



Null controllability for a singular heat equation with a memory term

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Abstract. In this paper we focus on the null controllability problem for the heat equation with the so-called inverse square potential and a memory term. To this aim, we first establish the null controllability for a nonhomogeneous singular heat equation by a new Carleman inequality with weights which do not blow up at $t = 0$. Then the null controllability property is proved for the singular heat equation with memory under a condition on the kernel, by means of Kakutani's fixed-point theorem.

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1 Introduction

In this paper we address the null controllability for the following singular heat equation with memory:

$$\begin{cases} y_t - y_{xx} - \frac{\mu}{x^2}y = \int_0^t a(t, r, x)y(r, x) dr + 1_\omega u, & (t, x) \in Q, \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

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where $y_0 \in L^2(0,1)$, $T > 0$ is fixed, μ is a real parameter, $Q := (0, T) \times (0, 1)$ and 1_ω stands for a characteristic function of a nonempty open subset ω of $(0, 1)$. Here y and u are the state variable and the control variable, respectively; a is a given L^∞ function defined on $(0, T) \times Q$.

The analysis of evolution equations involving memory terms is a topic in continuous development. In the last decades, many researchers have started devoting their attention to this branch of mathematics, motivated by many applications in modelling phenomena in which the processes are affected not only by its current state but also by its history. Indeed, there is a large spectrum of situations in which the presence of the memory may render the description of the phenomena more accurate. This is particularly the case for models such as heat conduction in materials with memory, viscoelasticity, theory of population dynamics and nuclear reactors, where one often needs to reflect the effects of the memory of the system (see for instance [4, 8, 32, 38]).

Controllability problems for evolution equations with memory terms have been extensively studied in the past. Among other contributions, we mention [5, 21, 24, 27, 28, 30, 33, 39, 42] which, as in our case, deal with parabolic type equations. We also refer to [37] for an overview of the bibliography on control problems for systems with persistent memory. The first results for a degenerate parabolic equation with memory can be found in [1].

In this work, for the first time to our knowledge, we study the null controllability for (1.1). We underline that here we consider not only a memory term but also a singular potential one. In other words, given any $y_0 \in L^2(0,1)$, we want to show that there exists a control function $u \in L^2(Q)$ such that the corresponding solution y to (1.1) satisfies $y(T, x) = 0$ for every $x \in [0, 1]$. First results in this direction are obtained in [46] in the absence of a memory term when $\mu \leq \frac{1}{4}$ (see also [45] for the wave and Schrödinger equations and [11] for boundary singularity). Indeed, for the equation

$$u_t - \Delta u - \mu \frac{1}{|x|^2} u = 0, \quad (t, x) \in (0, T) \times \Omega, \quad (1.2)$$

with associated Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the singularity $x = 0$ in the interior, the value of the parameter μ determines the behavior of the equation: if $\mu \leq 1/4$ (which is the optimal constant of the Hardy inequality, see [9]) global positive solutions exist, while, if $\mu > 1/4$, instantaneous and complete blow-up occurs (for other comments on this argument we refer to [44]). In the case of global positive solutions, hence if $\mu \leq \frac{1}{4}$, using Carleman estimates, it has been proved that such equations can be controlled (in any time $T > 0$) by a locally distributed control (see [46]). On the contrary, if $\mu > \frac{1}{4}$, the null controllability fails as shown in [14]. After these first results, several other works followed extending them in various situations (see for instance [6, 7, 11, 15–20, 36, 44]).

However, when $\mu = 0$ and $a = 1$, (1.1) reduces to the following control system associated to the classical heat equation with memory:

$$\begin{cases} y_t - y_{xx} = \int_0^t y(s) dr + 1_\omega u, & (t, x) \in Q, \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases} \quad (1.3)$$

In this case, as shown in [24, 49], there exists a set of initial conditions such that the null controllability property for (1.3) fails whenever the control region ω is fixed, independent of time. For some related works in this respect we also refer to [12, 28, 48].

Nevertheless, since the positive controllability results are important in real world applications, it is natural to analyze whether it is possible that control properties for (1.1) could be obtained. For this reason, under suitable conditions on the singularity parameter μ and on the kernel a (see (3.1)), we establish that (1.1) is null controllable.

Our approach is inspired from the techniques presented in the work [42] for the heat equation perturbed with a memory-type kernel, suitably adapted in order to deal with the additional inverse-square potential.

We recall that a natural technique for showing controllability results for parabolic equations is to prove an observability estimate for their adjoint systems by Carleman inequalities. However, this classical strategy does not seem to be appropriate for studying the controllability problem for integro-differential parabolic equations like (1.1). In fact, as in [10, 42], in this case we shall argue by a fixed point procedure. For this reason, we shall introduce a nonhomogeneous singular heat equation for which we prove a null controllability result by a modified Carleman inequality with weighted functions that do not blow up at $t = 0$. This is crucial in order to get the null controllability of the memory system (1.1) by weakening the assumptions on the kernel a . Finally, we mention that Carleman inequalities for singular equations without memory have been obtained in [44, 46], but the employment of a weight blowing up at $t = 0$ and $t = T$ in the Carleman inequality does not permit to consider a general kernel a .

The paper is organized as follows: Section 2 is devoted to the study of null controllability for a nonhomogeneous singular heat equation without memory via new Carleman estimates. In Section 3, the null controllability for the singular heat equation with memory (1.1) is proved.

A final comment on the notation: by C we shall denote universal positive constants, which are allowed to vary from line to line.

2 Nonhomogeneous singular heat equation

In this section, we prove the null controllability for a nonhomogeneous singular heat equation using a new modified Carleman inequality. This null controllability result is the key tool for the controllability of the heat equation with memory. Thus, as a first step, we consider the following problem:

$$\begin{cases} y_t - y_{xx} - \frac{\mu}{x^2}y = f + 1_\omega u(t), & (t, x) \in Q := (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \end{cases} \quad (2.1)$$

where $f \in L^2(Q)$ is a given source term.

Prior to null controllability is the well-posedness of (2.1), a question we address in the next subsection.

2.1 Functional framework and well-posedness

We analyze here existence and uniqueness of solutions for the heat problem (2.1). To simplify the presentation, we first focus on the well-posedness of the following inhomogeneous

singular problem

$$\begin{cases} y_t - y_{xx} - \frac{\mu}{x^2}y = f, & (t, x) \in Q, \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases} \quad (2.2)$$

In this framework, in order to deal with the singularity of the potential, a fundamental tool is the very famous Hardy inequality. To fix the ideas, we recall here the basic form of the Hardy inequality in dimension one (see, for example, [29, Theorem 327] or [13, Lemma 5.3.1]):

$$\frac{1}{4} \int_0^1 \frac{y^2}{x^2} dx \leq \int_0^1 y_x^2 dx, \quad (2.3)$$

which is valid for every $y \in H^1(0, 1)$ with $y(0) = 0$.

Now, for any $\mu \leq \frac{1}{4}$, we define

$$H_0^{1,\mu}(0, 1) := \left\{ y \in L^2(0, 1) \cap H_{loc}^1((0, 1]) \mid y(0) = y(1) = 0, \text{ and } \int_0^1 \left(y_x^2 - \mu \frac{y^2}{x^2} \right) dx < +\infty \right\}.$$

Note that $H_0^{1,\mu}(0, 1)$ is a Hilbert space obtained as the completion of $C_c^\infty(0, 1)$, or $H_0^1(0, 1)$, with respect to the norm

$$\|y\|_\mu := \left(\int_0^1 \left(y_x^2 - \mu \frac{y^2}{x^2} \right) dx \right)^{\frac{1}{2}}, \quad \forall y \in H_0^{1,\mu}(0, 1).$$

In the case of a sub-critical parameter $\mu < \frac{1}{4}$, thanks to the Hardy inequality (2.3), one can see that $\|\cdot\|_\mu$ is equivalent to the standard norm of $H_0^1(0, 1)$, and thus $H_0^{1,\mu}(0, 1) = H_0^1(0, 1)$. In the critical case $\mu = \frac{1}{4}$, it is proved (see [47]) that this identification does not hold anymore and the space $H_0^{1,\mu}(0, 1)$ is slightly (but strictly) larger than $H_0^1(0, 1)$.

Now, define the operator $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ corresponding to the heat equation with an inverse square potential in the following way:

$$Ay := y_{xx} + \frac{\mu}{x^2}y$$

$$\forall y \in D(A) := \left\{ y \in H_{loc}^2((0, 1]) \cap H_0^{1,\mu}(0, 1) : y_{xx} + \frac{\mu}{x^2}y \in L^2(0, 1) \right\}.$$

In this context, A is self-adjoint, nonpositive on $L^2(0, 1)$ and it generates an analytic semi-group of contractions in $L^2(0, 1)$ for the equation (2.2) (see [47]). Consequently, the singular heat equation (2.2) is well-posed. To be precise, the next result holds.

