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A class of fourth-order elliptic equations with concave and convex nonlinearities in $\mathbb{R}^N$

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Abstract. In this article, we study the multiplicity of solutions for a class of fourth-order elliptic equations with concave and convex nonlinearities in $\mathbb{R}^N$. Under the appropriate assumption, we prove that there are at least two solutions for the equation by Nehari manifold and Ekeland variational principle, one of which is the ground state solution.

Keywords: fourth-order elliptic equation, multiple solutions, Nehari manifold, Ekeland variational principle.

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1 Introduction and main results

In this article, we consider the multiplicity results of solutions of the following fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u - \Delta u + u = f(x)|u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N > 4$, $1 < q < 2 < p < 2_*(2_* = 2N/(N-4))$, the weight function f satisfies the following condition:

(F) $f \geq 0, f \in L^{r_q}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $r_q = \frac{r}{r-q}$ for some $r \in (2, 2_*)$.

Associated with (1.1), we consider the C^1-functional I_f, for each $u \in H^2(\mathbb{R}^N)$,

$$I_f(u) = \frac{1}{2}\|u\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx,$$

where $\|u\| = \left(\int_{\mathbb{R}^N}\left(|\Delta u|^2 + |\nabla u|^2 + u^2\right)dx\right)^{1/2}$ is the norm in $H^2(\mathbb{R}^N)$. It is well known that the solutions of (1.1) are the critical points of the energy functional I_f [14].

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In reality, elliptic equations with concave ang convex nonlinearities in bounded domains have been the focus of a great deal of research in recent years. Ambrosetti *et al.* [1], for example, considered the following equation:

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{p-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $1 < q < 2 < p < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3; 2^* = \infty$ if $N = 1, 2$) and $\lambda > 0$. They found that there is $\lambda_0 > 0$ such that (1.2) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Actually, many scholars have also obtained the same results in the unit ball $B^N(0; 1)$, see [2,6,10,13].

Furthermore, it is also an important subject to deal with elliptic equation with concave-convex nonlinearities when a bounded domain Ω is replaced by $\mathbb{R}^N$. Wu [18] studied the concave-convex elliptic problem:

$$\begin{cases} -\Delta u + u = f_\lambda(x) u^{q-1} + g_\mu(x) u^{p-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where $1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$),

$$f_\lambda = \lambda f_+ + f_- \ (f_\pm = \pm \max\{0, \pm f\} \neq 0)$$

is sign-changing, $g_\mu = a + \mu b$ and the parameters $\lambda, \mu > 0$. When the functions f_+, f_-, a, b satisfy appropriate hypotheses, author obtained the multiplicity of positive solutions for the problem (1.3). Hsu and Lin [9] dealt with the existence and multiplicity of positive solutions for the following semilinear elliptic equation:

$$\begin{cases} -\Delta u + u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where a, b are measurable functions and meet the right conditions. They obtained the result of multiple solutions of the equation (1.4).

Inspired by the existing literature [5,8,9,11,15,18–20], the main aim of this article is to study (1.1) involving concave-convex nonlinearities on the whole space $\mathbb{R}^N$. As far as we know, there are few articles dealing with this type of fourth-order elliptic equation (1.1) involving concave-convex nonlinearities. Using arguments similar to those used in [16], we will prove the existence of two nontrivial solutions by using Ekeland variational principle [7].

Let

$$\sigma = \left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{\frac{p(2-q)}{2(p-2)}} S_r^{\frac{q}{2}} > 0,$$

where S_p and S_r are the best Sobolev constant. Now, we state the main result.

Theorem 1.1. *Assume that* (F) *holds. If* $|f|_{r_q} \in (0, \sigma)$*, then* (1.1) *has at least two nontrivial solutions, one of which is the ground state solution.*

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we are concerned with the proof of Theorem 1.1.

2 Notations and preliminaries

We shall throughout use the Sobolev space $H^2(\mathbb{R}^N)$ with standard norm. The dual space of $H^2(\mathbb{R}^N)$ will be denoted by $H^{-2}(\mathbb{R}^N)$. $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $H^2(\mathbb{R}^N)$. $L^r(\mathbb{R}^N)$ is the usual Lebesgue space whose norms we denote by $|u|_r = \left(\int_{\mathbb{R}^N}|u|^r dx\right)^{1/r}$ for $1 \le p < \infty$. Moreover, we denote by S_r the best Sobolev constant for the embedding of $H^2(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$.

Now, we consider the Nehari minimization problem:

$$\alpha_f = \inf\{I_f(u)|u \in \mathcal{N}_f\},$$

where $\mathcal{N}_f = \{u \in H^2(\mathbb{R}^N)\backslash\{0\}|\langle I_f'(u),u\rangle = 0\}$. Define

$$\psi_f(u) = \langle I_f'(u),u\rangle = \|u\|^2 - \int_{\mathbb{R}^N} f(x)|u|^q dx - \int_{\mathbb{R}^N}|u|^p dx.$$

Then for $u \in \mathcal{N}_f$,

$$\begin{aligned}\langle\psi_f'(u),u\rangle &= \langle\psi_f'(u),u\rangle - \langle I_f'(u),u\rangle \\ &= \|u\|^2 - (q-1)\int_{\mathbb{R}^N} f(x)|u|^q dx - (p-1)\int_{\mathbb{R}^N}|u|^p dx.\end{aligned}$$

Similarly to the skill used in Tarantello [16], we split $\mathcal{N}_f$ into three parts:

$$\begin{aligned}\mathcal{N}_f^+ &= \{u \in \mathcal{N}_f \mid \langle\psi_f'(u),u\rangle > 0\}, \\ \mathcal{N}_f^0 &= \{u \in \mathcal{N}_f \mid \langle\psi_f'(u),u\rangle = 0\}, \\ \mathcal{N}_f^- &= \{u \in \mathcal{N}_f \mid \langle\psi_f'(u),u\rangle < 0\}\end{aligned}$$

and note that if $u \in \mathcal{N}_f$, that is, $\langle I_f'(u),u\rangle = 0$, then

$$\begin{aligned}\langle\psi_f'(u),u\rangle &= (2-p)\|u\|^2 - (q-p)\int_{\mathbb{R}^N} f(x)|u|^q dx \\ &= (2-q)\|u\|^2 - (p-q)\int_{\mathbb{R}^N}|u|^p dx.\end{aligned} \tag{2.1}$$

Then, we have the following results.

