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Bifurcation from zero or infinity in nonlinearizable Sturm–Liouville problems with indefinite weight

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Abstract. In this paper, we consider bifurcation from zero or infinity of nontrivial solutions of the nonlinear Sturm–Liouville problem with indefinite weight. This problem is mainly important because of it is related with a selection-migration model in genetic population. We show the existence of four families of unbounded continua of nontrivial solutions to this problem bifurcating from intervals of the line of trivial solutions or the line $\mathbb{R} \times \{\infty\}$ (these intervals are called bifurcation intervals). Moreover, these global continua have the usual nodal properties in some neighborhoods of bifurcation intervals.

Keywords: nonlinear Sturm–Liouville problem, indefinite weight, population genetics, selection-migration model, bifurcation point, bifurcation interval, global continua.

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1 Introduction

We consider the following nonlinear Sturm–Liouville eigenvalue problem

$$(\ell(u))(x) \equiv -(p(x)u'(x))' + q(x)u(x) = \lambda\rho(x)u(x) + h(x, u(x), u'(x), \lambda), \qquad x \in (0,1), \tag{1.1}$$

$$\alpha_0 u(0) - \beta_0 u'(0) = 0, \qquad \alpha_1 u(1) + \beta_1 u'(1) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p \in C^1([0,1];(0,+\infty))$, $q \in C([0,1];[0,+\infty))$, and $\rho \in C([0,1];\mathbb{R})$ such that there exist ς, $\xi \in [0,1]$ for which $\rho(\varsigma)\rho(\xi) < 0$, and α_i, β_i, $i = 0, 1$, are real

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constants such that $|\alpha_i| + |\beta_i| > 0$ and $\alpha_i \beta_i \geq 0$, $i = 0, 1$. The function h has the representation $h = f + g$, where the functions $f, g \in C([0,1] \times \mathbb{R}^3; \mathbb{R})$ and satisfy the conditions

$$u f(x,u,s,\lambda) \leq 0, \qquad u\, g(x,u,s,\lambda) \leq 0; \tag{1.3}$$

there exists a constant $M > 0$ such that

$$\left| \frac{f(x,u,s,\lambda)}{u} \right| \leq M, \qquad (x,u,s,\lambda) \in [0,1] \times \mathbb{R}^2 \times \mathbb{R}, \qquad u \neq 0. \tag{1.4}$$

Moreover, at various points in the paper, we will impose one or the other or both of the following conditions on the function g:

$$g(x,u,s,\lambda) = o(|u| + |s|), \quad \text{as } |u| + |s| \to \infty, \tag{1.5}$$

or

$$g(x,u,s,\lambda) = o(|u| + |s|), \quad \text{as } |u| + |s| \to 0, \tag{1.6}$$

uniformly for $x \in [0,1]$ and $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Nonlinear eigenvalue problems of the type (1.1), (1.2) have been intensively studied recently, as they arise from selection-migration models in population genetics (see for example [2, 3, 9, 10, 14, 16, 18] and references therein). Note that population genetics is one of the important branches of biology, which studies the genetic structure and evolution of populations. It has close ties to ecology, demography, epidemiology, phylogeny, genomics, and molecular evolution. Population genetics is mainly used in human genetics and medicine, as well as in animal and plant breeding. In the case of $p(x) \equiv 1$ and $h(x, u(x), u'(x), \lambda) = \lambda \rho(x)[u(x) - m(u(x))]$, where $m(u) = u(1-u)[h_0(1-u) + (1-h_0)u]$ and $h_0 \in (0,1)$, Eq. (1.1) is an one-dimensional reaction-diffusion equation, the interval [0,1] refers to the habitat of a species, the boundary conditions (1.2) for $\beta_0 = \beta_1 = 0$ means that no individuals cross the boundary of the habitat. Moreover, the weight function $\rho(x)$ represents either the selective strength of the environment on genes, or the intrinsic growth rate of the species at location x, and the real parameter λ corresponds to the reciprocal of the diffusion coefficient (see [10, 14, 18]).

If condition (1.6) is satisfied, then we consider bifurcation from $u = 0$, i.e. bifurcation from the line of trivial solutions $\mathcal{R}_0 = \mathbb{R} \times \{0\}$. In the case when $\rho(x) > 0$ for $x \in [0,1]$ the global bifurcation of solutions of the nonlinear eigenvalue problem (1.1), (1.2) under conditions (1.4) and (1.6) (but without the conditions $\alpha_i \beta_i \geq 0$, $i = 0, 1$) was considered in [8,11,12,22,23,25,26]. These papers it was shown the existence of two families of unbounded continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$, possessing the usual nodal properties and bifurcating from points and intervals of the line $\mathcal{R}_0$ corresponding to the eigenvalues of the linear problem obtained from (1.1), (1.2) by setting $h \equiv 0$. Similar results in nonlinear eigenvalue problems for ordinary differential equations of fourth order were established in the paper [1].

If condition (1.5) is satisfied, then problem (1.1), (1.2) for $f \equiv 0$ is asymptotically linear (see [17]), and, therefore, we must investigate the bifurcation from infinity, that is, the existence of non-trivial solutions to this problem with large norms. Note that in the case when $\rho(x) > 0$ for $x \in [0,1]$ the global bifurcation from infinity of nontrivial solutions of problem (1.1), (1.2) under conditions (1.4) and (1.5) (again without the conditions $\alpha_i \beta_i \geq 0$, $i = 0, 1$) was studied in [11, 12, 22, 24, 25, 27, 28], where in particular, it was shown that there are two families of global continua of nontrivial solutions of this problem bifurcating from points and intervals of the set $\mathcal{R}_\infty = \mathbb{R} \times \{\infty\}$ corresponding to the eigenvalues of the linear problem (1.1), (1.2) with $h \equiv 0$ and having usual nodal properties in some neighborhoods these points and intervals. Moreover, it was also established that these continua either contain other asymptotic bifurcation points and intervals, or intersect the line $\mathcal{R}_0$, or have an unbounded projection onto $\mathcal{R}_0$. Similar global results for fourth order nonlinear eigenvalue problems were obtained in the paper [6].

The problem (1.1), (1.2) in cases (i) $f \equiv 0$, and (ii) $g \equiv 0$ and f satisfies condition (1.4) for any $(x, u, s, \lambda) \in [0, 1] \times \mathbb{R}^2 \times \mathbb{R}$ such that $u \neq 0$ and $|u| + |s| \leq \tau_0$, where $\tau_0 > 0$ is some constant, was considered in [7, 21]. These papers prove the existence of four families of unbounded continua of solutions having the usual nodal properties and bifurcating from points and intervals of the line of trivial solutions corresponding to the positive and negative eigenvalues of linear problem (1.1), (1.2) with $h \equiv 0$.

The purpose of this paper is to study the location of bifurcation intervals in $\mathcal{R}_0$ and $\mathcal{R}_\infty$, and the structure of global continua of nontrivial solutions of problem (1.1), (1.2) emanating from these bifurcation intervals.