Theorem 2.1. *For all $f \in L^2(Q)$ and $y_0 \in L^2(0, 1)$, there exists a unique solution*

$$y \in \mathcal{W} := C([0, T]; L^2(0, 1)) \cap L^2(0, T; H_0^{1,\mu}(0, 1))$$

of (2.2) such that

$$\sup_{t \in [0, T]} \|y(t)\|_{L^2(0, 1)}^2 + \int_0^T \|y(t)\|_\mu^2 dt \leq C_T \left(\|y_0\|_{L^2(0, 1)}^2 + \|f\|_{L^2(Q)}^2 \right), \quad (2.4)$$

for some positive constant C_T . Moreover, if $y_0 \in H_0^{1,\mu}(0, 1)$, then

$$y \in \mathcal{Z} := H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A)) \cap C([0, T]; H_0^{1,\mu}(0, 1)), \quad (2.5)$$

and there exists a positive constant C such that

$$\sup_{t \in [0, T]} \left(\|y(t)\|_{\mu}^2 \right) + \int_0^T \left(\|y_t\|_{L^2(0,1)}^2 + \|y_{xx} + \frac{\mu}{x^2}y\|_{L^2(0,1)}^2 \right) dt \leq C \left(\|y_0\|_{\mu}^2 + \|f\|_{L^2(Q)}^2 \right). \quad (2.6)$$

Proof. In [47], the authors use semigroup theory to obtain the well-posedness result for the problem (2.2) (see also [36]). Thus, in the rest of the proof, we will prove only (2.4)–(2.6). First, being A the generator of a strongly continuous semigroup on $L^2(0,1)$, if $y_0 \in L^2(0,1)$, then the solution y of (2.2) belongs to $C([0, T]; L^2(0,1)) \cap L^2(0, T; H_0^{1,\mu}(0,1))$, while, if $y_0 \in D(A)$, then $y \in H^1(0, T; L^2(0,1)) \cap L^2(0, T; D(A))$.

Now, by a usual energy method we shall prove (2.5) and (2.6), from which the last required regularity property for y will follow by standard linear arguments. First, take $y_0 \in D(A)$ and multiply the equation of (2.2) by y . By the Cauchy–Schwarz inequality we obtain for every $t \in (0, T]$,

$$\frac{1}{2} \frac{d}{dt} \|y(t)\|_{L^2(0,1)}^2 + \|y(t)\|_{\mu}^2 \leq \frac{1}{2} \|f(t)\|_{L^2(0,1)}^2 + \frac{1}{2} \|y(t)\|_{L^2(0,1)}^2. \quad (2.7)$$

From (2.7) and using Gronwall’s inequality, we get

$$\|y(t)\|_{L^2(0,1)}^2 \leq e^T \left(\|y(0)\|_{L^2(0,1)}^2 + \|f\|_{L^2(Q)}^2 \right) \quad (2.8)$$

for every $t \leq T$. From (2.7) and (2.8) we immediately obtain

$$\int_0^T \|y(t)\|_{\mu}^2 dt \leq C_T \left(\|y(0)\|_{L^2(0,1)}^2 + \|f\|_{L^2(Q)}^2 \right) \quad (2.9)$$

for some universal constant $C_T > 0$. Thus, by (2.8) and (2.9), (2.4) follows if $y_0 \in D(A)$. Since $D(A)$ is dense in $L^2(0,1)$ (see [43, 47]), the same inequality holds if $y_0 \in L^2(0,1)$.

Now, multiplying the equation by $-y_{xx} - \frac{\mu}{x^2}y$, integrating on $(0,1)$ and using the Cauchy–Schwarz inequality, we easily get

$$\frac{d}{dt} \|y(t)\|_{\mu}^2 + \|y_{xx}(t) + \frac{\mu}{x^2}y(t)\|_{L^2(0,1)}^2 \leq \|f(t)\|_{L^2(0,1)}^2$$

for every $t \in [0, T]$, so that, as before, we find $C'_T > 0$ such that

$$\|y(t)\|_{\mu}^2 + \int_0^T \|y_{xx}(t) + \frac{\mu}{x^2}y(t)\|_{L^2(0,1)}^2 dt \leq C'_T \left(\|y(0)\|_{\mu} + \|f\|_{L^2(Q)}^2 \right) \quad (2.10)$$

for every $t \leq T$. Finally, from $y_t = y_{xx} + \frac{\mu}{x^2}y + f$, squaring and integrating on Q , we find

$$\int_0^T \|y_t(t)\|_{L^2(0,1)}^2 \leq C \left(\int_0^T \|y_{xx} + \frac{\mu}{x^2}y\|_{L^2(0,1)}^2 + \|f\|_{L^2(Q)}^2 \right),$$

and together with (2.10) we have

$$\int_0^T \|y_t(t)\|_{L^2(0,1)}^2 \leq C \left(\|y(0)\|_{\mu}^2 + \|f\|_{L^2(Q)}^2 \right). \quad (2.11)$$

In conclusion, (2.7), (2.8), (2.10) and (2.11) give (2.4) and (2.6). Notice that, (2.5) and (2.6) hold also if $y_0 \in H_0^{1,\mu}(0,1)$. \square

2.2 Carleman estimates for a singular problem

In this subsection we prove a new Carleman estimate for the adjoint parabolic equation associated to (2.1), which will provide that the nonhomogeneous singular heat equation (2.1) is null controllable. Hence, in the following, we concentrate on the next adjoint problem

$$\begin{cases} -z_t - z_{xx} - \frac{\mu}{x^2}z = g, & (t, x) \in Q, \\ z(t, 0) = z(t, 1) = 0, & t \in (0, T), \\ z(T, x) = z_T(x), & x \in (0, 1). \end{cases} \quad (2.12)$$

Following [46], for every $0 < \gamma < 2$, let us introduce the weight function

$$\varphi(t, x) := \theta(t)\psi(x), \quad (2.13)$$

where

$$\psi(x) := c(x^2 - d), \quad \theta(t) := \left(\frac{1}{t(T-t)} \right)^k, \quad k := 1 + \frac{2}{\gamma}, \quad (2.14)$$

$c > 0$ and $d > 1$. A more precise restriction on the parameters k, c and d will be needed later. Observe that $\lim_{t \rightarrow 0^+} \theta(t) = \lim_{t \rightarrow T^-} \theta(t) = +\infty$, and

$$\psi(x) < 0 \quad \text{for every } x \in [0, 1].$$

Using the previous weight functions and the following improved Hardy–Poincaré inequality given in [44]:

For all $\eta > 0$, there exists some positive constant $C = C(\eta) > 0$ such that, for all $z \in C_c^\infty(0, 1)$:

$$\int_0^1 x^\eta z_x^2 dx \leq C \int_0^1 \left(z_x^2 - \frac{1}{4} \frac{z^2}{x^2} \right) dx, \quad (2.15)$$

one can prove the following Carleman estimate for the case of a purely singular parabolic equation:

Lemma 2.2 ([44, Theorem 5.1]). *Assume that $\mu \leq \frac{1}{4}$. Then, there exists $C > 0$ and $s_0 > 0$ such that, for all $s \geq s_0$, every solution z of (2.12) satisfies*

$$\begin{aligned} & \iint_Q s^3 \theta^3 x^2 z^2 e^{2s\varphi} dx dt + \iint_Q s\theta \left(z_x^2 - \mu \frac{z^2}{x^2} \right) e^{2s\varphi} dx dt + \iint_Q s\theta \frac{z^2}{x^\gamma} e^{2s\varphi} dx dt \\ & \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta z_x^2(t, 1) e^{2s\varphi(t, 1)} dx dt \right). \end{aligned} \quad (2.16)$$

Observe that, if the term

$$\iint_Q s\theta \left(z_x^2 - \mu \frac{z^2}{x^2} \right) e^{2s\varphi} dx dt$$

is not positive, then the estimate (2.16) is not of great importance. In fact, the Hardy inequality (2.3) only ensures the positivity of the quantity

$$\iint_Q s\theta \left(z_x^2 - \mu \frac{z^2}{x^2} \right) dx dt.$$

However, from [44, Remark 3] and similarly as in [25], we will rewrite the result given in Lemma 2.2 in a more practical way.

Lemma 2.3. Assume that $\mu \leq \frac{1}{4}$. Then, there exist $C > 0$ and $s_0 > 0$ such that, for all $s \geq s_0$, every solution z of (2.12) satisfies

$$\mathfrak{J}_{\varphi,\eta,\gamma}(z) \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta z_x^2(t,1) e^{2s\varphi(t,1)} dx dt \right), \quad (2.17)$$

where

$$\mathfrak{J}_{\varphi,\eta,\gamma}(z) = \iint_Q s^3 \theta^3 x^2 z^2 e^{2s\varphi} dx dt + \iint_Q s\theta z_x^2 e^{2s\varphi} dx dt + \iint_Q s\theta \frac{z^2}{x^2} e^{2s\varphi} dx dt, \quad (2.18)$$

if $\mu < \frac{1}{4}$, and

$$\mathfrak{J}_{\varphi,\eta,\gamma}(z) = \iint_Q s^3 \theta^3 x^2 z^2 e^{2s\varphi} dx dt + \iint_Q s\theta x^\eta z_x^2 e^{2s\varphi} dx dt + \iint_Q s\theta \frac{z^2}{x^\gamma} e^{2s\varphi} dx dt, \quad (2.19)$$

if $\mu = \frac{1}{4}$. We recall that $0 < \gamma < 2$.