Lemma 2.1. *If $|f|_{r_q} \in (0,\sigma)$, then the submanifold $\mathcal{N}^0 = \emptyset$.*

Proof. Suppose the contrary. Then $\mathcal{N}_f^0 \ne \emptyset$, i.e., there exist $u \in \mathcal{N}_f$ such that $\langle\psi_f'(u),u\rangle = 0$. Then for $u \in \mathcal{N}^0$ by (2.1) and Sobolev inequality, we have

$$(2-q)\|u\|^2 = (p-q)\int_{\mathbb{R}^N}|u|^p dx \le (p-q)S_p^{-\frac{p}{2}}\|u\|^p,$$

and so

$$\|u\| \ge \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}}. \tag{2.2}$$

Similarly, using (2.1), Sobolev and Hölder inequalities, we have

$$(p-2)\|u\|^2 = (p-q)\int_{\mathbb{R}^N} f(x)|u|^q dx \le (p-q)|f|_{r_q} S_r^{-\frac{q}{2}}\|u\|^q,$$

which implies that

$$\|u\| \le \left(\frac{(p-q)|f|_{r_q}}{(p-2)S_r^{\frac{q}{2}}} \right)^{\frac{1}{2-q}}. \tag{2.3}$$

Combining (2.2) and (2.3) we deduce that

$$|f|_{r_q} \ge \left(\frac{p-2}{p-q} \right) \left(\frac{2-q}{p-q} \right)^{\frac{2-q}{p-2}} S_p^{\frac{p(2-q)}{2(p-2)}} S_r^{\frac{q}{2}} = \sigma,$$

which is a contradiction. This completes the proof. □

Lemma 2.2. *If $|f|_{r_q} \in (0, \sigma)$, then the set $\mathcal{N}_f^-$ is closed in $H^2(\mathbb{R}^N)$.*

Proof. Let $\{u_n\} \subset \mathcal{N}_f^-$ such that $u_n \to u$ in $H^2(\mathbb{R}^N)$. In the following we show $u \in \mathcal{N}_f^-$. In fact, by $\langle I_f'(u_n), u_n \rangle = 0$ and

$$\langle I_f'(u_n), u_n \rangle - \langle I_f'(u), u \rangle = \langle I_f'(u_n) - I_f'(u), u \rangle + \langle I_f'(u_n), u_n - u \rangle \to 0 \quad \text{as } n \to \infty,$$

we have $\langle I_f'(u), u \rangle = 0$. So $u \in \mathcal{N}_f$. For any $u \in \mathcal{N}_f^-$, that is, $\langle \psi_f'(u), u \rangle < 0$, from (2.1) we have

$$(2-q)\|u\|^2 < (p-q) \int_{\mathbb{R}^N} |u|^p dx \le (p-q) S_p^{-\frac{p}{2}} \|u\|^p,$$

and so

$$\|u\| > \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q} \right)^{\frac{1}{p-2}} > 0.$$

Hence $\mathcal{N}_f^-$ is bounded away from 0. Obviously, by (2.1), it follows that $\langle \psi_f'(u_n), u_n \rangle \to \langle \psi_f'(u), u \rangle$ as $n \to +\infty$. From $\langle \psi_f'(u_n), u_n \rangle < 0$, we have $\langle \psi_f'(u), u \rangle \le 0$. By Lemma 2.1, for $|f|_{r_q} \in (0, \sigma)$, $\mathcal{N}_f^0 = \emptyset$, then $\langle \psi_f'(u), u \rangle < 0$. Thus we deduce $u \in \mathcal{N}_f^-$. This completes the proof. □

Lemma 2.3. *The energy functional I_f is coercive and bounded below on $\mathcal{N}_f$.*

Proof. For $u \in \mathcal{N}_f$, then, by Sobolev and Hölder inequalities,

$$\begin{aligned} I_f(u) &= I_f(u) - \frac{1}{p} \langle I_f'(u), u \rangle \\ &= \frac{p-2}{2p} \|u\|^2 - \frac{p-q}{pq} \int_{\mathbb{R}^N} f(x)|u|^q dx \\ &\ge \frac{p-2}{2p} \|u\|^2 - \frac{p-q}{pq} |f|_{r_q} S_r^{-\frac{q}{2}} \|u\|^q. \end{aligned}$$

This completes the proof. □

The following lemma shows that the minimizers on $\mathcal{N}_f$ are "usually" critical points for I_f. The details of the proof can be referred to Brown and Zhang [4].

Lemma 2.4. *Suppose that $\hat{u}$ is a local minimizer for I_f on $\mathcal{N}_f$. Then, if $\hat{u} \notin \mathcal{N}_f^0$, $\hat{u}$ is a critical point of I_f.*

For each $u \in H^2(\mathbb{R}^N)\backslash\{0\}$, we write

$$t_{max} := \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N}|u|^p dx}\right)^{\frac{1}{p-2}} > 0.$$

Then, we have the following lemma.