In Section 2, we present the main properties of the eigenvalues and eigenfunctions of the linear problem (1.1), (1.2) with $h \equiv 0$. Here we introduce classes of functions in $\mathbb{R} \times C^1[0, 1]$ with a fixed oscillation counter and also possessing other properties of the eigenfunctions of this linear problem. Here we consider problem (1.1), (1.2) under conditions (1.3), (1.4) and (1.6). Then we find the bifurcation intervals of the line of trivial solutions with respect to the above-mentioned oscillation classes and establish that the connected components of solutions emanating from bifurcation intervals are contained in the corresponding oscillation classes, and are unbounded in $\mathbb{R} \times C^1[0, 1]$. In Section 3 and 4 we consider problem (1.1), (1.2) under conditions (1.3)- (1.5). In Section 3 developing the approximation technique from [8], we prove the existence of nontrivial solutions of problem (1.1), (1.2) with large norms contained in the classes with a fixed oscillation count. Moreover, we find intervals containing asymptotically bifurcation points of problem (1.1), (1.2) with respect to these classes. Note that the approximation equation introduced here is more natural than those introduced in [24, 25]. It is important to note that the solutions of problem (1.1), (1.2) from the global continuum emanating from the bifurcation interval of the set $\mathcal{R}_\infty$ with respect to a certain class of a fixed oscillation count and located outside a some neighborhood of this interval may not be included in this oscillation class. In Section 4, we present and prove the main result of this paper, namely, we show that there are four classes of unbounded continua of solutions of problem (1.1), (1.2) emanating from asymptotically bifurcation intervals which have usual nodal properties in a some neighborhoods of these intervals and for each of them one of the following statements holds: either contain other asymptotic bifurcation intervals, or intersect the line $\mathcal{R}_0$, or have an unbounded projection onto $\mathcal{R}_0$. Similar results in nonlinear eigenvalue problems for ordinary differential equations of fourth order and semi-linear elliptic partial differential equations with indefinite weight in the classes of positive and negative functions were obtained in recent papers [2–5]. In Section 5 we consider problem (1.1), (1.2) under both conditions (1.5) and (1.6). Here we manage to show that the continua emanating from asymptotically bifurcation intervals are contained in the corresponding oscillation classes and therefore they do not intersect other asymptotically bifurcation intervals. In Section 6 we consider problem (1.1), (1.2) in the case when the weight function $\rho(x) \geq 0$ for $x \in [0, 1]$. Note that in this case linear problem obtained from (1.1), (1.2) by setting $h \equiv 0$ has only one sequence of positive simple eigenvalues, and consequently, in this case problem (1.1), (1.2) has two families of global connected components emanating from bifurcation intervals of $\mathcal{R}_0$ or $\mathcal{R}_\infty$, and having the properties of global continua from Sections 2 and 3–4, respectively.

Note that similar results was obtained in [11,12] in the case of a special form of the nonlinear term f.

2 Preliminary

By (b.c.) we denote the set of functions satisfying the boundary conditions (1.2).

It is known [15, Ch. 10, § 10·61] that the spectrum of the linear eigenvalue problem

$$\begin{cases} (\ell(u))(x) = \lambda\rho(x)u(x), & x \in (0,1), \\ u \in (b.c.), \end{cases} \tag{2.1}$$

obtained from (1.1), (1.2) by setting $h \equiv 0$ consists of two sequences of real and simple eigenvalues

$$0 < \lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ \mapsto +\infty \quad \text{and} \quad 0 > \lambda_1^- > \lambda_2^- > \cdots > \lambda_k^- \mapsto -\infty;$$

for each $k \in \mathbb{N}$ the eigenfunctions u_k^+ and u_k^- corresponding to the eigenvalues λ_k^+ and λ_k^-, respectively, have exactly $k-1$ simple nodal zeros in $(0,1)$ (by a nodal zero, we mean that the function changes sign at the zero, and at a simple nodal zero, the derivative of the function is nonzero). Moreover, according to [21, formula (2.10)] for each $k \in \mathbb{N}$ the eigenfunctions u_k^+ and u_k^- satisfy the following relations

$$\int_0^1 \rho(x)(u_k^+(x))^2 dx > 0 \quad \text{and} \quad \int_0^1 \rho(x)(u_k^-(x))^2 dx < 0, \tag{2.2}$$

respectively.

Let $E = C^1[0,1] \cap (b.c.)$ be a Banach space with the usual norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$, where $\|u\|_\infty = \max_{x\in[0,1]} |u(x)|$.

From now on σ (ν respectively) will denote either $+$ or $-$.

For each $k \in \mathbb{N}$, each σ and each ν we denote by $\mathcal{S}_{k,\sigma}^\nu$ the set of functions $u \in E$ satisfying the following conditions:

(i) u has exactly $k-1$ simple nodal zeros in the interval $(0,1)$;

(ii) $\sigma \int_0^1 \rho(x)\, u^2(x) dx > 0$;

(iii) $\nu u(x)$ is positive in a deleted neighborhood of the point $x = 0$.

It follows from definition of the set $\mathcal{S}_{k,\sigma}^\nu$, $k \in \mathbb{N}$, that this set is open in E for each σ and each ν. Note that for any $(k,\sigma,\nu) \neq (k',\sigma',\nu')$ the following relation holds:

$$\mathcal{S}_{k,\sigma}^\nu \cap \mathcal{S}_{k',\sigma'}^{\nu'} = \emptyset.$$

Moreover, if $u \in \partial\mathcal{S}_{k,\sigma}^\nu$, then either

(i) $\int_0^1 \rho(x)\, u^2(x) dx = 0$, or

(ii) there exists $x_0 \in [0,1]$ that such $u(x_0) = u'(x_0) = 0$.

Remark 2.1. It follows from the above arguments that

$$u_k^\sigma \in \mathcal{S}_{k,\sigma} = \mathcal{S}_{k,\sigma}^+ \cup \mathcal{S}_{k,\sigma}^-, \qquad k \in \mathbb{N}.$$

Remark 2.2. If $(\lambda, u) \in \mathbb{R} \times E$ be a nontrivial solution of problem (1.1), (1.2), then

$$\lambda \int_0^1 \rho(x)\, u^2(x)\, dx \neq 0.$$

Indeed, multiplying both sides of (1.1) by $u(x)$, integrating the obtained equality in the range from 0 to 1, using the formula for integration by parts, and taking into account conditions (1.2), (1.3), we get

$$\int_0^1 \{p(x)u'^2(x) + q(x)\, u^2(x)\}\, dx + N[u] = \lambda \int_0^1 \rho(x)\, u^2(x)\, dx \\ + \int_0^1 f\left(x, u(x), u'(x), \lambda\right) u(x)\, dx + \int_0^1 g\left(x, u(x), u'(x), \lambda\right) u(x) dx, \quad (2.3)$$

where $N[u] = -\left(p(x)u'(x)u(x)\right)|_{x=0}^{x=1}$. Since the conditions $|\alpha_i| + |\beta| > 0$ and $\alpha_i\beta_i \geq 0$, $i = 0, 1$, are satisfied it follows that $N[u] \geq 0$ for any function $u \in E$. Consequently, the left hand side of (2.3) is positive, and if $\lambda \int_0^1 \rho(x)\, u^2(x)\, dx = 0$, then by conditions (1.3) the right hand side of this relation is non-positive, a contradiction.

Let $\{\lambda_{k,M}^+\}_{k=1}^\infty$ and $\{\lambda_{k,M}^-\}_{k=1}^\infty$ be sequences of positive and negative eigenvalues, respectively, of the following spectral problem

$$\begin{cases} (\ell(u))(x) + Mu(x) = \lambda\rho(x)u(x), \quad x \in (0,1), \\ u \in (b.c.), \end{cases} \quad (2.4)$$

which are simple (see [15, Ch. 10, § 10·61]).

We introduce the notations:

$$\mathcal{I}_k^+ = [\lambda_k^+, \lambda_{k,M}^+], \qquad I_k^- = [\lambda_{k,M}^-, \lambda_k^-],$$

$$\mathbb{R}^+ = (0, +\infty), \qquad \mathbb{R}^- = (-\infty, 0), \qquad \mathcal{R}_0^+ = \mathbb{R}^+ \times \{0\}, \qquad \mathcal{R}_0^- = \mathbb{R}^- \times \{0\},$$

and

$$\mathcal{R}_\infty^+ = \mathbb{R}^+ \times \{\infty\}, \qquad \mathcal{R}_\infty^- = \mathbb{R}^- \times \{\infty\}.$$

By $\mathcal{D}$ we denote the set of nontrivial solutions of the nonlinear eigenvalue problem (1.1), (1.2). For any $\lambda \in \mathbb{R}$, we say that a subset $\mathcal{C} \subset \mathcal{D}$ meets $(\lambda, 0)$ (respectively (λ, ∞)) if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathcal{C}$ such that $\lambda_n \to \lambda$ and $\|u_n\|_1 \to 0$ (respectively $\|u_n\|_1 \to +\infty$) as $n \to +\infty$. Furthermore, we will say that $\mathcal{C} \subset \mathcal{D}$ meets $(\lambda, 0)$ (respectively (λ, ∞)) with respect to the set $\mathbb{R} \times \mathcal{S}_{k,\sigma}^\nu$, if the sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty$ can be chosen so that $u_n \in \mathcal{S}_{k,\sigma}^\nu$ for all $n \in \mathbb{N}$. Moreover, we say $(\lambda, 0)$ (respectively (λ, ∞)) is a bifurcation point of problem (1.1), (1.2) with respect to the set $\mathbb{R} \times \mathcal{S}_{k,\sigma}^\nu$ if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathcal{D} \cap (\mathbb{R} \times \mathcal{S}_{k,\sigma}^\nu)$ such that $\lambda_n \to \lambda$ and $\|u_n\|_1 \to 0$ (respectively $\|u_n\|_1 \to +\infty$) as $n \to +\infty$. If $I \subset \mathbb{R}$ is a bounded interval we say that $\mathcal{C}$ meets $I \times \{0\}$ ($I \times \{\infty\}$ respectively) if C meets $(\lambda, 0)$ (respectively (λ, ∞)) for some $\lambda \in I$. Furthermore, we will say that $\mathcal{C}$ meets $I \times \{0\}$ ($I \times \{\infty\}$ respectively) with respect to the set $\mathbb{R} \times \mathcal{S}_{k,\sigma}^\nu$, if C meets $(\lambda, 0)$ (respectively (λ, ∞)), $\lambda \in I$, with respect to the set $\mathbb{R} \times \mathcal{S}_{k,\sigma}^\nu$ (see [1, 6, 25]).