Proof. Case 1: If $\mu < \frac{1}{4}$.

Let $Z = ze^{s\varphi}$. In order to prove [44, Theorem 5.1], the author has derived the following estimate

$$\begin{aligned} & \iint_Q s^3 \theta^3 x^2 Z^2 dx dt + \iint_Q s\theta \left(Z_x^2 - \mu \frac{Z^2}{x^2} \right) dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \\ & \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta Z_x^2(t,1) dx dt \right). \end{aligned} \quad (2.20)$$

Let $\delta < \inf(1, (1 - 4\mu))$ be a fixed positive constant. We have

$$\begin{aligned} \iint_Q s\theta \left(Z_x^2 - \mu \frac{Z^2}{x^2} \right) dx dt &= (1 - \delta) \iint_Q s\theta \left(Z_x^2 - \frac{1}{4} \frac{Z^2}{x^2} \right) dx dt \\ &+ \delta \iint_Q s\theta Z_x^2 dx dt + \left(\frac{1}{4}(1 - \delta) - \mu \right) \iint_Q s\theta \frac{Z^2}{x^2} dx dt. \end{aligned} \quad (2.21)$$

By (2.20) and (2.21), we obtain

$$\begin{aligned} & \iint_Q s^3 \theta^3 x^2 Z^2 dx dt + (1 - \delta) \iint_Q s\theta \left(Z_x^2 - \frac{1}{4} \frac{Z^2}{x^2} \right) dx dt + \delta \iint_Q s\theta Z_x^2 dx dt \\ &+ \left(\frac{1}{4}(1 - \delta) - \mu \right) \iint_Q s\theta \frac{Z^2}{x^2} dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \\ & \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta Z_x^2(t,1) dx dt \right). \end{aligned}$$

On the other hand, from (2.15), for all $\eta > 0$ there exists a constant $c_0 = c_0(\eta) > 0$ such that

$$\iint_Q s\theta \left(Z_x^2 - \frac{1}{4} \frac{Z^2}{x^2} \right) dx dt \geq c_0 \iint_Q s\theta x^\eta Z_x^2 dx dt. \quad (2.22)$$

Hence,

$$\begin{aligned} & \iint_Q s^3 \theta^3 x^2 Z^2 dx dt + (1 - \delta)c_0 \iint_Q s\theta x^\eta Z_x^2 dx dt + \delta \iint_Q s\theta Z_x^2 dx dt \\ &+ \left(\frac{1}{4}(1 - \delta) - \mu \right) \iint_Q s\theta \frac{Z^2}{x^2} dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \\ & \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta Z_x^2(t,1) dx dt \right). \end{aligned} \quad (2.23)$$

Using the definition of Z , we have

$$Z^2 = z^2 e^{2s\varphi}, \quad (2.24)$$

$$Z_x = z_x e^{s\varphi} + s\theta\psi_x Z \quad \text{and} \quad z_x^2 e^{2s\varphi} \leq 2Z_x^2 + cs^2\theta^2 x^2 Z^2, \quad (2.25)$$

for a positive constant c . Then,

$$\iint_Q s\theta z_x^2 e^{2s\varphi} dx dt \leq 2 \iint_Q s\theta Z_x^2 dx dt + c \iint_Q s^3\theta^3 x^2 Z^2 dx dt. \quad (2.26)$$

Combining (2.23)–(2.26), we obtain the desired estimate (2.17). Indeed, defining

$$a_0 = \min \left\{ \frac{1}{1+c}, \frac{\delta}{2}, \left(\frac{1}{4}(1-\delta) - \mu \right) \right\} > 0,$$

we have

$$\begin{aligned} & a_0 \left(\iint_Q s^3\theta^3 x^2 Z^2 e^{2s\varphi} dx dt + \iint_Q s\theta z_x^2 e^{2s\varphi} dx dt + \iint_Q s\theta \frac{z^2}{x^2} e^{2s\varphi} dx dt + \iint_Q s\theta \frac{z^2}{x^\gamma} e^{2s\varphi} dx dt \right) \\ & \leq a_0 \left((1+c) \iint_Q s^3\theta^3 x^2 Z^2 dx dt + 2 \iint_Q s\theta Z_x^2 dx dt + \iint_Q s\theta \frac{Z^2}{x^2} dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \right) \\ & \leq \iint_Q s^3\theta^3 x^2 Z^2 dx dt + \delta \iint_Q s\theta Z_x^2 dx dt + \left(\frac{1}{4}(1-\delta) - \mu \right) \iint_Q s\theta \frac{Z^2}{x^2} dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \\ & \leq \iint_Q s^3\theta^3 x^2 Z^2 dx dt + (1-\delta)c_0 \iint_Q s\theta x^\eta Z_x^2 dx dt + \delta \iint_Q s\theta Z_x^2 dx dt \\ & \quad + \left(\frac{1}{4}(1-\delta) - \mu \right) \iint_Q s\theta \frac{Z^2}{x^2} dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \\ & \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta Z_x^2(t,1) dx dt \right). \end{aligned}$$

Thus, the conclusion follows.

Case 2: If $\mu = \frac{1}{4}$.

As before, let $Z = z e^{s\varphi}$ and define

$$a_0 = \min \left\{ \frac{1}{1+c}, \frac{c_0}{2} \right\} > 0,$$

where c_0 and c are the constants of (2.22) and (2.25), respectively. Then, by (2.20), (2.22), (2.24) and (2.25), that still hold if $\mu = \frac{1}{4}$, we have

$$\begin{aligned} & a_0 \left(\iint_Q s^3\theta^3 x^2 Z^2 e^{2s\varphi} dx dt + \iint_Q s\theta x^\eta z_x^2 e^{2s\varphi} dx dt + \iint_Q s\theta \frac{z^2}{x^\gamma} e^{2s\varphi} dx dt \right) \\ & \leq a_0 \left(\iint_Q s^3\theta^3 x^2 Z^2 dx dt + 2 \iint_Q s\theta x^\eta Z_x^2 dx dt + c \iint_Q s^3\theta^3 x^2 Z^2 dx dt + \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \right) \\ & \leq a_0(1+c) \iint_Q s^3\theta^3 x^2 Z^2 dx dt + a_0 \frac{2}{c_0} \iint_Q s\theta \left(Z_x^2 - \frac{1}{4} \frac{Z^2}{x^2} \right) dx dt + a_0 \iint_Q s\theta \frac{Z^2}{x^\gamma} dx dt \quad (2.27) \\ & \text{(by (2.20))} \\ & \leq C \left(\iint_Q g^2 e^{2s\varphi} dx dt + \int_0^T s\theta z_x^2(t,1) e^{2s\varphi(t,1)} dx dt \right). \end{aligned}$$

Hence, also in this case the conclusion follows. \square

We point out that the Carleman estimates stated above are not appropriate to achieve our goal. In fact, all these estimates does not have the observation term in the interior of the domain. However, we use them to obtain the main Carleman estimate stated in Proposition 2.5. More precisely, from the boundary Carleman estimates (2.17), we will deduce a global Carleman estimate for the adjoint problem (2.12) with a distributed observation on a subregion

$$\omega' := (\alpha', \beta') \subset\subset \omega. \quad (2.28)$$

To do so, we recall the following weight functions associated to nonsingular Carleman estimates which are suited to our purpose:

$$\Phi(t, x) := \theta(t)\Psi(x)$$

where θ is defined in (2.14) and $\Psi(x) = e^{\rho\sigma} - e^{2\rho\|\sigma\|_\infty}$. Here $\rho > 0$, $\sigma \in C^2([0, 1])$ is such that $\sigma(x) > 0$ in $(0, 1)$, $\sigma(0) = \sigma(1) = 0$ and $\sigma_x(x) \neq 0$ in $[0, 1] \setminus \tilde{\omega}$, being $\tilde{\omega}$ an arbitrary open subset of ω .

In the following, we choose the constant \mathfrak{c} in (2.14) so that

$$\mathfrak{c} \geq \frac{e^{2\rho\|\sigma\|_\infty} - 1}{d - 1}.$$

By this choice one can prove that the function φ defined in (2.13) satisfies the next estimate

$$\varphi(t, x) \leq \Phi(t, x) \quad \text{for every } (t, x) \in [0, T] \times [0, 1]. \quad (2.29)$$

Thanks to this property, we can prove the main Carleman estimate of this paper whose proof is based also on the following Caccioppoli's inequality:

Proposition 2.4 (Caccioppoli's inequality). *Let ω' and ω'' be two nonempty open subsets of $(0, 1)$ such that $\overline{\omega''} \subset \omega'$ and $\phi(t, x) = \theta(t)\varrho(x)$, where $\varrho \in C^2(\overline{\omega'}, \mathbb{R})$. Then, there exists a constant $C > 0$ such that any solution z of (2.12) satisfies*

$$\iint_{Q_{\omega''}} z_x^2 e^{2s\phi} dx dt \leq C \iint_{Q_{\omega'}} (g^2 + s^2\theta^2 z^2) e^{2s\phi} dx dt, \quad (2.30)$$

where $Q_\omega := (0, T) \times \omega$.

