parts A, B in (1.1) are given by pairwise nonpermutable matrices. This does not allow to change the order when multiplying matrices and problem becomes much more difficult.

2 Delayed perturbation of discrete matrix exponential

The main tool in our study is the $\mathcal{Z}$-transform defined as

$$\mathcal{Z}\{f(k)\}(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k}$$

for $z \in \mathbb{R}$ and an exponentially bounded function $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^d$ such that $\|f(k)\| \leq c_1 c_2^k$ for all $k \in \mathbb{Z}_0^{\infty}$ and some constants $c_1, c_2 \in \mathbb{R}^+$. Note that if f is exponentially bounded, then $\mathcal{Z}\{f(k)\}(z)$ exists for all z sufficiently large. The $\mathcal{Z}$-transform is considered componentwisely. σ is the Heaviside step function defined as

$$\sigma(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

The next lemma gathers up some features of the $\mathcal{Z}$-transform.

Lemma 2.1 ([10]). *The following equalities are true for sufficiently large $z \in \mathbb{R}$ and exponentially bounded functions f, g:*

1. $\mathcal{Z}\{af(k)+bg(k)\} = a\mathcal{Z}\{f(k)\} + b\mathcal{Z}\{g(k)\}$, $a, b \in \mathbb{R}$;
2. $\mathcal{Z}^{-1}\{z^{-l}\}(k) = \delta(l,k)$ *for* $l \in \mathbb{Z}_0^{\infty}$, *where* δ *is the Kronecker delta,*
$$\delta(l,k) = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$
3. $\mathcal{Z}^{-1}\{F(z)G(z)\}(k) = (f * g)(k)$. *Here the convolution operator* $*$ *is given by*
$$(f*g)(k) = \sum_{j=0}^{k} f(j)\, g(k-j);$$

The next lemma is a corollary of the latter one.

Lemma 2.2. *The following identities are true for sufficiently large $z \in \mathbb{R}$:*

$$\mathcal{Z}^{-1}\left\{\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}(k) = Q(k-1;j), \tag{2.1}$$
$$\mathcal{Z}^{-1}\left\{\frac{1}{z^{mj+\gamma}}\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}(k) = Q(k-mj-\gamma-1;j), \tag{2.2}$$

where

$$Q(k;0) = A^{k}\sigma(k), \qquad Q(k;j) = \sum_{l=j}^{k} A^{k-l}BQ(l-1;j-1)\,\sigma(k-j).$$

Proof. To prove the formula (2.1) we recall the following identity

$$(I-C)^{j}\sum_{k=0}^{\infty}\binom{k+j-1}{j-1}C^{k} = I, \qquad \|C\| < 1.$$

Using this formula, we have

$$(zI-A)^{-j} = \frac{1}{z^{j}}\sum_{k=0}^{\infty}\binom{k+j-1}{j-1}\frac{1}{z^{k}}A^{k}.$$

We use the mathematical induction. For $j = 0$, we have

$$\begin{aligned}\mathcal{Z}^{-1}\left\{(zI-A)^{-1}\right\} &= \mathcal{Z}^{-1}\left\{\frac{1}{z^{1}}\right\} * \mathcal{Z}^{-1}\left\{\sum_{l=0}^{\infty}\frac{1}{z^{l}}A^{l}\right\} \\ &= \left(\delta(1,\cdot) * A^{\cdot}\right)(k) = \sum_{l=0}^{k}\delta(1,l)\,A^{k-l} = A^{k-1}\sigma(k-1) = Q(k-1;0).\end{aligned} \tag{2.3}$$

For $j = 1$, we have

$$\begin{aligned}&\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B(zI-A)^{-1}\right\}(k) = \mathcal{Z}^{-1}\left\{(zI-A)^{-1}B\right\} * \mathcal{Z}^{-1}\left\{(zI-A)^{-1}\right\}(k) \\ &= \left\{A^{\cdot-1}\sigma(\cdot-1)\,B * Q(\cdot-1;0)\right\}(k) = \sum_{j=0}^{k} A^{k-j-1}\sigma(k-j-1)\,BA^{j-1}\sigma(j-1) \\ &= \sum_{j=1}^{k-1} A^{k-j-1}BA^{j-1}\sigma(k-2) =: Q(k-1;1).\end{aligned}$$