Now we consider problem (1.1), (1.2) under the conditions (1.3), (1.4) and (1.6). Following the corresponding reasoning given in [7, 19, 21] (see also [1, 8]) and using Remark 2.2 we are convinced that the following results hold for this problem.

Lemma 2.3. *Let $(\lambda, u) \in \mathbb{R} \times E$ be a solution of problem (1.1), (1.2) such that $u \in \partial \mathcal{S}_{k,\sigma}^{\nu}$, $k \in \mathbb{N}$, $\sigma, \nu \in \{+,-\}$. Then $u \equiv 0$.*

Lemma 2.4. *For each $k \in \mathbb{N}$, each σ and each ν the set $\mathcal{B}_{k,\sigma}^{\nu}$ of bifurcation points (from zero) of problem (1.1), (1.2) with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k,\sigma}^{\nu}$ is nonempty. Furthermore, $\mathcal{B}_{k,\sigma}^{\nu} \subset \mathcal{I}_k^{\sigma} \times \{0\}$.*

For each $k \in \mathbb{N}$, each σ and each ν by $\mathcal{D}_{k,\sigma}^{\nu,*}$ we denote the union of all the components of $\mathcal{D}$ which meet $\mathcal{I}_k^{\sigma} \times \{0\}$ with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k,\sigma}^{\nu}$ ([25, Theorem 3.2] and Lemma 2.4 implies that $\mathcal{D}_{k,\sigma}^{\nu,*} \neq \emptyset$). The set $\mathcal{D}_{k,\sigma}^{\nu,*}$ may not be connected in $\mathbb{R}^{\sigma} \times E$, but joining the interval $\mathcal{I}_k^{\sigma} \times \{0\}$ to this set gives a connected set $\mathcal{D}_{k,\sigma}^{\nu} = \mathcal{D}_{k,\sigma}^{\nu,*} \cup (\mathcal{I}_k^{\sigma} \times \{0\})$.

Remark 2.5. By Lemma 2.4 it follows from Remark 2.2 that $\mathcal{D}_{k,\sigma}^{\nu} \subset \mathbb{R}^{\sigma} \times E$.

Theorem 2.6. *For each $k \in \mathbb{N}$, each σ and each ν the connected set $\mathcal{D}_{k,\sigma}^{\nu}$ which contains $I_k^{\sigma} \times \{0\}$ lies in $(\mathbb{R}^{\sigma} \times \mathcal{S}_{k,\sigma}^{\nu}) \cup (\mathcal{I}_k^{\sigma} \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

3 The existence of asymptotic bifurcation points of problem (1.1), (1.2) with respect to the set $S_{k,\sigma}^{\nu}$

In the next two sections, we will consider problem (1.1), (1.2) under the conditions (1.3)–(1.5).

To study the structure of the set of asymptotic bifurcation points of problem (1.1), (1.2), we introduce the following modified nonlinear eigenvalue problem

$$\begin{cases} \ell(u) = \lambda \rho(x) u + \frac{f(x,|u|^{\varepsilon} u, u', \lambda)}{(1+|u|+|u'|)^{2\varepsilon}} + g(x,u,u',\lambda), & x \in (0,1), \\ u \in (b.c.), \end{cases} \tag{3.1}$$

where $\varepsilon \in (0,1]$. It is seen that this problem in a sense approximates problem (1.1), (1.2) for ε near 0. Note that approximations similar to this one were previously used in [6, 25].

Since $f \in C([0,1] \times \mathbb{R}^3; \mathbb{R})$ it follows that $\frac{|f(x,|u|^{\varepsilon} u,s,\lambda)|}{(1+|u|+|s|)^{2\varepsilon}} \in C([0,1] \times \mathbb{R}^3; \mathbb{R})$ for any $\varepsilon \in (0,1]$. Moreover, by condition (1.4) we get

$$\frac{|f(x,|u|^{\varepsilon} u,s,\lambda)|}{(1+|u|+|s|)^{2\varepsilon}(|u|+|s|)} \leq \frac{M|u|^{\varepsilon}|u|}{(|u|+|s|)^{2\varepsilon}(|u|+|s|)} \leq \frac{M}{(|u|+|s|)^{\varepsilon}},$$

whence implies that for each bounded interval $\Lambda \subset \mathbb{R}$

$$\frac{f(x,|u|^{\varepsilon} u,s,\lambda)}{(1+|u|+|s|)^{2\varepsilon}} = o(|u|+|s|) \quad \text{as } |u|+|s| \to \infty,$$

uniformly for $(x,\lambda) \in [0,1] \times \Lambda$. Then, by conditions (1.5), it follows from [24, Theorem 2.4] that for each $k \in \mathbb{N}$, each σ and each ν there exist a neighborhood $\mathcal{P}_{k,\sigma}^{\nu}$ of the point $(\lambda_k^{\sigma}, \infty)$ and a continuum $\mathfrak{D}_{k,\sigma,\varepsilon}^{\nu} \subset \mathbb{R}^{\sigma} \times E$ of the set of solutions of problem (3.1) bifurcating from $(\lambda_k^{\sigma}, \infty)$ such that

(i) $(\mathfrak{D}_{k,\sigma,\varepsilon}^{\nu} \cap \mathcal{P}_{k,\sigma}^{\nu}) \subset \mathbb{R} \times \mathcal{S}_{k,\sigma}^{\nu}$;

(ii) either $\mathfrak{D}_{k,\sigma,\varepsilon}^{\nu} \setminus \mathcal{P}_{k,\sigma}^{\nu}$ is bounded in $\mathbb{R} \times E$, and in this case $\mathfrak{D}_{k,\sigma,\varepsilon}^{\nu} \setminus \mathcal{P}_{k,\sigma}^{\nu}$ meets $\mathcal{R}_0^{\sigma}$, or $\mathfrak{D}_{k,\sigma,\varepsilon}^{\nu} \setminus \mathcal{P}_{k,\sigma}^{\nu}$ is unbounded, and if in this case $\mathfrak{D}_{k,\sigma,\varepsilon}^{\nu} \setminus \mathcal{P}_{k,\sigma}^{\nu}$ has a bounded projection on $\mathcal{R}_0^{\sigma}$, then this set meets $(\lambda_{k'}^{\sigma}, \infty)$ with respect to $\mathcal{S}_{k',\sigma}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$.