The proof of the previous result is similar to the one given, for instance, in [3, Lemma 6.1], so we omit it.

Now, we are ready to prove the following result:

Proposition 2.5. *Assume that $\mu \leq \frac{1}{4}$. Then, there exist two positive constants C and s_0 such that, the solution z of equation (2.12) satisfies, for all $s \geq s_0$*

$$\mathfrak{J}_{\varphi, \eta, \gamma}(z) \leq C \left(\iint_Q g^2 e^{2s\Phi} dx dt + \iint_{Q_{\omega'}} s^3 \theta^3 z^2 e^{2s\Phi} dx dt \right). \quad (2.31)$$

Here $\mathfrak{J}_{\varphi, \eta, \gamma}(\cdot)$ is defined in (2.18) or (2.19).

Proof. Let us set $\omega'' = (\alpha'', \beta'') \subset\subset \omega'$ and consider a smooth cut-off function $\xi \in C^\infty([0, 1])$ such that $0 \leq \xi(x) \leq 1$ for $x \in (0, 1)$, $\xi(x) = 1$ for $x \in [0, \alpha'']$ and $\xi(x) = 0$ for $x \in [\beta'', 1]$. Define $w := \xi z$ where z is the solution of (2.12). Then, w satisfies the following problem:

$$\begin{cases} -w_t - w_{xx} - \frac{\mu}{x^2} w = \xi g - \xi_{xx} z - 2\xi_x z_x, & (t, x) \in Q, \\ w(t, 1) = w(t, 0) = 0, & t \in (0, T), \\ w(T, x) = \xi(x) z_T(x), & x \in (0, 1). \end{cases} \quad (2.32)$$

First of all, we prove the first intermediate Carleman estimate for z in $(0, T) \times (0, \alpha')$ (recall that $z \equiv w$ in $[0, \alpha']$):

$$\begin{aligned} \mathfrak{J}_{\varphi, \eta, \gamma}(w) &\leq C \left(\iint_Q \zeta^2 g^2 e^{2s\varphi} dx dt + \iint_{Q_{\omega'}} (g^2 + s^2 \theta^2 z^2) e^{2s\varphi} dx dt \right) \\ &\leq C \left(\iint_Q \zeta^2 g^2 e^{2s\Phi} dx dt + \iint_{Q_{\omega'}} (g^2 + s^2 \theta^2 z^2) e^{2s\Phi} dx dt \right). \end{aligned} \quad (2.33)$$

The second inequality in (2.33) follows by (2.29), thus it is sufficient to prove the first inequality of (2.33). Applying the Carleman estimate (2.17) to (2.32), we obtain

$$\mathfrak{J}_{\varphi, \eta, \gamma}(w) \leq C \iint_Q \left(\zeta^2 g^2 + (\zeta_{xx} z + 2\zeta_x z_x)^2 \right) e^{2s\varphi} dx dt. \quad (2.34)$$

From the definition of ζ and the Caccioppoli inequality (2.30), we obtain

$$\begin{aligned} \iint_Q (\zeta_{xx} z + 2\zeta_x z_x)^2 e^{2s\varphi} dx dt &\leq C \iint_{Q_{\omega''}} (z^2 + z_x^2) e^{2s\varphi} dx dt \\ &\leq C \iint_{Q_{\omega'}} (g^2 + s^2 \theta^2 z^2) e^{2s\varphi} dx dt. \end{aligned} \quad (2.35)$$

Combining (2.34) and (2.35) we obtain (2.33).

Now, using the non singular Carleman estimate of Corollary 5.2, we are going to show a second estimate of z in $(0, T) \times (\beta', 1)$. For this purpose, let $v = \zeta z$ where $\zeta := 1 - \xi$ (hence $z \equiv v$ in $[\beta', 1]$). Clearly, the function v is a solution of the uniformly parabolic equation

$$\begin{cases} -v_t - v_{xx} - \frac{\mu}{x^2} v = \zeta g - \zeta_{xx} z - 2\zeta_x z_x, & (t, x) \in (0, T) \times (\alpha', 1), \\ v(t, 1) = v(t, \alpha') = 0, & t \in (0, T), \\ v(T, x) = \zeta(x) z_T(x), & x \in (\alpha', 1). \end{cases} \quad (2.36)$$

Since ζ has its support in $[\alpha'', \beta'']$, by Corollary 5.2 we have

$$\begin{aligned} \iint_Q (s\theta v_x^2 + s^3 \theta^3 v^2) e^{2s\Phi} dx dt &= \int_0^T \int_{\alpha'}^1 (s\theta v_x^2 + s^3 \theta^3 v^2) e^{2s\Phi} dx dt \\ &\leq C \left(\int_0^T \int_{\alpha'}^1 (\zeta^2 g^2 + (\zeta_{xx} z + 2\zeta_x z_x)^2) e^{2s\Phi} dx dt + \iint_{Q_{\omega''}} s^3 \theta^3 v^2 e^{2s\Phi} dx dt \right) \\ &\leq C \left(\iint_Q \zeta^2 g^2 e^{2s\Phi} dx dt + \iint_{Q_{\omega''}} (z^2 + z_x^2) e^{2s\Phi} dx dt + \iint_{Q_{\omega''}} s^3 \theta^3 v^2 e^{2s\Phi} dx dt \right). \end{aligned}$$

Therefore, by the previous estimate, by (2.29) and using the Caccioppoli inequality (2.30), we deduce

$$\begin{aligned} \iint_Q (s\theta v_x^2 + s^3 \theta^3 v^2) e^{2s\varphi} dx dt &\leq \iint_Q (s\theta v_x^2 + s^3 \theta^3 v^2) e^{2s\Phi} dx dt \\ &\leq C \left(\iint_Q \zeta^2 g^2 e^{2s\Phi} dx dt + \iint_{Q_{\omega'}} (g^2 + s^3 \theta^3 z^2) e^{2s\Phi} dx dt \right). \end{aligned} \quad (2.37)$$

Thus, since $v = \zeta z$ has its support in $[0, T] \times [\alpha'', 1]$, that is far away from the singularity point $x = 0$, one can prove that there exists a constant $C > 0$ such that:

$$\begin{aligned} \mathfrak{J}_{\varphi, \eta, \gamma}(v) &\leq C \iint_Q \left(s\theta v_x^2 + s^3\theta^3 v^2 \right) e^{2s\varphi} dx dt \\ &\quad (\text{by (2.37)}) \\ &\leq C \left(\iint_Q \zeta^2 g^2 e^{2s\Phi} dx dt + \iint_{Q_{\omega'}} (g^2 + s^3\theta^3 z^2) e^{2s\Phi} dx dt \right). \end{aligned} \quad (2.38)$$

Note that

$$z^2 = (w + v)^2 \leq 2(w^2 + v^2) \quad \text{and} \quad z_x^2 = (w_x + v_x)^2 \leq 2(w_x^2 + v_x^2).$$

Therefore, adding (2.33) and (2.38), (2.31) follows immediately. \square

For our purposes in the next section, we concentrate now on a Carleman inequality for solutions of (2.12) obtained via weight functions not exploding at $t = 0$. To this end, we will apply a classical argument that can be found, for instance, in [22] and recently in [1] for a degenerate parabolic equation with memory. More precisely, let us consider the function:

$$v(t) = \begin{cases} \theta \left(\frac{T}{2} \right), & t \in \left[0, \frac{T}{2} \right], \\ \theta(t), & t \in \left[\frac{T}{2}, T \right], \end{cases} \quad (2.39)$$

and the following associated weight functions:

$$\begin{aligned} \tilde{\varphi}(t, x) &:= v(t)\psi(x), & \tilde{\Phi}(t, x) &:= v(t)\Psi(x), \\ \hat{\Phi}(t) &:= \max_{x \in [0, 1]} \tilde{\Phi}(t, x), & \hat{\varphi}(t) &:= \max_{x \in [0, 1]} \tilde{\varphi}(t, x) \quad \text{and} \quad \check{\varphi}(t) := \min_{x \in [0, 1]} \tilde{\varphi}(t, x). \end{aligned} \quad (2.40)$$

Now we are ready to state and prove this new modified Carleman estimate for the adjoint problem (2.12).