Lemma 2.5. *For each $u \in H^2(\mathbb{R}^N)\backslash\{0\}$ and $|f|_{r_q} \in (0,\sigma)$, we have*

(i) there exist unique $0 < t^+ := t^+(u) < t_{max} < t^- := t^-(u)$ such that $t^+u \in \mathcal{N}_f^+$, $t^-u \in \mathcal{N}_f^-$ and

$$I_f(t^+u) = \inf_{t_{max}\geq t\geq 0} I_f(tu), \quad I_f(t^-u) = \sup_{t\geq t_{max}} I_f(tu).$$

(ii) t^- is a continuous function for nonzero u.

(iii) $\mathcal{N}_f^- = \left\{u \in H^2(\mathbb{R}^N)\backslash\{0\} \,|\, \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}$.

Proof. *(i)* Fix $u \in H^2(\mathbb{R}^N)\backslash\{0\}$. Let

$$s(t) = t^{2-q}\|u\|^2 - t^{p-q}\int_{\mathbb{R}^N}|u|^p dx \quad \text{for } t \geq 0.$$

We have $s(0) = 0$, $s(t) \to -\infty$ as $t \to \infty$, $s(t)$ is concave and achieves its maximum at t_{max}. Moreover, for $|f|_{r_q} \in (0,\sigma)$,

$$\begin{aligned}
s(t_{max}) &= \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{R^N}|u|^p dx}\right)^{\frac{2-q}{p-2}}\|u\|^2 - \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{R^N}|u|^p dx}\right)^{\frac{p-q}{p-2}}\int_{\mathbb{R}^N}|u|^p dx \\
&= \|u\|^q\left(\frac{\|u\|^p}{\int_{R^N}|u|^p dx}\right)^{\frac{2-q}{p-2}}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\
&\geq \|u\|^q\left(\frac{\|u\|^p}{S_p^{-\frac{p}{2}}\|u\|^p}\right)^{\frac{2-q}{p-2}}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\
&= \|u\|^q\left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\
&> |f|_{r_q}S_r^{-\frac{q}{2}}\|u\|^q \\
&\geq \int_{\mathbb{R}^N} f(x)|u|^q dx > 0.
\end{aligned} \tag{2.4}$$

Hence, there are unique t^+ and t^- such that $0 < t^+ < t_{max} < t^-$,

$$s(t^+) = \int_{\mathbb{R}^N} f(x)|u|^q dx = s(t^-)$$

and

$$s'(t^+) > 0 > s'(t^-).$$

Note that

$$\langle I_f'(tu), tu\rangle = t^{q-1}\left(s(t) - \int_{\mathbb{R}^N} f(x)|u|^q dx\right)$$

and

$$\langle \psi_f'(tu), tu\rangle = t^{q+1}s'(t) \quad \text{for } tu \in \mathcal{N}_f.$$

We have $t^+u \in \mathcal{N}_f^+$, $t^-u \in \mathcal{N}_f^-$, and $I_f(t^-u) \geq I_f(tu) \geq I_f(t^+u)$ for each $t \in [t^+, t^-]$ and $I_f(t^+u) \geq I_f(tu)$ for each $t \in [0, t^+]$. Thus,

$$I_f(t^+u) = \inf_{t_{max}\geq t\geq 0} I_f(tu), \qquad I_f(t^-u) = \sup_{t\geq t_{max}} I_f(tu).$$

(ii) By the uniqueness of t^- and the external property of t^-, we have that t^- is a continuous function of $u \neq 0$.

(iii) For $u \in \mathcal{N}_f^-$, let $v = \frac{u}{\|u\|}$. By part (i), there is a unique $t^-(v) > 0$ such that $t^-(v)v \in \mathcal{N}_f^-$, that is $t^-(\frac{u}{\|u\|})\frac{u}{\|u\|} \in \mathcal{N}_f^-$. Since $u \in \mathcal{N}_f^-$, we have $t^-(\frac{u}{\|u\|})\frac{u}{\|u\|} = 1$, which implies

$$\mathcal{N}_f^- \subset \left\{u \in H^2(\mathbb{R}^N)\backslash\{0\} | \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}.$$

Conversely, let $u \in H^2(\mathbb{R}^N)\backslash\{0\}$ such that $\frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1$. Then $t^-\left(\frac{u}{\|u\|}\right)\frac{u}{\|u\|} \in \mathcal{N}_f^-$. Thus,

$$\mathcal{N}_f^- = \left\{u \in H^2(\mathbb{R}^N)\backslash\{0\} \;\middle|\; \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}.$$

This completes the proof. □

By Lemma 2.1, for $|f|_{r_q} \in (0, \sigma)$ we write $\mathcal{N}_f = \mathcal{N}_f^+ \cup \mathcal{N}_f^-$ and define

$$\alpha_f^+ = \inf_{u\in\mathcal{N}_f^+} I_f(u), \quad \alpha_f^- = \inf_{u\in\mathcal{N}_f^-} I_f(u).$$

Lemma 2.6. *For $|f|_{r_q} \in (0, \sigma)$, we have $\alpha_f \leq \alpha_f^+ < 0$.*

Proof. Let $u \in \mathcal{N}_f^+$. By (2.1) we have

$$\int_{\mathbb{R}^N} |u|^p dx < \frac{2-q}{p-q}\|u\|^2,$$

and so

$$\begin{aligned} I_f(u) &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N} |u|^p dx \\ &< \left[\left(\frac{1}{2} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{p}\right)\left(\frac{2-q}{p-q}\right)\right]\|u\|^2 \\ &= -\frac{(p-2)(2-q)}{2pq}\|u\|^2 < 0. \end{aligned}$$

Therefore, $\alpha_f \leq \alpha_f^+ < 0$. □

3 Proof of Theorem 1.1

First, we will use the idea of Ni and Takagi [12] to get the following lemmas.