Lemma 3.1. *For each $k \in \mathbb{N}$, each σ, each ν and any sufficiently large $R > 0$ there exists a solution $(\lambda^{\nu}_{k,\sigma,R}, u^{\nu}_{k,\sigma,R})$ of problem (1.1), (1.2) such that $\lambda^{\nu}_{k,\sigma,R} \in \mathbb{R}^{\sigma}$, $u^{\nu}_{k,\sigma,R} \in \mathcal{S}^{\nu}_{k,\sigma}$, and $\|u^{\nu}_{k,\sigma,R}\|_1 = R$.*

Proof. Let R be a sufficiently large positive number. Property (i) of the set $\mathfrak{D}^{\nu}_{k,\sigma,\varepsilon}$ implies that for any $\varepsilon \in (0,1)$ there exists a solution $(\lambda^{\nu}_{k,\sigma,R,\varepsilon}, u^{\nu}_{k,\sigma,R,\varepsilon})$ of problem (3.1) such that

$$\lambda^{\nu}_{k,\sigma,R,\varepsilon} \in \mathbb{R}^{\sigma}, \qquad u^{\nu}_{k,\sigma,R,\varepsilon} \in \mathcal{S}^{\nu}_{k,\sigma}, \qquad \|u^{\nu}_{k,\sigma,R,\varepsilon}\|_1 = R.$$

Then it follows from (3.1) that $(\lambda^{\nu}_{k,\sigma,R,\varepsilon}, u^{\nu}_{k,\sigma,R,\varepsilon})$ solves the following problem

$$\begin{cases} (\ell(u))(x) + \varphi^{\nu}_{k,\sigma,R,\varepsilon}(x)u(x) = \lambda\rho(x)u(x) + g(x,u(x),u'(x),\lambda), & x \in (0,1), \\ u \in (b.c.), \end{cases} \tag{3.2}$$

where

$$\varphi^{\nu}_{k,\sigma,R,\varepsilon}(x) = \begin{cases} -\frac{f(x,|u^{\nu}_{k,\sigma,R,\varepsilon}(x)|^{\varepsilon}u^{\nu}_{k,\sigma,R,\varepsilon}(x),(u^{\nu}_{k,\sigma,R,\varepsilon})'(x),\lambda^{\nu}_{k,\sigma,R,\varepsilon})}{u^{\nu}_{k,\sigma,R,\varepsilon}(x)\,(1+|u^{\nu}_{k,\sigma,R,\varepsilon}(x)|+|(u^{\nu}_{k,\sigma,R,\varepsilon})'(x)|)^{2\varepsilon}} & \text{if } u^{\nu}_{k,\sigma,R,\varepsilon}(x) \neq 0, \\ 0 & \text{if } u^{\nu}_{k,\sigma,R,\varepsilon}(x) = 0. \end{cases} \tag{3.3}$$

In view of conditions (1.3) and (1.4), by (3.3) we have

$$\varphi^{\nu}_{k,\sigma,R,\varepsilon}(x) \geq 0 \quad \text{and}$$

$$\begin{aligned} |\varphi^{\nu}_{k,\sigma,R,\varepsilon}(x)| &\leq \frac{M|u^{\nu}_{k,\sigma,R,\varepsilon}(x)|^{\varepsilon}}{(1+|u^{\nu}_{k,\sigma,R,\varepsilon}(x)|+|(u^{\nu}_{k,\sigma,R,\varepsilon})'(x)|)^{2\varepsilon}} \\ &\leq \frac{M}{(1+|u^{\nu}_{k,\sigma,R,\varepsilon}(x)|+|(u^{\nu}_{k,\sigma,R,\varepsilon})'(x)|)^{\varepsilon}} \leq M \quad \text{for } x \in [0,1]. \end{aligned} \tag{3.4}$$

Since $C[0,1]$ is dense in $L_1[0,1]$ and the function $u^{\nu}_{k,\sigma,R,\varepsilon}$ has a finite number of zeros in $(0,1)$, by relation (3.4), it follows from [15, Ch. 10, § 10·61] that the eigenvalues of problem

$$\begin{cases} (\ell(u))(x) + \varphi^{\nu}_{k,\sigma,R,\varepsilon}(x)u(x) = \lambda\rho(x)u(x), & x \in (0,1), \\ u \in (b.c.), \end{cases} \tag{3.5}$$

are real, simple and form a positive infinitely increasing and negative infinitely decreasing sequences $\{\lambda^{\nu,+}_{k,\sigma,R,\varepsilon}\}_{k=1}^{\infty}$ and $\{\lambda^{\nu,-}_{k,\sigma,R,\varepsilon}\}_{k=1}^{\infty}$ respectively. In this case, for each $k \in \mathbb{N}$ the function $u^{\nu,+}_{k,\sigma,R,\varepsilon}$ ($u^{\nu,-}_{k,\sigma,R,\varepsilon}$ respectively) corresponding to the eigenvalue $\lambda^{\nu,+}_{k,\sigma,R,\varepsilon}$ ($\lambda^{\nu,-}_{k,\sigma,R,\varepsilon}$ respectively) has $k-1$ simple nodal zeros in the interval $(0,1)$. Moreover, by [21, Lemma 2.2], the following relations hold:

$$\lambda^{+}_{k} \leq \lambda^{\nu,+}_{k,+,R,\varepsilon} \leq \lambda^{+}_{k,M} \quad \text{and} \quad \lambda^{-}_{k,M} \leq \lambda^{\nu,-}_{k,-,R,\varepsilon} \leq \lambda^{-}_{k}, \qquad k \in \mathbb{N}. \tag{3.6}$$

Hence it follows from (3.6) that

$$\lambda^{\nu,\sigma}_{k,\sigma,R,\varepsilon} \in \mathcal{I}^{\sigma}_{k}, \qquad k \in \mathbb{N}. \tag{3.7}$$

Let

$$\mathcal{I}^{+}_{k}(\delta) = [\lambda^{+}_{k} - \delta, \lambda^{+}_{k,M} + \delta], \qquad \mathcal{I}^{-}_{k}(\delta) = [\lambda^{-}_{k,M} - \delta, \lambda^{-}_{k} + \delta],$$

where δ is a positive number.

By [17, Ch. 4, §3, Theorem 3.1] for each $k \in \mathbb{N}$, each σ and each ν the point $(\lambda^{\nu,\sigma}_{k,\sigma,R,\varepsilon}, \infty)$ is an unique asymptotic bifurcation point of the nonlinear eigenvalue problem (3.2) with respect to the set $\mathbb{R}^\sigma \times \mathcal{S}^\nu_{k,\sigma}$. Then for any sufficiently large $R > 0$ there exists a sufficiently small $\tau_{R,\varepsilon}$ such that

$$\lambda^\nu_{k,\sigma,R,\varepsilon} \in [\lambda^{\nu,\sigma}_{k,\sigma,R,\varepsilon} - \tau_{R,\varepsilon}, \lambda^{\nu,\sigma}_{k,\sigma,R,\varepsilon} + \tau_{R,\varepsilon}] \subseteq \mathcal{I}^\sigma_k(\tau_{R,\varepsilon}) \subset \mathbb{R}^\sigma.$$

Let $\tau_0 = \sup_{R,\varepsilon} \tau_{R,\varepsilon}$. Hence it follows from last relation that

$$\lambda^\nu_{k,\sigma,R,\varepsilon} \in \mathcal{I}^\sigma_k(\tau_0) \subset \mathbb{R}^\sigma. \tag{3.8}$$

Since $\|u^\nu_{k,\sigma,R,\varepsilon}\|_1 = R$ and $f, g \in C([0,1] \times \mathbb{R}^3; \mathbb{R})$, by relation (3.8), it follows from (3.1) that

$$\|u^\nu_{k,\sigma,R,\varepsilon}\|_2 \leq \text{const},$$

where $\|u\|_2 = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$. Then by the Arzelà–Ascoli theorem the set $\{u^\nu_{k,\sigma,R,\varepsilon}\}_{\varepsilon \in (0,1]}$ is precompact in E. Hence we can choose the sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ converging to 0 as $n \to \infty$ such that

$$(\lambda^\nu_{k,\sigma,R,\varepsilon_n}, u^\nu_{k,\sigma,R,\varepsilon_n}) \to (\lambda^\nu_{k,\sigma,R}, u^\nu_{k,\sigma,R}) \quad \text{as } n \to \infty \text{ in } \mathbb{R} \times E,$$

and by (3.1) the sequence $\{(\lambda^\nu_{k,\sigma,R,\varepsilon_n}, u^\nu_{k,\sigma,R,\varepsilon_n})\}_{n=1}^\infty$ is convergent in $\mathbb{R} \times C^2[0,1]$. Putting $(\lambda^\nu_{k,\sigma,R,\varepsilon_n}, u^\nu_{k,\sigma,R,\varepsilon_n})$ instead of (λ, u) in (3.1) and passing to the limit (as $n \to \infty$) in this relation we obtain that $(\lambda^\nu_{k,\sigma,R}, u^\nu_{k,\sigma,R})$ is a solution of problem (1.1), (1.2). It is obvious that $(\lambda^\nu_{k,\sigma,R}, u^\nu_{k,\sigma,R})$ has the following properties

$$\lambda^\nu_{k,\sigma,R} \in \mathcal{I}^\sigma_k(\tau_0), \qquad \|u^\nu_{k,\sigma,R}\|_1 = R, \quad \text{and} \quad u^\nu_{k,\sigma,R} \in \overline{\mathcal{S}^\nu_{k,\sigma}} = \mathcal{S}^\nu_{k,\sigma} \cup \partial\mathcal{S}^\nu_{k,\sigma}.$$

If $u^\nu_{k,\sigma,R} \in \partial\mathcal{S}^\nu_{k,\sigma}$, then either

(i) $\int_0^1 \rho(x)\,(u^\nu_{k,\sigma,R}(x))^2 dx = 0$, or

(ii) there exists $x_0 \in [0,1]$ that such $u^\nu_{k,\sigma,R}(x_0) = (u^\nu_{k,\sigma,R})'(x_0) = 0$.