For $j=2$, we get

$$\begin{aligned}
&\mathcal{Z}^{-1}\left\{(zI-A)^{-2}B^{2}(zI-A)^{-1}\right\}(k)\\
&=\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B\right\}*\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B(zI-A)^{-1}\right\}(k)\\
&=\left\{A^{\cdot-1}\sigma(\cdot-1)B*Q(\cdot-1;1)\sigma(\cdot-2)\right\}(k)\\
&=\sum_{j=0}^{k}A^{k-j-1}\sigma(k-j-1)BQ(j-1;1)\sigma(j-2)\\
&=\sum_{j=2}^{k-1}A^{k-j-1}\sigma(k-j-1)BQ(j-1;1)\sigma(j-2)\\
&=\sum_{j=2}^{k-1}A^{k-j-1}BQ(j-1;1)\sigma(k-3)=:Q(k-1;2).
\end{aligned}$$

Now, suppose that it holds for $j=n$. Then convolution property yields

$$\begin{aligned}
&\mathcal{Z}^{-1}\left\{(zI-A)^{-(n+1)}B^{n+1}(zI-A)^{-1}\right\}(k)\\
&=\left(\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B\right\}*\mathcal{Z}^{-1}\left\{(zI-A)^{-n}B^{n}(zI-A)^{-1}\right\}\right)(k)\\
&=\left\{A^{\cdot-1}\sigma(\cdot-1)B*Q(\cdot-1,n)\right\}(k)=\sum_{j=0}^{k}A^{k-j-1}\sigma(k-j-1)BQ(j-1;n)\sigma(j-n-1)\\
&=\sum_{j=n+1}^{k-1}A^{k-j-1}\sigma(k-j-1)BQ(j-1;n):=Q(k-1;n+1).
\end{aligned}$$

what was to be proved.

The identity (2.2) is obvious:

$$\begin{aligned}
&\mathcal{Z}^{-1}\left\{\frac{1}{z^{mj+\gamma}}\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}(k)\\
&=\mathcal{Z}^{-1}\left\{\frac{1}{z^{mj+\gamma}}\right\}*\mathcal{Z}^{-1}\left\{\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}\\
&=\left(\delta(mj+\gamma,\cdot)Q(\cdot-1;j)\sigma(\cdot-j-1)\right)\\
&=\sum_{s=0}^{k}\delta(mj+\gamma,s)Q(k-s-1;j)\sigma(k-s-j-1)\\
&=Q(k-mj-\gamma-1;j).
\end{aligned}$$

□

Lemma 2.3. *Let $m\geq 1$, A,B be a constant $d\times d$ matrices, $\varphi:\mathbb{Z}_{-m}^{0}\to\mathbb{R}^{d}$ be given function. Assume that $f:\mathbb{Z}_{0}^{\infty}\to\mathbb{R}^{d}$ is exponentially bounded. Then the solution of Cauchy problem (1.1), (1.2) is exponentially bounded.*

For given matrices A,B and delay m, we define delayed perturbation of discrete matrix exponential $X_{m}^{A,B}(k)$ by the following definition.

Definition 2.4. Let $m\geq 1$, A,B be a constant $d\times d$ matrices. Delayed perturbation of discrete matrix exponential is defined as

$$X_{m}^{A,B}(k)=\sum_{j=0}^{\left\lfloor\frac{k+m}{m+1}\right\rfloor}Q(k+m-mj;j):\mathbb{Z}_{0}^{\infty}\to\mathbb{R}^{d\times d},$$

where

$$Q(k;j) = \begin{cases} 0, & j \in \mathbb{Z}_{-\infty}^{-1}, \\ A^k \sigma(k), & j = 0, \\ \sum_{l=j}^{k} A^{k-l} B Q(l-1; j-1) \sigma(k-j), & j \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{2.4}$$

Remark 2.5. It should be stressed out that $Q(k;j)$ was used in [17] to define delayed perturbation of Mittag-Leffler functions. Using the definition (2.4) of $Q(k;j)$ one may show that

	$j=0$	$j=1$	$j=2$	$j=3$	$\cdots$	$j=p,$
$Q(0,j)$	I	Θ	Θ			
$Q(1,j)$	A	B	Θ	Θ	$\cdots$	$\Theta,$
$Q(2,j)$	A^2	$AB+BA$	B^2	Θ	$\cdots$	$\Theta,$
$Q(3,j)$	A^3	$A(AB+BA)+BA^2$	$AB^2+B(AB+BA)$	B^3	$\cdots$	$\Theta,$
					$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\Theta,$
$Q(p,j)$	A^p	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$B^p.$

From the above table, it is easily seen that, in the case of commutativity $AB = BA$, we have $Q(k;j) := \binom{k}{j} A^{k-j} B^j \sigma(k-j), \quad k, j \in \mathbb{Z}_0^{\infty}$.