Lemma 3.1. *If $|f|_{r_q} \in (0,\sigma)$, then for every $u \in \mathcal{N}_f$, there exist $\epsilon > 0$ and a differentiable function $g : B_\epsilon(0) \subset H^2(\mathbb{R}^N) \to \mathbb{R}^+ := (0,+\infty)$ such that*

$$g(0) = 1, \quad g(\omega)(u-\omega) \in \mathcal{N}_f, \quad \forall \omega \in B_\epsilon(0)$$

and

$$\langle g'(0), v\rangle = \frac{2(u,v) - q\int_{\mathbb{R}^N} f(x)|u|^{q-2}uvdx - p\int_{\mathbb{R}^N}|u|^{p-2}uvdx}{\langle \psi_f'(u), u\rangle} \tag{3.1}$$

for all $v \in H^2(\mathbb{R}^N)$. Moreover, if $0 < C_1 \le \|u\| \le C_2$, then there exists $C > 0$ such that

$$|\langle g'(0), v\rangle| \le C\|v\|. \tag{3.2}$$

Proof. We define $F : \mathbb{R} \times H^2(\mathbb{R}^N) \to \mathbb{R}$ by

$$F(t,\omega) = t\|u-\omega\|^2 - t^{q-1}\int_{\mathbb{R}^N} f(x)|u-\omega|^q dx - t^{p-1}\int_{\mathbb{R}^N}|u-\omega|^p dx,$$

it is easy to see F is differentiable. Since $F(1,0) = 0$ and $F_t(1,0) = \langle \psi_f'(u), u\rangle \ne 0$, we apply the implicit function theorem at point $(1,0)$ to get the existence of $\epsilon > 0$ and differentiable function $g : B_\epsilon(0) \to \mathbb{R}^+$ such that $g(0) = 1$ and $F(g(\omega),\omega) = 0$ for $\forall \omega \in B_\epsilon(0)$. Thus,

$$g(\omega)(u-\omega) \in \mathcal{N}_f, \quad \forall \omega \in B_\epsilon(0).$$

Also by the differentiability of the implicit function theorem, for all $v \in H^2(\mathbb{R}^N)$, we know that

$$\langle g'(0), v\rangle = -\frac{\langle F_\omega(1,0), v\rangle}{F_t(1,0)}.$$

Note that

$$-\langle F_\omega(1,0), v\rangle = 2(u,v) - q\int_{\mathbb{R}^N} f(x)|u|^{q-2}uvdx - p\int_{\mathbb{R}^N}|u|^{p-2}uvdx$$

and $F_t(1,0) = \langle \psi_f'(u), u\rangle$. So (3.1) holds.

Moreover, by (3.1), $0 < C_1 \le \|u\| \le C_2$ and Hölder's inequality, we have

$$|\langle g'(0), v\rangle| \le \frac{\widetilde{C}\|v\|}{\langle \psi_f'(u), u\rangle}$$

for some $\widetilde{C} > 0$. To prove (3.2), therefore, we only need to show that $|\langle \psi_f'(u), u\rangle| > d$ for some $d > 0$. We argue by contradiction. Assume that there exists a sequence $\{u_n\} \in \mathcal{N}_f$, $C_1 \le \|u_n\| \le C_2$, we have $\langle \psi_f'(u_n), u_n\rangle = o_n(1)$. Then by (2.1) and Sobolev's inequality, we have

$$\begin{aligned}(2-q)\|u_n\|^2 &= (p-q)\int_{\mathbb{R}^N}|u_n|^p dx + o_n(1)\\ &\le (p-q)S_p^{-\frac{p}{2}}\|u_n\|^p + o_n(1),\end{aligned}$$

and so

$$\|u_n\| \ge \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}} + o_n(1). \tag{3.3}$$

Similarly, using (2.1) and Hölder and Sobolev inequalities, we have

$$(p-2)\|u_n\|^2 = (p-q)\int_{\mathbb{R}^N} f(x)|u_n|^q dx + o_n(1)$$
$$\leq (p-q)|f|_{r_q} S_r^{-\frac{q}{2}} \|u_n\|^q + o_n(1),$$

which implies that

$$\|u_n\| \leq \left(\frac{(p-q)|f|_{r_q}}{(p-2)S_r^{\frac{q}{2}}}\right)^{\frac{1}{2-q}} + o_n(1). \tag{3.4}$$

Combining (3.3) and (3.4) as $n \to +\infty$, we deduce that

$$|f|_{r_q} \geq \left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{\frac{p(2-q)}{2(p-2)}} S_r^{\frac{q}{2}} = \sigma,$$

which is a contradiction. Thus if $0 < C_1 \leq \|u\| \leq C_2$, there exists $C > 0$ such that

$$|\langle g'(0), v\rangle| \leq C\|v\|.$$

This completes the proof. □

Lemma 3.2. *If $|f|_{r_q} \in (0,\sigma) \in (0,\sigma)$, then for every $u \in \mathcal{N}_f^-$, there exist $\epsilon > 0$ and a differentiable function $g^- : B_\epsilon(0) \subset H^2(\mathbb{R}^N) \to \mathbb{R}^+$ such that*

$$g^-(0) = 1, \quad g^-(\omega)(u-\omega) \in \mathcal{N}_f^-, \quad \forall \omega \in B_\epsilon(0)$$

and

$$\langle (g^-)'(0), v\rangle = \frac{2(u,v) - q\int_{\mathbb{R}^N} f(x)|u|^{q-2}uvdx - p\int_{\mathbb{R}^N} |u|^{p-2}uvdx}{\langle \psi_f'(u), u\rangle} \tag{3.5}$$

for all $v \in H^2(\mathbb{R}^N)$. Moreover, if $0 < C_1 \leq \|u\| \leq C_2$, then there exists $C > 0$ such that

$$|\langle (g^-)'(0), v\rangle| \leq C\|v\|. \tag{3.6}$$

Proof. Similar to the argument in Lemma 3.2, there exist $\epsilon > 0$ and a differentiable function $g^- : B_\epsilon(0) \to \mathbb{R}^+$ such that $g^-(0) = 1$ and $g^-(\omega)(u-\omega) \in \mathcal{N}_f$ for all $\omega \in B_\epsilon(0)$. By $u \in \mathcal{N}_f^-$, we have

$$\langle \psi_f'(u), u\rangle = \|u\|^2 - (q-1)\int_{\mathbb{R}^N} f(x)|u|^q dx - (p-1)\int_{\mathbb{R}^N} |u|^p dx < 0.$$