Lemma 2.6. *Assume that $\mu \leq \frac{1}{4}$. Then, there exist two positive constants C and s_0 such that every solution z of (2.12) satisfies, for all $s \geq s_0$*

$$\begin{aligned} \|e^{s\hat{\varphi}(0)} z(0)\|_{L^2(0, 1)}^2 + \iint_Q v z^2 e^{2s\tilde{\varphi}} dx dt \\ \leq C e^{2s[\hat{\varphi}(0) - \check{\varphi}(\frac{5T}{8})]} \left(\iint_Q g^2 e^{2s\tilde{\Phi}} dx dt + \iint_{Q_{\omega'}} s^3 v^3 z^2 e^{2s\tilde{\Phi}} dx dt \right). \end{aligned} \quad (2.41)$$

Proof. By the definitions of v and $\tilde{\varphi}$ and using Proposition 2.5, it results that there exists a positive constant C such that all the solutions to equation (2.12) satisfy

$$\begin{aligned} \int_{\frac{T}{2}}^T \int_0^1 v z^2 e^{2s\tilde{\varphi}} dx dt &= \int_{\frac{T}{2}}^T \int_0^1 \theta z^2 e^{2s\varphi} dx dt \leq C \int_{\frac{T}{2}}^T \int_0^1 s\theta \frac{z^2}{x^\gamma} e^{2s\varphi} dx dt \\ &\leq C \left(\iint_Q g^2 e^{2s\Phi} dx dt + \iint_{Q_{\omega'}} s^3 \theta^3 z^2 e^{2s\Phi} dx dt \right). \end{aligned} \quad (2.42)$$

Let us introduce a function $\tau \in C^1([0, T])$ such that $\tau = 1$ in $[0, \frac{T}{2}]$ and $\tau \equiv 0$ in $[\frac{5T}{8}, T]$. Denote $\tilde{\tau} = e^{s\hat{\varphi}(0)} \sqrt{v}\tau$, where $e^{s\hat{\varphi}(0)} = \max_{0 \leq t \leq T} e^{s\hat{\varphi}(t)}$.

Let $\tilde{z} = \tilde{\tau}z$, then \tilde{z} satisfies

$$\begin{cases} -\tilde{z}_t - \tilde{z}_{xx} - \frac{\mu}{x^2}\tilde{z} = -\tilde{\tau}_t z + \tilde{\tau}g, & (t, x) \in Q, \\ \tilde{z}(t, 0) = \tilde{z}(t, 1) = 0, & t \in (0, T), \\ \tilde{z}(T, x) = 0, & x \in (0, 1). \end{cases} \quad (2.43)$$

Thanks to the estimate of $\sup_{t \in [0, T]} \|\tilde{z}(t)\|_{L^2(0,1)}^2$ (see the energy estimate (2.4)), we have

$$\|\tilde{z}(0)\|_{L^2(0,1)}^2 + \|\tilde{z}\|_{L^2(Q)}^2 \leq C \iint_Q (\tilde{\tau}_t z + \tilde{\tau}g)^2 dx dt,$$

which implies

$$v(0)\|e^{s\hat{\phi}(0)}z(0)\|_{L^2(0,1)}^2 + \|e^{s\hat{\phi}(0)}\sqrt{v}\tau z\|_{L^2(Q)}^2 \leq C \iint_Q (\tilde{\tau}_t z + \tilde{\tau}g)^2 dx dt.$$

By using the boundedness of θ in $[\frac{T}{2}, \frac{5T}{8}]$, the definitions of τ and of v in $[0, \frac{5T}{8}]$ and the fact that $v_t(t) = 0$ in $[0, \frac{T}{2}]$ and $\tau(t) = 0$ in $[\frac{5T}{8}, T]$, it holds that

$$\begin{aligned} & \bar{c} \left(\|e^{s\hat{\phi}(0)}z(0)\|_{L^2(0,1)}^2 + \int_0^{\frac{5T}{8}} \int_0^1 v\tau^2 z^2 e^{2s\hat{\phi}} dx dt \right) \\ & \leq v(0)\|e^{s\hat{\phi}(0)}z(0)\|_{L^2(0,1)}^2 + \int_0^{\frac{5T}{8}} \int_0^1 v\tau^2 z^2 e^{2s\hat{\phi}} dx dt \\ & \leq C \left(\int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 (\theta^2(t) + \theta(t))z^2 e^{2s\hat{\phi}(0)} dx dt + \int_0^{\frac{5T}{8}} \int_0^1 v g^2 e^{2s\hat{\phi}(0)} dx dt \right) \\ & \leq C \left(\int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 z^2 e^{2s\hat{\phi}(0)} dx dt + \int_0^{\frac{5T}{8}} \int_0^1 g^2 e^{2s\hat{\phi}(0)} dx dt \right), \end{aligned}$$

where $\bar{c} := \min\{v(0), 1\}$. That is,

$$\begin{aligned} & \|e^{s\hat{\phi}(0)}z(0)\|_{L^2(0,1)}^2 + \int_0^{\frac{T}{2}} \int_0^1 v z^2 e^{2s\hat{\phi}} dx dt \\ & \leq C \left(\int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 z^2 e^{2s(\hat{\phi}(0) - \bar{\phi})} e^{2s\bar{\phi}} dx dt + \int_0^{\frac{5T}{8}} \int_0^1 g^2 e^{2s(\hat{\phi}(0) - \bar{\phi})} e^{2s\bar{\phi}} dx dt \right). \end{aligned}$$

Observe that

$$\bar{\phi} \left(\frac{5T}{8} \right) \leq \bar{\varphi} \quad \text{in} \quad \left(0, \frac{5T}{8} \right) \times (0, 1)$$

so that,

$$\begin{aligned} & \|e^{s\hat{\phi}(0)}z(0)\|_{L^2(0,1)}^2 + \int_0^{\frac{T}{2}} \int_0^1 v z^2 e^{2s\bar{\phi}} dx dt \\ & \leq C e^{2s(\hat{\phi}(0) - \bar{\varphi}(\frac{5T}{8}))} \left(\int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 z^2 e^{2s\bar{\phi}} dx dt + \int_0^{\frac{5T}{8}} \int_0^1 g^2 e^{2s\bar{\phi}} dx dt \right). \end{aligned} \quad (2.44)$$

As in (2.42), one can prove that there exists a positive constant C such that

$$\int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 z^2 e^{2s\bar{\phi}} dx dt \leq C \left(\iint_Q g^2 e^{2s\Phi} dx dt + \iint_{Q_\omega} s^3 \theta^3 z^2 e^{2s\Phi} dx dt \right).$$

Using this last inequality in (2.44), we have

$$\begin{aligned} & \|e^{s\hat{\varphi}(0)}z(0)\|_{L^2(0,1)}^2 + \int_0^{\frac{T}{2}} \int_0^1 \nu z^2 e^{2s\hat{\varphi}} dx dt \\ & \leq C e^{2s(\hat{\varphi}(0) - \hat{\varphi}(\frac{5T}{8}))} \left(\iint_Q g^2 e^{2s\hat{\Phi}} dx dt + \iint_{Q_\omega} s^3 \theta^3 z^2 e^{2s\hat{\Phi}} dx dt + \int_0^{\frac{5T}{8}} \int_0^1 g^2 e^{2s\hat{\varphi}} dx dt \right). \end{aligned} \quad (2.45)$$

From (2.29) and by the definition of the modified weights, notice that, in particular $\tilde{\varphi} \leq \tilde{\Phi}$ and $\tilde{\Phi} \leq \tilde{\Phi}$ in Q . This, together with (2.42) and (2.45), implies that

$$\begin{aligned} & \|e^{s\hat{\varphi}(0)}z(0)\|_{L^2(0,1)}^2 + \int_0^T \int_0^1 \nu z^2 e^{2s\tilde{\varphi}} dx dt \\ & \leq C e^{2s(\hat{\varphi}(0) - \tilde{\varphi}(\frac{5T}{8}))} \left(\iint_Q g^2 e^{2s\tilde{\Phi}} dx dt + \iint_{Q_\omega} s^3 \theta^3 z^2 e^{2s\tilde{\Phi}} dx dt \right). \end{aligned} \quad (2.46)$$

To conclude, it suffices to remark that for $c > 0$, the function $s \mapsto s^3 e^{-cs}$ is nonincreasing for s sufficiently large. So, since $\nu(t) \leq \theta(t)$ by taking s large enough, one has

$$s^3 \theta^3 e^{2s\tilde{\Phi}} \leq s^3 \nu^3 e^{2s\tilde{\Phi}},$$

which, together with (2.46), provides the desired inequality. \square

2.3 Null controllability result

Following the classical method as in [22], with the modified Carleman inequality proved in the previous subsection, we can get a null controllability result for (2.1). However, as explained in [42], this null controllability result cannot help to solve the controllability for integro-differential equations. Indeed, we will need to prove the null controllability of the singular heat equation (2.1), for more regular solutions. For this reason, to formulate our results we introduce the following function space where the controllability will be solved:

$$X_s := \{y \in \mathcal{Z} : e^{-s\tilde{\Phi}} y \in L^2(Q)\}$$

equipped with the norm

$$\|y\|_{X_s} := \|e^{-s\tilde{\Phi}} y\|_{L^2(Q)}.$$

Observe that, since $\tilde{\Phi} < 0$, we have that the function $e^{-s\tilde{\Phi}}$ tends to $+\infty$ for $t \rightarrow T^-$. Therefore, $y \in X_s$ requires that the solution y has more regularity than the one in Lemma 2.1. Moreover,

$$\text{if } y \in X_s \text{ then } y(T, x) = 0 \text{ in } (0, 1). \quad (2.47)$$

From now on, we denote by s_0 the parameter defined in Lemma 2.6. Our first result, stated as follows, ensures the null controllability for (2.1).