Since $\|u^\nu_{k,\sigma,R}\|_1 = R$ it follows from Remark 2.2 that $\int_0^1 \rho(x)\,(u^\nu_{k,\sigma,R}(x))^2 dx \neq 0$.

Now let there exists $x_0 \in [0,1]$ such that $u^\nu_{k,\sigma,R}(x_0) = (u^\nu_{k,\sigma,R})'(x_0) = 0$. By (1.1), (1.2) we have

$$\begin{aligned}\ell(u^\nu_{k,\sigma,R}) = \lambda^\nu_{k,\sigma,R}\,\rho(x)\,u^\nu_{k,\sigma,R} + f(x, u^\nu_{k,\sigma,R}\,(u^\nu_{k,\sigma,R})', \lambda^\nu_{k,\sigma,R})& \\ + g(x, u^\nu_{k,\sigma,R}, (u^\nu_{k,\sigma,R})', \lambda^\nu_{k,\sigma,R}), \quad x \in (0,1), \quad u^\nu_{k,\sigma,R} \in (b.c.).&\end{aligned} \tag{3.9}$$

Dividing both sides of (3.9) by $\|u^\nu_{k,\sigma,R}\|_1$ and setting $v^\nu_{k,\sigma,R} = \frac{u^\nu_{k,\sigma,R}}{\|u^\nu_{k,\sigma,R}\|_1}$ we get

$$\begin{aligned}\ell(v^\nu_{k,\sigma,R}) = \lambda^\nu_{k,\sigma,R}\,\rho(x)\,v^\nu_{k,\sigma,R} + \frac{f(x, u^\nu_{k,\sigma,R}\,(u^\nu_{k,\sigma,R})', \lambda^\nu_{k,\sigma,R})}{\|u^\nu_{k,\sigma,R}\|_1}& \\ + \frac{g(x, u^\nu_{k,\sigma,R}, (u^\nu_{k,\sigma,R})', \lambda^\nu_{k,\sigma,R})}{\|u^\nu_{k,\sigma,R}\|_1}, \quad x \in (0,1).&\end{aligned} \tag{3.10}$$

In view of (1.4) we have

$$\left| \frac{f(x, u^\nu_{k,\sigma,R}\,(u^\nu_{k,\sigma,R})', \lambda^\nu_{k,\sigma,R})}{\|u^\nu_{k,\sigma,R}\|_1} \right| \leq M|v^\nu_{k,\sigma,R}|. \tag{3.11}$$

By virtue of condition (1.5) for any sufficiently small fixed $\epsilon > 0$ there exists a sufficiently large $\delta_\epsilon > 0$ such that

$$|g(x,u,s,\lambda)| < \varepsilon(|u|+|s|)/2 \quad \text{for any } (x,u,s,\lambda) \in [0,1]\times\mathbb{R}^2\times\Lambda,\ |u|+|s| > \delta_\epsilon, \tag{3.12}$$

where $\Lambda \in \mathbb{R}^\sigma$ is a bounded interval. In other hand, since $g \in C([0,1]\times\mathbb{R}^3;\mathbb{R})$ it follows that there exists a positive number K_ϵ such that

$$|g(x,u,s,\lambda)| \le K_\epsilon \quad \text{for any } (x,u,s,\lambda) \in [0,1]\times\mathbb{R}^2\times\Lambda,\ |u|+|s| \le \delta_\epsilon. \tag{3.13}$$

We choose R large enough to satisfy the relations

$$R > \delta_\epsilon \quad \text{and} \quad K_\epsilon < R\,\epsilon/2.$$

Then by (3.12) and (3.13) we have

$$\begin{aligned}
&\left|\frac{g(x,u^\nu_{k,\sigma,R}(x)\ (u^\nu_{k,\sigma,R})'(x),\ \lambda^\nu_{k,\sigma,R})}{\|u^\nu_{k,\sigma,R}\|_1}\right| \\
&\quad = \frac{1}{R}\left|g(x,u^\nu_{k,\sigma,R}(x)\ (u^\nu_{k,\sigma,R})'(x),\ \lambda^\nu_{k,\sigma,R})\right| \\
&\quad \le \frac{1}{R}\left\{\max_{\left\{x\in[0,1]\,:\,|u^\nu_{k,\sigma,R}(x)|+\left|(u^\nu_{k,\sigma,R})'(x)\right|\le\delta_\epsilon\right\}}\left|g\left(x,u^\nu_{k,\sigma,R}(x),\left(u^\nu_{k,\sigma,R}\right)'(x),\lambda^\nu_{k,\sigma,R}\right)\right|\right. \\
&\qquad \left. + \max_{\left\{x\in[0,1]\,:\,|u^\nu_{k,\sigma,R}(x)|+\left|(u^\nu_{k,\sigma,R})'(x)\right|>\delta_\epsilon\right\}}\left|g\left(x,u^\nu_{k,\sigma,R}(x),\left(u^\nu_{k,\sigma,R}\right)'(x),\lambda^\nu_{k,\sigma,R}\right)\right|\right\} \\
&\quad \le \frac{1}{R}\left\{K_\epsilon + \epsilon R/2\right\} = \frac{K_\epsilon}{R} + \epsilon/2 < \epsilon/2+\epsilon/2 = \epsilon.
\end{aligned} \tag{3.14}$$

Taking into account (3.11) and (3.14), from (3.10) we obtain

$$\begin{aligned}
p_0|(v^\nu_{k,\sigma,R})''(x)| &\le |p(x)(v^\nu_{k,\sigma,R})''(x)| \\
&\le \left(|\lambda^\nu_{k,\sigma,R}||\rho(x)| + |q(x)| + M\right)|v^\nu_{k,\sigma,R}(x)| + |p'(x)||(v^\nu_{k,\sigma,R})'(x)| + \epsilon,
\end{aligned}$$

which implies that

$$|(v^\nu_{k,\sigma,R})''(x)| \le c_0\left(|v^\nu_{k,\sigma,R}(x)| + |(v^\nu_{k,\sigma,R})'(x)|\right) + \frac{\epsilon}{p_0}, \tag{3.15}$$

where

$$c_0 = \frac{1}{p_0}\max_{x\in[0,1]}\left\{\max\left\{\lambda^+_{k,M}, |\lambda^-_{k,M}|\right\}|\rho(x)| + |q(x)| + M, |p'(x)|\right\}, \qquad p_0 = \min_{x\in[0,1]} p(x).$$

Let $w^\nu_{k,\sigma,R} = \begin{pmatrix} v^\nu_{k,\sigma,R} \\ (v^\nu_{k,\sigma,R})' \end{pmatrix} \in \mathbb{R}^2$ with the norm that given by

$$|w^\nu_{k,\sigma,R}|_2 = |v^\nu_{k,\sigma,R}| + |(v^\nu_{k,\sigma,R})'|.$$

Then it follows from (3.15) that

$$|(w^\nu_{k,\sigma,R})'(x)|_2 \le c_1|w^\nu_{k,\sigma,R}(x)|_2 + \frac{\epsilon}{p_0},$$

where $c_1 = c_0 + 1$. Integrating the last relation in the range from x_0 to x we have

$$\left| \int_{x_0}^{x} |(w^{\nu}_{k,\sigma,R})'(t)|_2 dt \right| \le c_1 \left| \int_{x_0}^{x} |w^{\nu}_{k,\sigma,R}(t)|_2 dt \right| + \frac{\epsilon}{p_0}. \tag{3.16}$$