Since $g^-(\omega)(u-\omega)$ is continuous with respect to ω, when ϵ is small enough, we know for $\omega \in B_\epsilon(0)$

$$\|g^-(\omega)(u-\omega)\|^2 - (q-1)\int_{\mathbb{R}^N} f(x)|g^-(\omega)(u-\omega)|^q dx - (p-1)\int_{\mathbb{R}^N} |g^-(\omega)(u-\omega)|^p dx < 0.$$

Thus, $g^-(\omega)(u-\omega) \in \mathcal{N}_f^-$, $\forall \omega \in B_\epsilon(0)$. Moreover, the proof details of (3.5) and (3.6) are similar to Lemma 3.1. □

Lemma 3.3. *If $|f|_{r_q} \in (0,\sigma)$, then*

(i) there exists a minimizing sequence $\{u_n\} \in \mathcal{N}_f$ such that

$$I_f(u_n) = \alpha_f + o_n(1),$$
$$I_f'(u_n) = o_n(1) \quad \text{in } H^{-2}(\mathbb{R}^N);$$

(ii) there exists a minimizing sequence $\{u_n\} \in \mathcal{N}_f^-$ *such that*

$$I_f(u_n) = \alpha_f^- + o_n(1),$$
$$I_f'(u_n) = o_n(1) \quad \text{in } H^{-2}(\mathbb{R}^N).$$

Proof. (i) By Lemma 2.3 and the Ekeland variational principle on $\mathcal{N}_f$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_f$ such that

$$\alpha_f \leq I_f(u_n) < \alpha_f + \frac{1}{n} \tag{3.7}$$

and

$$I_f(u_n) \leq I_f(v) + \frac{1}{n}\|v - u_n\| \quad \text{for each } v \in \mathcal{N}_f. \tag{3.8}$$

And we can show that there exists $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$. Indeed, if not, that is, $u_n \to 0$ in $H^2(\mathbb{R}^N)$, then $I_f(u_n)$ would converge to zero, which contradict with $I_f(u_n) \to \alpha_f < 0$. Moreover, by Lemma 2.3 we know that $I_f(u)$ is coercive on $\mathcal{N}_f$, $\{u_n\}$ is bounded in $\mathcal{N}_f$.

Now, we show that

$$\|I_f'(u_n)\|_{H^{-2}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

Applying Lemma 3.1 with u_n to obtain the functions $g_n(\omega) : B_{\epsilon_n}(0) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that

$$g_n(0) = 1, \quad g_n(\omega)(u_n - \omega) \in \mathcal{N}_f, \quad \forall \omega \in B_{\epsilon_n}(0).$$

We choose $0 < \rho < \epsilon_n$. Let $u \in H^2(\mathbb{R}^N)\backslash\{0\}$ and $\omega_\rho = \frac{\rho u}{\|u\|}$. Since $g_n(\omega_\rho)(u_n - \omega_\rho) \in \mathcal{N}_f$, we deduce from (3.8) that

$$\begin{aligned}
&\frac{1}{n}[|g_n(\omega_\rho) - 1|\|u_n\| + \rho g_n(\omega_\rho)] \\
&\quad \geq \frac{1}{n}\|g_n(\omega_\rho)(u_n - \omega_\rho) - u_n\| \\
&\quad \geq I_f(u_n) - I_f(g_n(\omega_\rho)(u_n - \omega_\rho)) \\
&\quad = \frac{1}{2}\|u_n\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u_n|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dx - \frac{1}{2}\left(g_n(\omega_\rho)\right)^2 \|u_n - \omega_\rho\|^2 \\
&\qquad + \frac{1}{q}\left(g_n(\omega_\rho)\right)^q \int_{\mathbb{R}^N} f(x)|u_n - \omega_\rho|^q dx + \frac{1}{p}\left(g_n(\omega_\rho)\right)^p \int_{\mathbb{R}^N} |u_n - \omega_\rho|^p dx \\
&\quad = -\frac{\left(g_n(\omega_\rho)\right)^2 - 1}{2}\|u_n - \omega_\rho\|^2 - \frac{1}{2}(\|u_n - \omega_\rho\|^2 - \|u_n\|^2) \\
&\qquad + \frac{\left(g_n(\omega_\rho)\right)^q - 1}{q}\int_{\mathbb{R}^N} f(x)|u_n - \omega_\rho|^q dx \\
&\qquad + \frac{1}{q}\left(\int_{\mathbb{R}^N} f(x)|u_n - \omega_\rho|^q dx - \int_{\mathbb{R}^N} f(x)|u_n|^q dx\right) \\
&\qquad + \frac{\left(g_n(\omega_\rho)\right)^p - 1}{p}\int_{\mathbb{R}^N} |u_n - \omega_\rho|^p dx + \frac{1}{p}\left(\int_{\mathbb{R}^N} |u_n - \omega_\rho|^p dx - \int_{\mathbb{R}^N} |u_n|^p dx\right).
\end{aligned} \tag{3.9}$$

Note that

$$\lim_{\rho \to 0^+} \frac{g_n(\omega_\rho) - 1}{\rho} = \lim_{\rho \to 0^+} \frac{g_n(0 + \rho\frac{u}{\|u\|}) - g_n(0)}{\rho} = \left\langle (g_n)'(0), \frac{u}{\|u\|} \right\rangle.$$