Theorem 2.7. *Assume that $\mu \leq \frac{1}{4}$ and $y_0 \in H_0^{1,\mu}(0,1)$. If $e^{-s\hat{\varphi}} f \in L^2(Q)$ with $s \geq s_0$, then there exists a control function $u \in L^2(Q)$, such that the associated solution y of (2.1) belongs to X_s .*

Moreover, there exists a positive constant C such that y satisfies the following estimate:

$$\begin{aligned} & \iint_Q y^2 e^{-2s\tilde{\Phi}} dx dt + \iint_{Q_\omega} s^{-3} \nu^{-3} u^2 e^{-2s\tilde{\Phi}} dx dt \\ & \leq C e^{2s[\hat{\varphi}(0) - \hat{\varphi}(\frac{5T}{8})]} \left(\iint_Q f^2 e^{-2s\hat{\varphi}} dx dt + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)}^2 \right). \end{aligned} \quad (2.48)$$

Proof. Following the ideas in [10,42], fixed $s \geq s_0$, let us consider the functional

$$J(y, u) = \left(\iint_Q y^2 e^{-2s\Phi} dx dt + \iint_{Q_\omega} s^{-3} v^{-3} u^2 e^{-2s\Phi} dx dt \right), \quad (2.49)$$

where (y, u) satisfies

$$\begin{cases} y_t - y_{xx} - \frac{\mu}{x^2} y = f + 1_\omega u(t), & (t, x) \in Q, \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \quad y(T, x) = 0, & x \in (0, 1), \end{cases} \quad (2.50)$$

with $u \in L^2(Q)$.

By means of standard arguments, it is easy to prove (see [34,35]) that J attains its minimizer at a unique point denoted as (\bar{y}, \bar{u}) .

We set

$$L_\mu y := y_t - y_{xx} - \frac{\mu}{x^2} y \quad \text{in } Q.$$

We will first prove that there exists a dual variable \bar{z} such that

$$\begin{cases} \bar{y} = e^{2s\Phi} L_\mu^* \bar{z}, & \text{in } Q, \\ \bar{u} = -s^3 v^3 e^{2s\Phi} \bar{z}, & \text{in } (0, T) \times \omega, \\ \bar{z} = 0, & \text{on } (0, T) \times \{0, 1\}, \end{cases} \quad (2.51)$$

where L_μ^* is the (formally) adjoint operator of L_μ .

Let us start by introducing the following linear space

$$\mathcal{P}_0 = \{z \in C^\infty(\bar{Q}) : z = 0 \text{ on } (0, T) \times \{0, 1\}\},$$

and introduce the bilinear form a :

$$a(z_1, z_2) = \iint_Q e^{2s\Phi} L_\mu^* z_1 L_\mu^* z_2 dx dt + \iint_{Q_\omega} s^3 v^3 e^{2s\Phi} z_1 z_2 dx dt, \quad \forall z_1, z_2 \in \mathcal{P}_0.$$

Then, if the functions \bar{y} and \bar{u} given by (2.51) satisfy the parabolic problem (2.50), we must have

$$a(\bar{z}, z) = \iint_Q f z dx dt + \int_0^1 y_0 z(0) dx, \quad \forall z \in \mathcal{P}_0. \quad (2.52)$$

The key idea in this proof is to show that there exists exactly one \bar{z} satisfying (2.52) in an appropriate class. We will then define \bar{y} and \bar{u} using (2.51) and we will check that the couple (\bar{y}, \bar{u}) fulfills the desired properties.

Observe that the modified Carleman inequality (2.41) holds for all $z \in \mathcal{P}_0$. Consequently,

$$\|e^{s\hat{\varphi}(0)} z(0)\|_{L^2(0,1)}^2 + \iint_Q v z^2 e^{2s\hat{\varphi}} dx dt \leq C e^{2s[\hat{\varphi}(0) - \hat{\varphi}(\frac{5T}{8})]} a(z, z). \quad (2.53)$$

In particular, $a(\cdot, \cdot)$ is a strictly positive and symmetric bilinear form, that is, $a(\cdot, \cdot)$ is a scalar product in \mathcal{P}_0 .

Denote by \mathcal{P} the Hilbert space which is the completion of \mathcal{P}_0 with respect to the norm associated to $a(\cdot, \cdot)$ (which we denote by $\|\cdot\|_{\mathcal{P}}$). Let us now consider the linear form l , given by

$$l(z) = \iint_Q f z dx dt + \int_0^1 y_0 z(0) dx, \quad \forall z \in \mathcal{P}.$$

By the Cauchy–Schwarz inequality and in view of (2.53), we have that

$$\begin{aligned} |l(z)| &\leq \left\| f \frac{e^{-s\hat{\varphi}}}{\sqrt{v}} \right\|_{L^2(Q)} \left\| z \sqrt{v} e^{s\hat{\varphi}} \right\|_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)} \|z(0) e^{s\hat{\varphi}(0)}\|_{L^2(0,1)} \\ &\leq C e^{s[\hat{\varphi}(0) - \check{\varphi}(\frac{5T}{8})]} \left(\|f e^{-s\hat{\varphi}}\|_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)} \right) \|z\|_{\mathcal{P}}, \end{aligned}$$

and then l is a linear continuous form on \mathcal{P} . Hence, in view of Lax–Milgram’s Lemma, there exists one and only one $\bar{z} \in \mathcal{P}$ satisfying

$$a(\bar{z}, z) = l(z), \quad \forall z \in \mathcal{P}. \quad (2.54)$$

Moreover, we have

$$\|\bar{z}\|_{\mathcal{P}} \leq C e^{s[\hat{\varphi}(0) - \check{\varphi}(\frac{5T}{8})]} \left(\|f e^{-s\hat{\varphi}}\|_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)} \right). \quad (2.55)$$

Let us set

$$\bar{y} = e^{2s\check{\Phi}} L_{\mu}^* \bar{z} \quad \text{and} \quad \bar{u} = -1_{\omega} s^3 v^3 e^{2s\check{\Phi}} \bar{z}. \quad (2.56)$$

With these definitions and by (2.55), it is easy to check that \bar{y} and \bar{u} satisfy

$$\begin{aligned} &\iint_Q \bar{y}^2 e^{-2s\check{\Phi}} dx dt + \iint_{Q_{\omega}} s^{-3} v^{-3} \bar{u}^2 e^{-2s\check{\Phi}} dx dt \\ &\leq C e^{2s[\hat{\varphi}(0) - \check{\varphi}(\frac{5T}{8})]} \left(\|f e^{-s\hat{\varphi}}\|_{L^2(Q)}^2 + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)}^2 \right), \end{aligned} \quad (2.57)$$

which implies (2.48).

It remains to check that \bar{y} is the solution of (2.50) corresponding to \bar{u} . First of all, it is immediate that $\bar{y} \in X_s$ and $\bar{u} \in L^2(Q)$. Denote by \tilde{y} the (weak) solution of (2.1) associated to the control function $u = \bar{u}$, then \tilde{y} is also the unique solution of (2.1) defined by transposition. In other words, \tilde{y} is the unique function in $L^2(Q)$ satisfying

$$\iint_Q \tilde{y} h dx dt = \iint_Q 1_{\omega} \bar{u} z dx dt + \iint_Q f z dx dt + \int_0^1 y_0 z(0) dx, \quad \forall h \in L^2(Q), \quad (2.58)$$

where z is the solution to

$$\begin{cases} -z_t - z_{xx} - \frac{\mu}{x^2} z = h, & (t, x) \in Q, \\ z(t, 0) = z(t, 1) = 0, & t \in (0, T), \\ z(T, x) = 0 & x \in (0, 1). \end{cases}$$

According to (2.54) and (2.56), we see that \bar{y} also satisfies (2.58). Therefore, $\bar{y} = \tilde{y}$. Consequently, the control $\bar{u} \in L^2(\omega \times (0, T))$ drives the state $\bar{y} \in X_s$ exactly to zero at time T . \square

3 Singular heat equation with memory

Prior to null controllability is the well-posedness of problem (1.1). From the results in [23], we recall that in the nonsingular case ($\mu = 0$), it is well known that the heat operator with memory gives rise to well-posed Cauchy–Dirichlet problems. Likewise in [23], by an application of the Contraction Mapping Principle and invoking Theorem 2.1, we have that (1.1) is well-posed in the following sense:

Proposition 3.1. *Assume that $\mu \leq \frac{1}{4}$. If $y_0 \in L^2(0,1)$ and $u \in L^2(Q)$, then there exists a unique solution y of (1.1) such that*

$$y \in C([0, T]; L^2(0,1)) \cap L^2(0, T; H_0^{1,\mu}(0,1)).$$

Now, we pass to derive our main result, which concerns the null controllability of the singular heat equation with memory (1.1). Hence, in what follows, we assume that the function a satisfies

$$e^{\frac{4^k s c d}{T^k (T-t)^k}} a \in L^\infty((0, T) \times Q), \quad (3.1)$$

where c, d, k are the constants defined in (2.14) and s is the same of Theorem 2.7.