Using the relation $v^{\nu}_{k,\sigma,R}(x_0) = (v^{\nu}_{k,\sigma,R})'(x_0) = 0$ and inequality (3.16) we get

$$|w^{\nu}_{k,\sigma,R}(x)|_2 = \left| \int_{x_0}^{x} |(w^{\nu}_{k,\sigma,R})'(t)|_2 dt \right| \le c_1 \left| \int_{x_0}^{x} |w^{\nu}_{k,\sigma,R}(t)|_2 dt \right| + \frac{\epsilon}{p_0},$$

whence, with regard the Gronwall's inequality, we get

$$|w^{\nu}_{k,\sigma,R}(x)|_2 \le \frac{\epsilon}{p_0} e^{c_1|x-x_0|} \le \frac{\epsilon}{p_0} e^{c_1} < 1, \qquad x \in [0,1], \tag{3.17}$$

(in advance we could choose ϵ so small enough that the inequality $\epsilon < \frac{p_0}{e^{c_1}}$ holds). Then it follows from (3.17) that $\|v^{\nu}_{k,\sigma,R}\|_1 < 1$ which contradicts the condition $\|v^{\nu}_{k,\sigma,R}\|_1 = 1$. Therefore, we have $u^{\nu}_{k,\sigma,R} \in \mathcal{S}^{\nu}_{k,\sigma}$. □

Corollary 3.2. *For each $k \in \mathbb{N}$, each σ and each ν there exists a sufficiently large positive number $R^{\nu}_{k,\sigma}$ such that for any $R \ge R^{\nu}_{k,\sigma}$ problem (1.1), (1.2) has a solution (λ, u) which satisfies the following properties:*

$$\lambda \in \mathcal{I}^{\sigma}_k(\tau_0), \qquad u \in \mathcal{S}^{\nu}_{k,\sigma} \quad \text{and} \quad \|u\|_1 = R.$$

Recall that (λ, ∞), $\lambda \in \mathbb{R}^{\sigma}$, is an asymptotic bifurcation point of problem (1.1), (1.2) with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma}$, $k \in \mathbb{N}$, if for any sufficiently small $r > 0$ there exists a solution $(\lambda^{\nu}_{k,\sigma,r}, u^{\nu}_{k,\sigma,r}) \in \mathbb{R}^{\sigma} \times E$ such that

$$|\lambda^{\nu}_{k,\sigma,r} - \lambda| < r, \qquad \|u^{\nu}_{k,\sigma,r}\|_1 > r^{-1} \quad \text{and} \quad u^{\nu}_{k,\sigma,r} \in \mathcal{S}^{\nu}_{k,\sigma}.$$

Remark 3.3. We add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to the space $\mathbb{R} \times E$ and define an appropriate topology on the resulting set.

For each $k \in \mathbb{N}$, each σ and each ν by $\mathfrak{B}^{\nu}_{k,\sigma}$, $k \in \mathbb{N}$, we denote the set of asymptotic bifurcation of (1.1), (1.2) with respect to $\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma}$.

The following result immediately follows from Lemma 3.1 and Corollary 3.2.

Corollary 3.4. *For each $k \in \mathbb{N}$, each σ and each ν the set $\mathfrak{B}^{\nu}_{k,\sigma}$ is nonempty. Furthermore, $\mathfrak{B}^{\nu}_{k,\sigma} \subset \mathcal{I}^{\sigma}_k \times \{\infty\}$.*

4 Structures of global continua emanating from the asymptotic bifurcation points of problem (1.1), (1.2)

Let conditions (1.3)–(1.5) hold.

For each $k \in \mathbb{N}$, each σ and each ν we define the set $\mathfrak{D}^{\nu,*}_{k,\sigma}$ as the union of all the components of $\mathcal{D}$ bifurcating from $\mathcal{I}^{\sigma}_k \times \{\infty\}$ with respect to the set $\mathbb{R} \times \mathcal{S}^{\nu}_{k,\sigma}$. It follows from Corollary 3.4 that the set $\mathfrak{D}^{\nu,*}_{k,\sigma}$ is nonempty. This set may not be connected in $\mathbb{R} \times E$, but the set $\mathfrak{D}^{\nu}_{k,\sigma} = \mathfrak{D}^{\nu,*}_{k,\sigma} \cup (\mathcal{I}^{\sigma}_k \times \{\infty\})$ will be connected in this space (see Remark 3.3).

The main result of this paper is the following theorem.

Theorem 4.1. *For each $k \in \mathbb{N}$, each σ and each ν the set $\mathfrak{D}^{\nu}_{k,\sigma}$ is contained in $\mathbb{R}^{\sigma} \times E$ and for this set at least one of the following statements holds:*

(i) there exists $(k', \nu') \neq (k, \nu)$ *such that* $\mathfrak{D}^{\nu}_{k,\sigma}$ *meets* $\mathcal{I}^{\sigma}_{k'} \times \{\infty\}$ *with respect to the set* $\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu'}_{k',\sigma}$*;*

(ii) there exists $\lambda \in \mathbb{R}^{\sigma}$ *such that* $\mathfrak{D}^{\nu}_{k,\sigma}$ *meets* $\mathcal{R}^{\sigma}_0$ *at the point* $(\lambda, 0)$*;*

(iii) the projection $\mathcal{P}_{\mathcal{R}^{\sigma}_0}(\mathfrak{D}^{\nu}_{k,\sigma})$ *of* $\mathfrak{D}^{\nu}_{k,\sigma}$ *on* $\mathcal{R}^{\sigma}_0$ *is unbounded.*

Proof. Let $(\lambda, u) \in \mathcal{D}$ and $v = \frac{u}{\|u\|_1^2}$. Then we have $\|v\|_1 = \frac{1}{\|u\|_1}$ and $u = \frac{v}{\|v\|_1^2}$.

Dividing both sides of (1.1), (1.2) by $\|u\|_1^2$ we obtain

$$\begin{cases} (\ell(v))(x) = \lambda\rho(x)v(x) + \frac{f(x,u(x),u'(x),\lambda)}{\|u\|_1^2} + \frac{g(x,u(x),u'(x),\lambda)}{\|u\|_1^2}, & x \in (0,1), \\ v \in (b.c.). \end{cases} \tag{4.1}$$

We set

$$\hat{f}(x, v(x), v'(x), \lambda) = \begin{cases} \|v\|_1^2 f\left(x, \frac{v(x)}{\|v\|_1^2}, \frac{v'(x)}{\|v\|_1^2}, \lambda\right) & \text{if } v(x) \neq 0, \\ 0 & \text{if } v(x) = 0, \end{cases} \tag{4.2}$$

and

$$\hat{g}(x, v(x), v'(x), \lambda) = \begin{cases} \|v\|_1^2 g\left(x, \frac{v(x)}{\|v\|_1^2}, \frac{v'(x)}{\|v\|_1^2}, \lambda\right) & \text{if } v(x) \neq 0, \\ 0 & \text{if } v(x) = 0. \end{cases} \tag{4.3}$$

By conditions (1.4) for any $x \in [0,1]$ with $v(x) \neq 0$, and $\lambda \in \mathbb{R}$ the following estimates hold:

$$\begin{aligned} \frac{|\hat{f}(x, v(x), v'(x), \lambda)|}{|v(x)|} &= \frac{\|v\|_1^2 \left|f\left(x, \frac{v(x)}{\|v\|_1^2}, \frac{v'(x)}{\|v\|_1^2}, \lambda\right)\right|}{|v(x)|} = \frac{\left|f\left(x, \frac{v(x)}{\|v\|_1^2}, \frac{v'(x)}{\|v\|_1^2}, \lambda\right)\right|}{\left|\frac{v(x)}{\|v\|_1^2}\right|} \\ &= \frac{|f(x, u(x), u'(x), \lambda)|}{|u(x)|} \leq M. \end{aligned} \tag{4.4}$$