3 Representation of a solution

Below using the $\mathcal{Z}$-transform we prove the main result of the paper on the representation of solution of the problem (1.1), (1.2) in terms of the delayed perturbation of discrete matrix exponential.

Theorem 3.1. *Let $m \geq 1$, A, B be a constant $d \times d$ matrices, $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^d$ be given function. Assume that $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^d$ is exponentially bounded. The solution $x(k)$ of the Cauchy problem* (1.1), (1.2) *has the following form*

$$x(k) = X_m^{A,B}(k-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i) B\varphi(i) + \sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1),$$

for $k \in \mathbb{Z}_{-m}^{\infty}$.

Proof. We recall that existence of $\mathcal{Z}$-transform of $f(k)$ and $x(k)$ is guaranteed by Lemma 2.3. Thus we may apply the $\mathcal{Z}$-transform to the equation (1.1) to get

$$\sum_{k=0}^{\infty} \frac{x(k+1)}{z^k} = A \sum_{k=0}^{\infty} \frac{x(k)}{z^k} + B \sum_{k=0}^{\infty} \frac{x(k-m)}{z^k} + \sum_{k=0}^{\infty} \frac{f(k)}{z^k},$$

$$z(X(z) - \varphi(0)) = AX(z) + \frac{B}{z^m}\left(X(z) + \sum_{k=-m}^{-1} \frac{\varphi(k)}{z^k}\right) + F(z),$$

$$\left(zI - A - \frac{B}{z^m}\right) X(z) = z\varphi(0) + \frac{B}{z^m} \sum_{k=-m}^{-1} \frac{\varphi(k)}{z^k} + F(z)$$

$$X(z) = z\left(zI - A - \frac{B}{z^m}\right)^{-1} \varphi(0) + \left(zI - A - \frac{B}{z^m}\right)^{-1} \sum_{k=-m}^{-1} B \frac{\varphi(k)}{z^{k+m}}$$

$$+ \left(zI - A - \frac{B}{z^m}\right)^{-1} F(z). \tag{3.1}$$

On the other hand, for sufficiently large $z \in \mathbb{R}$ so that $\left\| (zI - A)^{-1} \frac{B}{z^m} \right\| < 1$

$$\left(zI - A - \frac{B}{z^m} \right)^{-1} = \left(I - (zI - A)^{-1} \frac{B}{z^m} \right)^{-1} (zI - A)^{-1}$$
$$= \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1}. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$X(z) = \sum_{j=0}^{\infty} \frac{z}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} \varphi(0)$$
$$+ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} \sum_{k=-m}^{-1} B \frac{\varphi(k)}{z^{k+m}}$$
$$+ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} F(z),$$

for sufficiently large z. Taking the inverse $\mathcal{Z}$-transform, we have

$$x(k) = A_0(k) + \sum_{i=-m}^{-1} A_i(k) + A_f(k),$$

where

$$A_0(k) = \mathcal{Z}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j \frac{1}{z^{-1}} (zI - A)^{-1} \varphi(0) \right\}(k),$$
$$A_i(k) = \mathcal{Z}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j \frac{1}{z^{i+m}} (zI - A)^{-1} B\varphi(i) \right\}(k), \quad i \in \mathbb{Z}_{-m}^{-1},$$
$$A_f(k) = \mathcal{Z}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} F(z) \right\}(k).$$

By Lemma 2.2, we have

$$x(k) = \sum_{j=0}^{\left\lfloor \frac{k}{m+1} \right\rfloor} Q(k - jm; j) \varphi(0) + \sum_{i=-m}^{-1} \sum_{j=0}^{\left\lfloor \frac{k-i}{m+1} + 1 \right\rfloor} Q(k - jm - i - m - 1; j) B\varphi(i)$$
$$+ \sum_{l=1}^{k} \sum_{j=0}^{\left\lfloor \frac{k-l}{m+1} \right\rfloor} Q(k - l - jm; j) f(l - 1).$$

□

Lemma 3.2. *Matrix $Q(k;j)$ has the following properties*

(i) $Q(k+1;j) = AQ(k;j) + BQ(k;j-1), \ k, j \in \mathbb{Z}_0^{\infty}$.