Remark 3.2. It is worth mentioning that, from the results in Guerrero and Imanuvilov [24], it seems that the null controllability property of parabolic equations with memory may fail without any additional conditions on the kernel. On the other hand, observe that the condition (3.1) may appear as a quite strong restriction on the admissible function a , but it is a natural one. Indeed, the only thing that we are asking is its integrability with respect to the Carleman weight: it just restricts the function a very near T , which is due to the fact that the function v blows up only at $t = T$ (see also [6]).

For our proof, we are going to employ a fixed point strategy. For $R > 0$, we define

$$X_{s,R} = \{w \in X_s : \|e^{-s\Phi} w\|_{L^2(Q)} \leq R\},$$

which is a bounded, closed, and convex subset of $L^2(Q)$.

For any $w \in X_{s,R}$, let us consider the control problem

$$\begin{cases} y_t - y_{xx} - \frac{\mu}{x^2} y = \int_0^t a(t, r, x) w(r, x) dr + 1_\omega u, & (t, x) \in Q, \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases} \quad (3.2)$$

By Theorem 2.7 we first derive a null controllability result for (3.2); then, as a second step, we will obtain the same controllability result for (1.1) applying Kakutani's fixed point Theorem.

Our main result is thus the following.

Theorem 3.3. *Assume that $\mu \leq \frac{1}{4}$. If the function a satisfies (3.1), then for any $y_0 \in H_0^{1,\mu}(0,1)$, there exists a control function $u \in L^2(Q)$ such that the associated solution $y \in \mathcal{Z}$ of (1.1) satisfies*

$$y(T, \cdot) = 0 \quad \text{in } (0,1). \quad (3.3)$$

Proof. Setting $C_0 := \frac{4^k c d}{T^k}$, by (3.1) and the estimate $e^{-s\Phi} \leq e^{\frac{s C_0}{(T-t)^k}}$, we get that

$$\begin{aligned} \iint_Q \left(e^{-s\Phi} \int_0^t a(t, r, x) w(r, x) dr \right)^2 dx dt &\leq C \iint_Q \int_0^t e^{\frac{2C_0 s}{(T-t)^k}} a^2(t, r, x) w^2(r, x) dr dx dt \\ &\leq C \iint_Q w^2 dx dt \leq C \left(\sup_{(t,x) \in \bar{Q}} e^{2s\Phi} \right) \iint_Q e^{-2s\Phi} w^2 dx dt \leq C R^2 < +\infty. \end{aligned}$$

(recall that $w \in X_{s,R}$). Thus, the result in Theorem 2.7 holds for the equation (3.2), i.e. for any $y_0 \in H_0^{1,\mu}(0,1)$, there exists a control function $u \in L^2(Q)$ such that the associated solution y of (3.2) is in X_s and

$$y(T, \cdot) = 0 \quad \text{in } (0,1).$$

Let us now introduce, for every $w \in X_{s,R}$, the multivalued map

$$\Lambda : X_{s,R} \subset X_s \rightarrow 2^{X_s}$$

with

$$\Lambda(w) = \left\{ y \in X_s : \text{for some } u \in L^2(Q) \text{ satisfying} \right. \\ \left. \iint_{Q_\omega} s^{-3} v^{-3} u^2 e^{-2s\tilde{\Phi}} dx dt \leq C e^{2s[\hat{\Phi}(0) - \check{\Phi}(\frac{5T}{8})]} \left(R^2 + \int_0^1 y_0^2 e^{-2s\hat{\Phi}(0)} dx \right) \right. \\ \left. y \text{ solves (3.2)} \right\}.$$

Observe that if $y \in \Lambda(w)$, then $y(T, \cdot) = 0$ in $(0, 1)$ via (2.47).

To achieve our goal, it will suffice to show that Λ possesses at least one fixed point. To this purpose, we shall apply Kakutani's fixed point Theorem (see [10, Theorem 2.3]).

It is readily seen that $\Lambda(w)$ is a nonempty, closed and convex subset of $L^2(Q)$ for every $w \in X_{s,R}$. Then, we prove that $\Lambda(X_{s,R}) \subset X_{s,R}$ with sufficiently large $R > 0$. By (2.48) and condition (3.1), and arguing as before we have

$$\begin{aligned} & \iint_Q y^2 e^{-2s\tilde{\Phi}} dx dt + \iint_{Q_\omega} s^{-3} v^{-3} u^2 e^{-2s\tilde{\Phi}} dx dt \\ & \leq C e^{2s[\hat{\Phi}(0) - \check{\Phi}(\frac{5T}{8})]} \left(\iint_Q e^{-2s\tilde{\Phi}} \left(\int_0^t a(t, r, x) w(r, x) dr \right)^2 dx dt + e^{-2s\hat{\Phi}(0)} \int_0^1 y_0^2 dx \right) \\ & \leq C e^{2s[\hat{\Phi}(0) - \check{\Phi}(\frac{5T}{8})]} \left(\iint_Q w^2(t, x) dx dt + e^{-2s\hat{\Phi}(0)} \int_0^1 y_0^2 dx \right) \\ & \leq C e^{2s[\hat{\Phi}(0) - \check{\Phi}(\frac{5T}{8})]} \left(\sup_{(t,x) \in \bar{Q}} e^{2s\tilde{\Phi}} \right) \left(\iint_Q e^{-2s\tilde{\Phi}(t,x)} w^2(t, x) dx dt \right) + C e^{-2s\check{\Phi}(\frac{5T}{8})} \int_0^1 y_0^2 dx. \end{aligned}$$

By virtue of $\hat{\Phi}(0) \leq \hat{\Phi}(0)$ and $\tilde{\Phi} \leq \hat{\Phi}(0)$ in Q , we get

$$\begin{aligned} & \iint_Q y^2 e^{-2s\tilde{\Phi}} dx dt + \iint_{Q_\omega} s^{-3} v^{-3} u^2 e^{-2s\tilde{\Phi}} dx dt \\ & \leq C e^{s[2\hat{\Phi}(0) - 2\check{\Phi}(\frac{5T}{8}) + 2\hat{\Phi}(0)]} \iint_Q e^{-2s\tilde{\Phi}(t,x)} w^2(t, x) dx dt + C e^{-2s\check{\Phi}(\frac{5T}{8})} \int_0^1 y_0^2 dx \\ & \leq C e^{s[4\hat{\Phi}(0) - 2\check{\Phi}(\frac{5T}{8})]} R^2 + C e^{-2s\check{\Phi}(\frac{5T}{8})} \int_0^1 y_0^2 dx. \end{aligned} \tag{3.4}$$

Now, choosing the constant c (see (2.14)) in the interval

$$\left(\frac{e^{2\rho\|\sigma\|_\infty} - 1}{d-1}, \frac{16 e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}}{15(d-1)} \right),$$

which is not empty for ρ sufficiently large, we have

$$\begin{aligned} 2\hat{\Phi}(0) - \check{\Phi}\left(\frac{5T}{8}\right) &= \left(\frac{4}{T^2}\right)^k \left[2(e^{\rho\|\sigma\|_\infty} - e^{2\rho\|\sigma\|_\infty}) + cd \left(\frac{16}{15}\right)^k \right] \\ &< \left(\frac{4}{T^2}\right)^k \left(-2 + \frac{d}{d-1} \left(\frac{16}{15}\right)^{k+1} \right) (e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}). \end{aligned}$$

Therefore, taking the parameters d and k defined in (2.14) in such a way that $d > 3$ and $2 < k < \frac{\ln(4/3)}{\ln(16/15)} - 1$, we infer that

$$2\hat{\Phi}(0) - \check{\varphi}\left(\frac{5T}{8}\right) < 0.$$

Hence for s sufficiently large, increasing the parameter s_0 if necessary, we obtain

$$\iint_Q y^2 e^{-2s\check{\Phi}} dx dt + \iint_{Q_\omega} s^{-3} v^{-3} u^2 e^{-2s\check{\Phi}} dx dt \leq \frac{1}{2}R^2 + Ce^{-2s\check{\varphi}\left(\frac{5T}{8}\right)} \int_0^1 y_0^2 dx.$$

Then, for s and R large enough, we obtain

$$\iint_Q y^2 e^{-2s\check{\Phi}} dx dt \leq R^2.$$

It follows that $\Lambda(X_{s,R}) \subset X_{s,R}$. Furthermore, let $\{w_n\}$ be a sequence of $X_{s,R}$. The regularity assumption on y_0 and Theorem 2.1, imply that the associated solutions $\{y_n\}$ are bounded in $H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A))$. Therefore, $\Lambda(X_{s,R})$ is a relatively compact subset of $L^2(Q)$ by the Aubin–Lions Theorem [41].