We choose $\delta_{\epsilon,1} > \delta_\epsilon$ so that (see Section 3)

$$\frac{K_\epsilon}{\delta_{\epsilon,1}} < \frac{\epsilon}{2}.$$

Let $(\lambda, u) \in \mathbb{R} \times E$ such that $\lambda \in \Lambda$ and $\|u\|_1 > \delta_{\epsilon,1}$, where $\Lambda \subset \mathbb{R}$ is any fixed bounded interval. Then for any $x \in [0,1]$ we have

$$\begin{aligned} &|g(x, u(x)\ u'(x),\ \lambda)| \\ &\quad \leq \left\{ \max_{\{x\in[0,1]\,:\,|u(x)|+|u'(x)|\leq\delta_\epsilon\}} |g(x, u(x), u'(x), \lambda)| \right. \\ &\qquad \left. + \max_{\{x\in[0,1]\,:\,|u(x)|+|u'(x)|>\delta_\epsilon\}} |g(x, u(x), u'(x), \lambda)| \right\} \\ &\quad \leq K_\epsilon + \frac{\epsilon}{2}\{|u(x)| + |u'(x)|\} \leq \frac{\epsilon}{2}\delta_{\epsilon,1} + \frac{\epsilon}{2}\|u\|_1 \leq \frac{\epsilon}{2}\|u\|_1 + \frac{\epsilon}{2}\|u\|_1 = \epsilon\|u\|_1. \end{aligned} \tag{4.5}$$

By (4.5) for any $x \in [0,1]$, $v \in E$ with $0 < \|v\|_1 < \frac{1}{\delta_{\epsilon,1}}$, and $\lambda \in \Lambda$ we get

$$\begin{aligned} |\hat{g}(x, v(x), v'(x), \lambda)| &= \|v\|_1^2 \left|g\left(x, \frac{v(x)}{\|v\|_1^2}, \frac{v'(x)}{\|v\|_1^2}, \lambda\right)\right| = \frac{\|v\|_1}{\|u\|_1}\left|g\left(x, \frac{v(x)}{\|v\|_1^2}, \frac{v'(x)}{\|v\|_1^2}, \lambda\right)\right| \\ &= \|v\|_1 \cdot \frac{1}{\|u\|_1}\ |g(x,\ u(x),\ u'(x),\ \lambda)| \leq \epsilon\|v\|_1, \end{aligned}$$

which shows that

$$|\hat{g}(x,v(x),v'(x),\lambda)|_\infty = o(\|v\|_1) \quad \text{as } \|v\|_1 \to 0, \tag{4.6}$$

uniformly for $(x,\lambda) \in [0,1] \times \Lambda$.

By (4.2) and (4.3) it follows from (4.1) that (λ, v) solves the nonlinear eigenvalue problem

$$\begin{cases} (\ell(v))(x) = \lambda\rho(x)v(x) + \hat{f}(x,v(x),v'(x),\,\lambda) + \hat{g}(x,v(x),v'(x),\lambda), \;\; x \in (0,1), \\ v \in (b.c.). \end{cases} \tag{4.7}$$

Conditions (4.4) and (4.6) show that the inversion

$$(\lambda, u) \to \left(\lambda, \frac{u}{\|u\|_1^2}\right) = (\lambda, v) \tag{4.8}$$

transforms "bifurcation at infinity" problem (1.1), (1.2) into "bifurcation from zero" problem (4.7).

It follows from Lemma 2.4 that the set of bifurcation points of problem (4.7) with respect to the set $\mathbb{R}^\sigma \times \mathcal{S}_{k,\sigma}^\nu$ is nonempty; moreover, if $(\lambda, 0)$ is a bifurcation point of this problem with respect to $\mathbb{R}^\sigma \times \mathcal{S}_{k,\sigma}^\nu$, then $\lambda \in \mathcal{I}_k^\sigma$.

Let $\hat{\mathcal{D}}$ be the set of nontrivial solutions of problem (4.7). It is obvious that inversion (4.8) transforms $\mathcal{D}$ into $\hat{\mathcal{D}}$. By $\hat{\mathcal{D}}_{k,\sigma}^{\nu,*}$ we denote the union of all the components of $\hat{\mathcal{D}}$ that meet $\mathcal{I}_k^\sigma \times \{0\}$ by the set $\mathbb{R}^\sigma \times \mathcal{S}_{k,\sigma}^\nu$, and let $\hat{\mathcal{D}}_{k,\sigma}^\nu = \hat{\mathcal{D}}_{k,\sigma}^{\nu,*} \cup (\mathcal{I}_k^\sigma \times \{0\})$. Then the set $\mathfrak{D}_{k,\sigma}^\nu$ is the inverse image of the set $\hat{\mathcal{D}}_{k,\sigma}^\nu$ under inversion (4.8).

Let $\mathcal{Q}_{k,\sigma}^\nu$ be some neighborhood of the set $\mathcal{I}_k^\sigma(\tau_0) \times (E \setminus B_{R_{k,\sigma}^\nu})$, where $B_{R_{k,\sigma}^\nu}$ is a ball in $\mathbb{R} \times E$ with center 0 and radius $R_{k,\sigma}^\nu$. Note that inversion (4.8) transforms the set $\mathcal{I}_k^\sigma(\tau_0) \times (E \setminus B_{R_{k,\sigma}^\nu})$ into the set $\mathcal{I}_k^\sigma(\tau_0) \times B_{\frac{1}{R_{k,\sigma}^\nu}}$, and the set $\mathcal{Q}_{k,\sigma}^\nu$ into the set $\mathcal{T}_{k,\sigma}^\nu$ which is some neighborhood of $\mathcal{I}_k^\sigma(\tau_0) \times B_{\frac{1}{R_{k,\sigma}^\nu}}$. Then it follows from Corollary 3.2 that $(\hat{\mathcal{D}}_{k,\sigma}^\nu \cap (\mathcal{I}_k^\sigma(\tau_0) \times B_{\frac{1}{R_{k,\sigma}^\nu}})) \subset (\mathbb{R}^\sigma \times \mathcal{S}_{k,\sigma}^\nu) \cup (\mathcal{I}_k^\sigma(\tau_0) \times \{0\})$.

Remark 4.2. If $(\lambda, u) \in (\mathbb{R} \times E) \setminus (\bigcup_{k \in \mathbb{N},\sigma,\nu} \mathcal{Q}_{k,\sigma}^\nu)$ is a solution of (1.1), (1.2) such that $u \in \partial\mathcal{S}_{k,\sigma}^\nu$, then it is seen from the proof of Lemma 3.1 that, in contrast to Lemma 2.3, the relation $u \equiv 0$ may not hold. Consequently, if $(\lambda, u) \in \mathbb{R}^\sigma \times E$ is a solution of (1.1), (1.2) outside $\bigcup_{k \in \mathbb{N},\sigma,\nu} \mathcal{Q}_{k,\sigma}^\nu$ such that $u \in \partial\mathcal{S}_{k,\sigma}^\nu$, then the relation $u \equiv 0$ may not hold.

Due to Remark 4.2, it need not be the case that $(\hat{\mathcal{D}}_{k,\sigma}^\nu \setminus \mathcal{T}_{k,\sigma}^\nu) \subset \mathbb{R}^\sigma \times \mathcal{S}_{k,\sigma}^\nu$ (see for example [24, Remark 2.12]). Hence the assertions of Theorem 2.6 will not hold for $\hat{\mathcal{D}}_{k,\sigma}^\nu$. However, using Remark 2.2, the above arguments and the techniques of [6, 25], and combining it with the global results from [13] and [23], we can show that for each $k \in \mathbb{N}$, each σ and each ν the set $\hat{\mathcal{D}}_{k,\sigma}^\nu$ lies in $\mathbb{R}^\sigma \times E$ and for this set at least one of the following hold:

(a) there exists $(k', \nu') \neq (k, \nu)$ such that $\hat{\mathcal{D}}_{k,\sigma}^\nu$ meets $\mathcal{I}_{k'}^\sigma \times \{0\}$ with respect to $\mathbb{R}^\sigma \times \mathcal{S}_{k',\sigma}^{\nu'}$;

(b) $\hat{\mathcal{D}}_{k,\sigma}^\nu$ is unbounded in $\mathbb{R} \times E$.