(ii) If $AB = BA$, then we have

$$Q(k;j) := \binom{k}{j} A^{k-j} B^j \sigma(k-j), \qquad k, j \in \mathbb{Z}_0^{\infty}.$$

Proof. (i) follows directly from the definition (2.4) of $Q(k;j)$. To show (ii) we use the definition of $Q(k;j)$:

$$Q(k,0) = A^k\sigma(k), \qquad Q(k;j) = \sum_{l=j}^{k} A^{k-l}BQ(l-1;j-1)\,\sigma(k-j), \qquad j \geq 1.$$

For $j = 0, 1$, we have

$$Q(k,0) = A^k\sigma(k), \qquad Q(k,1) = \sum_{l=1}^{k} A^{k-l}BA^{l-1} = kA^{k-1}B = \binom{k}{1} A^{k-1}B.$$

Assume that it is true for $j = n$, and let us prove it for $j = n+1$:

$$\begin{aligned} Q(k;n+1) &= \sum_{l=n+1}^{k} A^{k-l}BQ(l-1;n)\,\sigma(k-n-1) \\ &= \sum_{l=n+1}^{k} A^{k-l}B\binom{l-1}{n} A^{l-1-n}B^n\sigma(k-n-1)\,\sigma(l-n-1) \\ &= A^{k-n-1}B^{n+1}\sum_{l=n+1}^{k}\binom{l-1}{n}\sigma(k-n-1) \\ &= \binom{k}{n+1} A^{k-n-1}B^{n+1}\sigma(k-n-1). \end{aligned}$$

□

Lemma 3.3. *We have the following special cases:*

(i) If $A = I$, then $X_m^{A,B}(k) = e_m^{Bk}$;

(ii) If $B = \Theta$, then $X_m^{A,\Theta}(k) = A^{k+m}$.

Proof. It follows

$$Q(k-jm;j) = \binom{k-jm}{j} A^{k-jm-j}B^j$$

(i) It follows that

$$X_m^{I,B}(k) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} Q(k+m-mj;j) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} \binom{k+m-jm}{j} B^j = e_m^{Bk}.$$

(ii) $B = \Theta$:

$$X_m^{A,\Theta}(k) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} Q(k+m-mj;j) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} \binom{k+m-jm}{j} A^{k+m-jm-j}B^j = A^{k+m}.$$

□

Lemma 3.4 ([21]). *Let $l \in \mathbb{Z}_0^{\infty}$. $k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}$ if and only if*

$$l = \left\lfloor \frac{k-1}{m+1} \right\rfloor + 1 = \left\lfloor \frac{k+m}{m+1} \right\rfloor.$$

Proof. Indeed, for this l,

$$(l-1)(m+1)+1=\left\lfloor \frac{k-1}{m+1}\right\rfloor (m+1)+1\leq k$$

and

$$l(m+1)=\left\lfloor \frac{k+m}{m+1}\right\rfloor (m+1)=\left\lceil \frac{k}{m+1}\right\rceil (m+1)\geq k.$$

On the other hand, if $k\in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}$ for some $l\in \mathbb{Z}_0^{\infty}$, then $l\leq \frac{k+m}{m+1}$ and $\frac{k}{m+1}\leq l$. Hence, $l\leq \left\lfloor \frac{k+m}{m+1}\right\rfloor$ and $\left\lceil \frac{k}{m+1}\right\rceil \leq l$, i.e. $l=\left\lfloor \frac{k+m}{m+1}\right\rfloor$. □

Using this lemma, one can easily show that

$$X_m^{A,B}(k)=\begin{cases}\Theta, & k\in \mathbb{Z}_{-\infty}^{-m-1},\\ A^{k+m}+\sum\limits_{j=1}^{l}Q(k+m-mj;j), & k\in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)},\ l\in \mathbb{Z}_0^{\infty}.\end{cases}$$

Lemma 3.5. *$X_m^{A,B}(k)$ is a solution of*

$$X_m^{A,B}(k+1)=AX_m^{A,B}(k)+BX_m^{A,B}(k-m),$$
$$X_m^{A,B}(k)=A^{k+m},\qquad k\in \mathbb{Z}_{-m}^{0},\ X_m^{A,B}(k)=\Theta,\ k\in \mathbb{Z}_{-\infty}^{-m-1}.$$