If we divide the ends of (3.9) by ρ and let $\rho \to 0^+$, we have

$$\begin{aligned}
&\frac{1}{n}\left[\left|\left\langle (g_n)'(0), \frac{u}{\|u\|}\right\rangle\right| \|u_n\| + 1\right] \\
&\geq -\left\langle (g_n)'(0), \frac{u}{\|u\|}\right\rangle \|u_n\|^2 - \int_{\mathbb{R}^N} \Delta u_n \Delta\left(-\frac{u}{\|u\|}\right) + \nabla u_n \nabla\left(-\frac{u}{\|u\|}\right) + u_n\left(-\frac{u}{\|u\|}\right) dx \\
&\quad + \left\langle (g_n)'(0), \frac{u}{\|u\|}\right\rangle \int_{\mathbb{R}^N} f(x)|u_n|^q dx + \int_{\mathbb{R}^N} f(x)|u_n|^{q-2}u_n\left(-\frac{u}{\|u\|}\right) dx \\
&\quad + \left\langle (g_n)'(0), \frac{u}{\|u\|}\right\rangle \int_{\mathbb{R}^N} |u_n|^p dx + \int_{\mathbb{R}^N} |u_n|^{p-2}u_n\left(-\frac{u}{\|u\|}\right) dx \\
&= -\left\langle (g_n)'(0), \frac{u}{\|u\|}\right\rangle \left(\|u_n\|^2 - \int_{\mathbb{R}^N} f(x)|u_n|^q dx - \int_{\mathbb{R}^N} |u_n|^p dx\right) - \frac{1}{\|u\|}\int_{\mathbb{R}^N} |u_n|^{p-2}u_n u dx \\
&\quad + \frac{1}{\|u\|}\int_{\mathbb{R}^N} (\Delta u_n \Delta u + \nabla u_n \nabla u + u_n u) dx - \frac{1}{\|u\|}\int_{\mathbb{R}^N} f(x)|u_n|^{q-2}u_n u dx \\
&= -\left\langle (g_n)'(0), \frac{u}{\|u\|}\right\rangle \langle I_f'(u_n), u_n\rangle + \frac{1}{\|u\|}\langle I_f'(u_n), u\rangle \\
&= \frac{1}{\|u\|}\left\langle I_f'(u_n), u\right\rangle,
\end{aligned}$$

that is,

$$\frac{1}{n}\left[|\langle (g_n)'(0), u\rangle| \|u_n\| + \|u\|\right] \geq \langle I_f'(u_n), u\rangle.$$

By the boundedness of $\|u_n\|$ and Lemma 3.2, there exists $\hat{C} > 0$ such that

$$\frac{\hat{C}}{n} \geq \left\langle I_f'(u_n), \frac{u}{\|u\|}\right\rangle.$$

Therefore, we have

$$\|I_f'(u_n)\|_{H^{-2}(\mathbb{R}^N)} = \sup_{u \in H^2(\mathbb{R}^N)\setminus\{0\}} \frac{\langle I_f'(u_n), u\rangle}{\|u\|} \leq \frac{\hat{C}}{n},$$

that is, $I_f'(u_n) = o_n(1)$ as $n \to +\infty$. This completes the proof of (i).

(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit the details here. □

Now, we establish the existence of minimum for I_f on $\mathcal{N}_f^+$.

Theorem 3.4. *Assume that* (F) *holds. If* $|f|_{r_q} \in (0, \sigma)$, *then the functional* I_f *has a minimizer* u^+ *in* $\mathcal{N}_f^+$ *and it satisfies*

(i) $I_f(u^+) = \alpha_f = \alpha_f^+$;

(ii) u^+ *is a solution of equation* (1.1).

Proof. From Lemma 3.3, let $\{u_n\}$ be a $(PS)_{\alpha_f}$ sequence for I_f on $\mathcal{N}_f$, i.e.,

$$I_f(u_n) = \alpha_f + o_n(1), \quad I_f'(u_n) = o_n(1) \quad \text{in } H^{-2}(\mathbb{R}^N). \tag{3.10}$$

Then it follows from Lemma 2.3 that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. Hence, up to a subsequence, there exists $u^+ \in H^2(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u^+ & \text{in } H^2(\mathbb{R}^N); \\ u_n \to u^+ & \text{in } L^s_{loc}(\mathbb{R}^N)\ (2 \le s < 2_*); \\ u_n(x) \to u^+(x) & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.11}$$

By (F), Hölder inequality and (3.11), we can infer that

$$\int_{\mathbb{R}^N} f(x)|u_n|^q dx = \int_{\mathbb{R}^N} f(x)|u^+|^q dx + o_n(1) \quad \text{as } n \to \infty. \tag{3.12}$$

In fact, for any $\epsilon > 0$, there exists M sufficiently large such that

$$\left(\int_{|x|>M} |f(x)|^{r_q} dx\right)^{\frac{1}{r_q}} < \epsilon.$$

And from $\{u_n\} \subset \mathcal{N}_f$ in $H^2(\mathbb{R}^N)$ is bounded, we obtain that $\left(\int_{\mathbb{R}^N} |u_n - u^+|^r dx\right)^{\frac{q}{r}}$ is bounded. Therefore, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |f(x)(|u_n|^q - |u^+|^q)| dx &\le \int_{\mathbb{R}^N} f(x)|u_n - u^+|^q dx \\ &= \int_{|x|\le M} f(x)|u_n - u^+|^q dx + \int_{|x|>M} f(x)|u_n - u^+|^q dx \\ &\le \left(\int_{|x|\le M} |f(x)|^{r_q} dx\right)^{\frac{1}{r_q}} \left(\int_{|x|\le M} |u_n - u^+|^r dx\right)^{\frac{q}{r}} \\ &\quad + \left(\int_{|x|>M} |f(x)|^{r_q} dx\right)^{\frac{1}{r_q}} \left(\int_{|x|>M} |u_n - u^+|^r dx\right)^{\frac{q}{r}} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