In order to conclude, we have to prove that Λ is upper-semicontinuous under the L^2 topology. First, observe that for any $w \in X_{s,R}$, we have at least $u \in L^2(Q)$ such that the corresponding solution $y \in X_{s,R}$. Hence, taking $\{w_n\}$ a sequence in $X_{s,R}$, we can find a sequence of controls $\{u_n\}$ such that the corresponding solutions $\{y_n\}$ is in $L^2(Q)$. Thus, let $\{w_n\}$ be a sequence satisfying $w_n \rightarrow w$ in $X_{s,R}$ and $y_n \in \Lambda(w_n)$ such that $y_n \rightarrow y$ in $L^2(Q)$. We must prove that $y \in \Lambda(w)$. For every n , we have a control $u_n \in L^2(Q)$ such that the system

$$\begin{cases} y_{n,t} - y_{n,xx} - \frac{\mu}{x^2} y_n = \int_0^t a(t, r, x) w_n(r, x) dr + 1_\omega u_n, & (t, x) \in Q, \\ y_n(t, 0) = y_n(t, 1) = 0, & t \in (0, T), \\ y_n(0, x) = y_0(x), & x \in (0, 1) \end{cases} \quad (3.5)$$

has a least one solution $y_n \in L^2(Q)$ that satisfies

$$y_n(T, \cdot) = 0 \quad \text{in } (0, 1).$$

From Theorem 2.1 and (3.4), it follows (at least for a subsequence) that

$$\begin{aligned} u_n &\rightarrow u \quad \text{weakly in } L^2(Q), \\ y_n &\rightarrow y \quad \text{weakly in } H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A)), \\ &\quad \text{strongly in } C(0, T; L^2(0, 1)). \end{aligned}$$

Passing to the limit in (3.5), we obtain a control $u \in L^2(Q)$ such that the corresponding solution y to (3.2) satisfies (3.3). This shows that $y \in \Lambda(w)$ and, therefore, the map Λ is upper-semicontinuous.

Hence, the multivalued map Λ possesses at least one fixed point, i.e., there exists $y \in X_{s,R}$ such that $y \in \Lambda(y)$. By the definition of Λ , this implies that there exists at least one pair (y, u) satisfying the conditions of Theorem 3.3. The uniqueness of y follows by Proposition 3.1. This ends the proof of Theorem 3.3. \square

As a consequence of the previous theorem one has the next result.

Theorem 3.4. Assume that $\mu \leq \frac{1}{4}$. If the function a satisfies (3.1), then for any $y_0 \in L^2(0,1)$, there exists a control function $u \in L^2(Q)$ such that the associated solution $y \in \mathcal{W}$ of (1.1) satisfies

$$y(T, \cdot) = 0 \quad \text{in } (0,1).$$

Proof. Consider the following singular parabolic problem:

$$\begin{cases} w_t - w_{xx} - \frac{\mu}{x^2}w = \int_0^t a(t,r,x)w(r,x) dr, & (t,x) \in \left(0, \frac{T}{2}\right) \times (0,1), \\ w(t,0) = w(t,1) = 0, & t \in \left(0, \frac{T}{2}\right), \\ w(0,x) = y_0(x), & x \in (0,1), \end{cases}$$

where $y_0 \in L^2(0,1)$ is the initial condition in (1.1).

By Theorem 2.1, the solution of this system belongs to

$$\mathcal{W} \left(0, \frac{T}{2}\right) := L^2 \left(0, \frac{T}{2}; H_0^{1,\mu}(0,1)\right) \cap C \left(\left[0, \frac{T}{2}\right]; L^2(0,1)\right).$$

Then, there exists $t_0 \in (0, \frac{T}{2})$ such that $w(t_0, \cdot) := \tilde{w}(\cdot) \in H_0^{1,\mu}(0,1)$.

Now, we consider the following controlled parabolic system:

$$\begin{cases} z_t - z_{xx} - \frac{\mu}{x^2}z = \int_0^t a(t,r,x)z(r,x) dr + 1_\omega h & (t,x) \in (t_0, T) \times (0,1), \\ z(t,0) = z(t,1) = 0, & t \in (t_0, T), \\ z(t_0, x) = \tilde{w}(x), & x \in (0,1). \end{cases}$$

We start by observing that, since Theorem 3.3 holds also in a general domain $(t_0, T) \times (0,1)$ with suitable changes, we can see that there exists a control function $h \in L^2((t_0, T) \times (0,1))$ such that the associated solution

$$z \in \mathcal{Z}(t_0, T) := L^2(t_0, T; D(A)) \cap H^1(t_0, T; L^2(0,1)) \cap C([t_0, T]; H_0^{1,\mu}(0,1))$$

satisfies

$$z(T, \cdot) = 0 \quad \text{in } (0,1).$$

Finally, setting

$$y := \begin{cases} w, & \text{in } [0, t_0], \\ z, & \text{in } [t_0, T] \end{cases} \quad \text{and} \quad u := \begin{cases} 0, & \text{in } [0, t_0], \\ h, & \text{in } [t_0, T], \end{cases}$$

one can prove that $y \in \mathcal{W}$ is the solution to the system (1.1) corresponding to u and satisfies

$$y(T, \cdot) = 0 \quad \text{in } (0,1).$$

Hence, our assertion is proved. □

4 Conclusions and perspectives

In this work, we have addressed the problem of null controllability for a class of one dimensional heat equations with an inverse square potential and a memory type-kernel. Using Carleman-based techniques and a fixed point argument, we have proved that under suitable decaying conditions on the memory kernel, the null controllability of the system is ensured by means of a distributed control.

In what follows, we highlight a few possible directions related with the topics addressed in this work.

Memory-type null controllability of singular parabolic equation: This work addresses only the null controllability property for system (1.1). It would be of interest to consider the problem of memory-type controllability (see [12] for the corresponding definition). The goal is then not only to drive the solution to rest at some time-instant, but also to require the memory term to vanish at the same time, ensuring that the whole process reaches the equilibrium. In the spirit of previous results in [12,31], it would be interesting to analyse this memory-type null controllability problem for system (1.1), provided the support of the control moves, covering the whole domain where the equation evolves.

Coupled singular parabolic systems with memory: Inspired by the results in [2,26,40], it would be quite interesting to consider the null controllability of coupled system of parabolic equations with singular potentials and memory effects, with less controls than equations (and ideally only one control if possible).

Degenerate and singular parabolic equation with memory: In [1], the null controllability for a one-dimensional degenerate heat equation is investigated. So, in this regard, following the method of proof used in this paper (see also [1]), we think that it is possible to combine the techniques in both papers and obtain a result for the degenerate/singular equation with memory-type kernel.

5 Appendix

In this section, we recall a classical Carleman estimate for the following nonsingular heat equation

$$\begin{cases} y_t - y_{xx} - by = f, & (t, x) \in Q, \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \end{cases} \quad (5.1)$$

where $b \in L^\infty(Q)$ and $f \in L^2(Q)$.

Following [22], we introduce the weight functions

$$\tilde{\Phi}(t, x) := \theta(t)(e^{-\rho\sigma(x)} - e^{2\rho\|\sigma\|_\infty}) \quad \text{and} \quad \phi(t, x) := \theta(t)e^{\rho\sigma(x)},$$

where θ , ρ and σ are defined in Subsection 2.2. Then, [22, Lemma 1.2] gives the following.

Lemma 5.1. *There exists a positive constant ρ_0 such that for an arbitrary $\rho \geq \rho_0$ there exists $s_0(\rho_0) > 0$ such that for each $s \geq s_0(\rho_0)$ the solutions of (5.1) satisfy the inequality*

$$\begin{aligned} & \iint_Q (s\phi v_x^2 + s^3\phi^3 v^2) (e^{2s\Phi} + e^{2s\tilde{\Phi}}) dx dt \\ & \leq C \left(\iint_Q f^2 (e^{2s\Phi} + e^{2s\tilde{\Phi}}) dx dt + \iint_{Q_\omega} s^3\phi^3 v^2 (e^{2s\Phi} + e^{2s\tilde{\Phi}}) dx dt \right). \end{aligned}$$

Since $0 \leq \sigma(x) \leq \|\sigma\|_\infty$, one has $\tilde{\Phi} \leq \Phi$ and $\theta(t) \leq \phi(t, x) \leq \theta(t)e^{\|\sigma\|_\infty}$ for all $(t, x) \in Q$. Hence, from Lemma 5.1 one can easily deduce the following result.

Corollary 5.2. *There exists a positive constant ρ_0 such that for an arbitrary $\rho \geq \rho_0$ there exists $s_0(\rho_0) > 0$ such that for each $s \geq s_0(\rho_0)$ the solutions of (5.1) satisfy*

$$\iint_Q (s\theta v_x^2 + s^3\theta^3 v^2) e^{2s\Phi} dx dt \leq C \left(\iint_Q f^2 e^{2s\Phi} dx dt + \iint_{Q_\omega} s^3\theta^3 v^2 e^{2s\Phi} dx dt \right).$$

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