Proof. By Lemma 3.2, we have

$$\begin{aligned}X_m^{A,B}(k+1)&=\sum_{j=0}^{\left\lfloor \frac{k+1+m}{m+1}\right\rfloor}Q(k+1+m-mj;j)\\&=\sum_{j=0}^{\left\lfloor \frac{k+m}{m+1}\right\rfloor}AQ(k+m-mj;j)+\sum_{j=1}^{\left\lfloor \frac{k+1+m}{m+1}\right\rfloor}BQ(k+m-mj;j-1)\\&=AX_m^{A,B}(k)+B\sum_{j=0}^{\left\lfloor \frac{k+m}{m+1}\right\rfloor}Q(k-mj;j)\\&=AX_m^{A,B}(k)+BX_m^{A,B}(k-m).\end{aligned}$$

□

It should be stressed out that the assumption on the exponential boundedness of the function f can be omitted.

Theorem 3.6. *The solution of initial value problem (1.1), (1.2) can be written in the following form*

$$x(k)=X_m^{A,B}(k-m)\varphi(0)+\sum_{i=-m}^{-1}X_m^{A,B}(k-1-2m-i)B\varphi(i)$$
$$+\sum_{i=1}^{k}X_m^{A,B}(k-m-i)f(i-1),\qquad k\in \mathbb{Z}_0^{\infty}. \tag{3.3}$$

Proof. If $k\in \mathbb{Z}_0^{m-1}$, then $k-m\in \mathbb{Z}_{-m}^{1}$ and

$$X_m^{A,B}(k-1-2m-i)=\begin{cases}\Theta, & i\in \mathbb{Z}_{k-m}^{-1},\ (k-1-2m-i\leq -m-1)\\ A^k, & i\in \mathbb{Z}_{-m}^{k-m-1}\ \ (-m\leq k-1-2m-i\leq 0).\end{cases}$$

Thus (3.3) gives

$$x(k) = A^k \varphi(0) + \sum_{i=-m}^{k-m-1} A^{k-1-m-i} B\varphi(i) + \sum_{i=1}^{k} A^{k-i} f(i-1)$$

and

$$\begin{aligned} x(k+1) &= A^{k+1}\varphi(0) + \sum_{i=-m}^{k-m} A^{k-m-i} B\varphi(i) + \sum_{i=1}^{k+1} A^{k+1-i} f(i-1) \\ &= A\left(A^k \varphi(0) + \sum_{i=-m}^{k-m-1} A^{k-1-m-i} B\varphi(i) + \sum_{i=1}^{k} A^{k-i} f(i-1)\right) + B\varphi(k-m) + f(k) \\ &= Ax(k) + B\varphi(k-m) + f(k). \end{aligned}$$

For $k \in \mathbb{Z}_m^{\infty}$:

$$\begin{aligned} x(k+1) = {} & X_m^{A,B}(k+1-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-2m-i) B\varphi(i) \\ & + \sum_{i=1}^{k+1} X_m^{A,B}(k+1-m-i) f(i-1) \\ = {} & AX_m^{A,B}(k-m)\varphi(0) + BX_m^{A,B}(k-2m)\varphi(0) \\ & + A\sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i) B\varphi(i) + B\sum_{i=-m}^{-1} X_m^{A,B}(k-1-3m-i) B\varphi(i) \\ & + A\sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1) + B\sum_{i=1}^{k} X_m^{A,B}(k-2m-i) f(i-1) \\ & + X_m^{A,B}(-m) f(k) \\ = {} & Ax(k) + Bx(k-m) + f(k). \end{aligned}$$

For $k \in \mathbb{Z}_{-m}^{-1}$:

$$\begin{aligned} x(k) = {} & X_m^{A,B}(k-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i) B\varphi(i) \\ & + \sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1). \end{aligned}$$

□

4 Conclusion

The paper solves a problem of representation of solution for discrete linear delay system using the delayed perturbation of discrete matrix exponential. In [2,3] discrete delayed matrix exponential is suggested to express solutions of delayed equations with first-order differences: $x(k+1) = Ax(k) + Bx(k-m) + f(k)$. These results are obtained under the commutativity of A and B, and under the condition $\det A \neq 0$. Commutativity condition was omitted in [15]. In this paper we drop the condition of existence of a matrix A^{-1}. The result has been obtained by defining the new delayed perturbation of discrete matrix exponential and employing the $\mathcal{Z}$-transform.

One possible direction in which to generalise the results of this paper is by looking at higher-order linear delay difference equations. It would be interesting to see how the theorems proved above can be extended to these cases. Another direction in which we would like to extend is to consider the classical, fractional and discrete linear systems containing multiple delays.

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