First, we can claim that u^+ is a nontrivial solution of (1.1). Indeed, by (3.10) and (3.11), it is easy to see that u^+ is a solution of (1.1). Next we show that u^+ is nontrivial. From $u_n \in \mathcal{N}_f$, we have that

$$I_f(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(x)|u_n|^q dx. \tag{3.13}$$

Let $n \to \infty$ in (3.13), we can get

$$\alpha_f \ge -\frac{p-q}{pq} \int_{\mathbb{R}^N} f(x)|u^+|^q dx.$$

In view of Lemma 2.6, we have $0 > \alpha_f^+ \ge \alpha_f$, which implies $\int_{\mathbb{R}^N} f(x)|u^+|^q dx > 0$. Thus, u^+ is a nontrivial solution of (1.1). Now we prove that $u_n \to u^+$ strongly in $H^2(\mathbb{R}^N)$ and $I_f(u^+) = \alpha$. In fact, by $u_n, u \in \mathcal{N}_f$, (3.12) and weak lower semicontinuity of norm, we have

$$\begin{aligned} \alpha_f \le I_f(u^+) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u^+\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(x)|u^+|^q dx \\ &\le \varliminf_{n\to\infty} \left(\left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(x)|u_n|^q dx\right) \\ &= \lim_{n\to\infty} I_f(u_n) = \alpha_f, \end{aligned}$$

which implies that $I_f(u^+) = \alpha_f$ and $\lim_{n\to\infty} \|u_n\|^2 = \|u^+\|^2$. Noting that $u_n \rightharpoonup u^+$ in $H^2(\mathbb{R}^N)$, so $u_n \to u^+$ strongly in $H^2(\mathbb{R}^N)$. Furthermore, we have $u^+ \in \mathcal{N}_f^+$. On the contrary, if $u^+ \in \mathcal{N}_f^-$, then by Lemma 2.5 (i), there are unique t^+ and t^- such that $t^+u^+ \in \mathcal{N}_f^+$ and $t^-u^+ \in \mathcal{N}_f^-$. In particular, we have $t^+ < t^- = 1$ and so $I_f(t^+u^+) < I_f(t^-u^+) = I_f(u^+) = \alpha_f$, which is a contradiction. By Lemma 2.4 we may assume that u^+ is a solution of (1.1). This completes the proof. □

In order to obtain the existence of the second local minimum, we consider the following minimization problem:

$$S_0 = \inf\{I_0(u) \mid u \in H^2(\mathbb{R}^N)\backslash\{0\}, I_0'(u) = 0\},$$

where

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

From [17,21], we know that S_0 is achieved at $u_0 \in H^2(\mathbb{R}^N)$. Moreover,

$$S_0 = I_0(u_0) = \sup_{t\geq 0} I_0(tu_0).$$

Then, we have the following lemma.

Lemma 3.5. *If $|f|_{r_q} \in (0,\sigma)$, then $\alpha_f^- < \alpha_f + S_0$.*

Proof. From Lemma 2.5 (iii), $\mathcal{N}_f^-$ disconnects $H^2(\mathbb{R}^N)\backslash\{0\}$ in exactly two components:

$$\Lambda_1 = \left\{u \,\middle|\, \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) > 1\right\},$$
$$\Lambda_2 = \left\{u \,\middle|\, \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) < 1\right\},$$

and $\mathcal{N}_f^+ \subset \Lambda_1$. Moreover, there exists t_1 such that $u^+ + t_1u_0 \in \Lambda_2$. Indeed, denote $t_0 = t^-((u^+ + tu_0)/\|u^+ + tu_0\|)$. Since

$$t^-\left(\frac{u^+ + tu_0}{\|u^+ + tu_0\|}\right)\left(\frac{u^+ + tu_0}{\|u^+ + tu_0\|}\right) \in \mathcal{N}_f^-,$$

we have

$$0 \leq \frac{t_0^q\int_{R^N} f(x)|u^+ + tu_0|^q dx}{\|u^+ + tu_0\|^q} = t_0^2 - \frac{t_0^p\int_{R^N} |u^+ + tu_0|^p dx}{\|u^+ + tu_0\|^p}.$$

Thus

$$t_0 \leq \left[\frac{\|u^+/t + u_0\|}{\left(\int_{R^N} |u^+/t + u_0|^p\right)^{1/p}}\right]^{p/(p-2)} \to \|u_0\| \quad \text{as } t \to \infty.$$

Therefore, there exists $t_2 > 0$ such that $t_0 < l\|u_0\|$, for some $l > 1$ and $t \geq t_2$. Set $t_1 > t_2 + l$, then

$$\begin{aligned}\left(t^-\left(\frac{u^+ + t_1u_0}{\|u^+ + t_1u_0\|}\right)\right)^2 &< l^2\|u_0\|^2 \\ &\leq \|u^+\|^2 + t_1^2\|u_0\|^2 + 2t_1\int_{\mathbb{R}^N}(\Delta u^+\Delta u_0 + \nabla u^+\nabla u_0 + u^+u_0)dx \\ &= \|u^+ + t_1u_0\|^2,\end{aligned}$$

that is, $u^+ + t_1u_0 \in \Lambda_2$. So there exists $k \in (0,1)$ such that $u^+ + kt_1u_0 \in \mathcal{N}_f^-$. Furthermore, we have

$$\begin{aligned}
\alpha_f^- &\le I_f(u^+ + kt_1u_0) \\
&= \frac{1}{2}\|u^+ + kt_1u_0\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u^+ + kt_1u_0|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |u^+ + kt_1u_0|^p dx \\
&< I_f(u^+) + \frac{1}{2}\|kt_1u_0\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |kt_1u_0|^p dx \\
&= I_f(u^+) + I_0(kt_1u_0) \\
&\le \alpha_f + I_0(u_0) \\
&= \alpha_f + S_0.
\end{aligned}$$

This completes the proof. □

Next, we establish the existence of minimum for I_f on $\mathcal{N}_f^-$.

Theorem 3.6. *Assume that* (F) *holds. If $|f|_{r_q} \in (0, \sigma)$, then the functional I_f has a minimizer u^- in $\mathcal{N}_f^-$ and it satisfies*

(i) $I_f(u^-) = \alpha_f^-$;

(ii) u^- *is a solution of equation* (1.1).

Proof. From Lemma 3.3, let $\{u_n\}$ be a $(PS)_{\alpha_f^-}$ sequence for I_f on $\mathcal{N}_f^-$, i.e.,

$$I_f(u_n) = \alpha_f^- + o_n(1), \quad I_f'(u_n) = o_n(1) \quad \text{in } H^{-2}(\mathbb{R}^N). \tag{3.14}$$

From Lemma 2.3 we have $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. Hence, up to a subsequence, there exists $u^- \in H^2(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u^- & \text{in } H^2(\mathbb{R}^N); \\ u_n \to u^- & \text{in } L^s_{loc}(\mathbb{R}^N)\ (2 \le s < 2_*); \\ u_n(x) \to u^-(x) & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.15}$$

From (3.14) and (3.15), we have $\langle I_f'(u^-), v\rangle = 0$, $\forall v \in H^2(\mathbb{R}^N)$, that is, u^- is a weak solution of (1.1) and $u^- \in \mathcal{N}_f$. Let $v_n = u_n - u^-$. Then

$$\begin{cases} v_n \rightharpoonup 0 & \text{in } H^2(\mathbb{R}^N); \\ v_n \to 0 & \text{in } L^s_{loc}(\mathbb{R}^N)\ (2 \le s < 2_*); \\ v_n(x) \to 0 & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.16}$$

Now we prove that $u_n \to u^-$ strongly in $H^2(\mathbb{R}^N)$, that is, $v_n \to 0$ strongly in $H^2(\mathbb{R}^N)$. Arguing by contradiction, we assume that there is $c > 0$ such that $\|v_n\| \ge c > 0$. By the Brézis–Lieb theorem [3],

$$\begin{aligned}
I_f(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u_n|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dx \\
&= I_f(u^-) + \frac{1}{2}\|v_n\|^2 - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|v_n|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx + o_n(1) \qquad (3.17) \\
&= I_f(u^-) + \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx + o_n(1),
\end{aligned}$$

where $\int_{\mathbb{R}^N} f(x)|v_n|^q dx \to 0$ as $n \to \infty$. In fact, for any $\epsilon > 0$, there exists M sufficiently large such that

$$\left(\int_{|x|>M} |f(x)|^{r_q} dx\right)^{\frac{1}{r_q}} < \epsilon.$$

By (F), Hölder's inequality and (3.16), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f(x)|v_n|^q dx &= \int_{|x|\le M} f(x)|v_n|^q dx + \int_{|x|>M} f(x)|v_n|^q dx \\ &\le \left(\int_{|x|\le M} |f(x)|^{r_q} dx\right)^{\frac{1}{r_q}} \left(\int_{|x|\le M} |v_n|^r dx\right)^{\frac{q}{r}} \\ &\quad + \left(\int_{|x|>M} |f(x)|^{r_q} dx\right)^{\frac{1}{r_q}} \left(\int_{|x|>M} |v_n|^r dx\right)^{\frac{q}{r}} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Moreover,

$$\begin{aligned} o_n(1) = \langle I_f'(u_n), u_n\rangle &= \|u_n\|^2 - \int_{\mathbb{R}^N} f(x)|u_n|^q dx - \int_{\mathbb{R}^N} |u_n|^p dx \\ &= \langle I_f'(u^-), u^-\rangle + \|v_n\|^2 - \int_{\mathbb{R}^N} f(x)|v_n|^q dx - \int_{\mathbb{R}^N} |v_n|^p dx + o_n(1) \\ &= \|v_n\|^2 - \int_{\mathbb{R}^N} |v_n|^p dx + o_n(1). \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\|v_n\|^2 - \int_{\mathbb{R}^N} |v_n|^p dx = o_n(1), \quad I_f(u_n) \ge \alpha_f + \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx + o_n(1).$$

Since $\|v_n\| \ge c > 0$, we can get a sequence k_n, $k_n > 0$, $k_n \to 1$ as $n \to \infty$, such that $s_n = k_n v_n$ satisfying $\|s_n\|^2 - \int_{\mathbb{R}^N} |s_n|^p dx = 0$. Thus

$$I_f(u_n) \ge \alpha_f + \frac{1}{2}\|s_n\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |s_n|^p dx + o_n(1) \ge \alpha_f + S_0 + o_n(1),$$

that is, $\alpha_f^- \ge \alpha_f + S_0$, contradicting Lemma 3.5. Hence $u_n \to u^-$ strongly in $H^2(\mathbb{R}^N)$. This implies

$$I_f(u_n) \to I_f(u^-) = \alpha_f^- \quad \text{as } n \to \infty.$$

Furthermore, from Lemma 2.2, $\mathcal{N}_f^-$ is closed set and bounded away from 0. We have $u^- \in \mathcal{N}_f^-$ and u^- is nontrivial. By Lemma 2.4 we may assume that u^- is a solution of (1.1). This completes the proof. □

Proof of Theorem 1.1. By Theorems 3.4 and 3.6, for (1.1) there exist two solutions u^+ and u^- such that $u^+ \in \mathcal{N}_f^+$, $u^- \in \mathcal{N}_f^-$. Since $\mathcal{N}_f^+ \cap \mathcal{N}_f^- = \emptyset$, this implies that u^+ and u^- are different. Moreover, u^+ is the ground state solution. It completes the proof of Theorem 1.1. □

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