Now alternative (i) of Theorem 4.1 for $\mathfrak{D}_{k,\sigma}^\nu$ is obtained from the alternative (a) for $\hat{\mathcal{D}}_{k,\sigma}^\nu$ by the inversion (4.8). The alternatives (ii) and (iii) of this theorem for $\mathfrak{D}_{k,\sigma}^\nu$ correspond, via the inversion (4.8), to the various ways of the alternative (b) in which $\hat{\mathcal{D}}_{k,\sigma}^\nu$ can be unbounded. □

Now suppose that along with the constant $M > 0$ there exists a sufficiently large constant $\chi > 0$ such that

$$\left|\frac{f(x,u,s,\lambda)}{u}\right| \leq M, \qquad (x,u,s,\lambda) \in [0,1] \times \mathbb{R}^3,\ u \neq 0,\ |u| + |s| \geq \chi. \tag{4.9}$$

Then the following result holds.

Lemma 4.3. *Let conditions* (1.3), (1.5) *and* (4.9) *hold. Then there are functions* $f_1, g_1 \in C([0,1] \times \mathbb{R}^3; \mathbb{R})$ *such that* f_1 *satisfies condition* (1.4) *for any* $(x,u,s,\lambda) \in [0,1] \times \mathbb{R}^3$ *with* $u \neq 0$ *and* g_1 *satisfies the conditions* (1.5) *uniformly for* $(x,\lambda) \in [0,1] \times \Lambda$*, and the function* h *can also have a representation* $h = f_1 + g_1$.

The proof of this lemma is similar to that of [6, Lemma 5.1].

Remark 4.4. Let conditions (1.3), (1.5) and (4.9) hold. Then, in this case, for problem (1.1), (1.2), Theorem 4.1 again holds.

5 Global bifurcation of solutions of problem (1.1), (1.2) under both conditions (1.5) and (1.6)

In the case when both conditions (1.5) and (1.6) are satisfied, Theorems 2.6 and 4.1 can be improved as follows.

Theorem 5.1. *Let both conditions* (1.5) *and* (1.6) *be satisfied. Then for each* $k \in \mathbb{N}$*, each* σ *and each* ν, $(\mathfrak{D}^{\nu}_{k,\sigma} \setminus (\mathcal{I}^{\sigma}_k \times \{\infty\})) \subset \mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma}$*, and alternative* (i) *of Theorem* 4.1 *cannot hold. Moreover, if* $\mathfrak{D}^{\nu}_{k,\sigma}$ *meets* $\mathcal{R}^{\sigma}_0$ *for some* $\lambda \in \mathbb{R}^{\sigma}$*, then* $\lambda \in \mathcal{I}^{\sigma}_k$*. Similarly, if* $\mathcal{D}^{\nu}_{k,\sigma}$ *meets* $\mathcal{R}^{\sigma}_{\infty}$ *for some* $\lambda \in \mathbb{R}^{\sigma}$*, then* $\lambda \in \mathcal{I}^{\sigma}_k$.

Proof. If (1.6) holds, then by Lemma 2.3 we have

$$\mathcal{D} \cap (\mathbb{R}^{\sigma} \times \partial\mathcal{S}^{\nu}_{k,\sigma}) = \emptyset,$$

whence implies that the sets

$$\mathcal{D} \cap (\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma}) \quad \text{and} \quad \mathcal{D} \setminus (\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma})$$

are mutually separated in $\mathbb{R} \times E$. Hence, in view of [29, Corollary 26.6], every component of $\mathcal{D}$ must be a subset of $\mathcal{D} \cap (\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma})$ or $\mathcal{D} \setminus (\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma})$. Recall that $\mathfrak{D}^{\nu,*}_{k,\sigma} = \mathfrak{D}^{\nu}_{k,\sigma} \setminus (\mathcal{I}^{\sigma}_k \times \{\infty\})$ is the union of all components of the set $\mathcal{D}$ which intersect the set $\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma}$. Therefore, each of these components must be contained in $\mathbb{R}^{\sigma} \times \mathcal{S}^{\nu}_{k,\sigma}$, and consequently, $\mathfrak{D}^{\nu,*}_{k,\sigma} \subset \mathbb{R}^{\sigma} \times S^{\nu}_{k,\sigma}$. Then, by virtue of Theorem 4.1, its alternative (i) will not hold.

Now let $\mathfrak{D}^{\nu}_{k,\sigma}$ meets $\mathcal{R}^{\sigma}_0$ for some $\lambda \in \mathbb{R}^{\sigma}$. Then it follows from Lemma 2.4 that $\lambda \in \mathcal{I}^{\sigma}_k$. Similarly, if $\mathcal{D}^{\nu}_{k,\sigma}$ meets $\mathcal{R}^{\sigma}_{\infty}$ for some $\lambda \in \mathbb{R}^{\sigma}$, then Corollary 3.4 implies that $\lambda \in \mathcal{I}^{\sigma}_k$. □

6 Bifurcation of problem (1.1), (1.2) in the case $\rho(x) \geq 0$

In this section we consider problem (1.1), (1.2) in the case when weight function $\rho(x) \geq 0$ on $[0,1]$ and $\rho(x) \not\equiv 0$ on any subinterval of $[0,1]$. Then it follows from [15, Ch. 10, §10·6 and

10·61] that the spectrum of the linear spectral problem consists of one sequence of positive and simple eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k \mapsto +\infty;$$

for each $k \in \mathbb{N}$ the eigenfunctions u_k corresponding to the eigenvalues λ_k has exactly $k-1$ simple nodal zeros in $(0,1)$. Moreover, it follows from (2.3) with $f \equiv 0$ and $g \equiv 0$ that for each $k \in \mathbb{N}$ the eigenfunction $u_k(x)$ satisfies the condition

$$\int_0^1 \rho(x) u_k^2(x) dx > 0.$$

By following the arguments in Sections 2–4 in this case we can justify the following results.

Theorem 6.1. *Let the condition (1.6) holds. Then for each $k \in \mathbb{N}$ and each ν there exists a connected component $\mathcal{D}_k^\nu$ of the closure of the set of nontrivial solutions of (1.1), (1.2) which contains $\mathcal{I}_k \times \{0\}$ lies in $(\mathbb{R}^+ \times \mathcal{S}_{k,+}^\nu) \cup (\mathcal{I}_k \times \{0\})$ and is unbounded in $\mathbb{R} \times E$, where $\mathcal{I}_k = [\lambda_k, \lambda_{k,M}]$, $k \in \mathbb{N}$, and $\lambda_{k,M}$ is the kth eigenvalue of problem (2.4).*

Theorem 6.2. *Let the condition (1.5) holds. Then for each $k \in \mathbb{N}$ and each ν there exists a connected component $\mathfrak{D}_k^\nu$ of the closure of the set of nontrivial solutions of (1.1), (1.2) which contains $I_k \times \{\infty\}$ is contained in $\mathbb{R}^+ \times E$ and for this set at least one of the following statements holds:*

(i) there exists $(k', \nu') \neq (k, \nu)$ such that $\mathfrak{D}_k^\nu$ meets $\mathcal{I}_{k'} \times \{\infty\}$ with respect to the set $\mathbb{R} \times \mathcal{S}_{k',+}^{\nu'}$;

(ii) there exists $\lambda \in \mathbb{R}^+$ such that $\mathfrak{D}_k^\nu$ meets $\mathcal{R}_0^+$ at the point $(\lambda, 0)$;

(iii) the projection $\mathcal{P}_{\mathcal{R}_0^+}(\mathfrak{D}_k^\nu)$ of $\mathfrak{D}_k^\nu$ on $\mathcal{R}_0^+$ is unbounded.

Theorem 6.3. *Let both conditions (1.5) and (1.6) be satisfied. Then for each $k \in \mathbb{N}$ and each ν, $(\mathfrak{D}_k^\nu \setminus (\mathcal{I}_k \times \{\infty\})) \subset \mathbb{R}^+ \times \mathcal{S}_{k,+}^\nu$, and alternative (i) of Theorem 6.2 cannot hold. Moreover, if $\mathfrak{D}_k^\nu$ meets $\mathcal{R}_0^+$ for some $\lambda \in \mathbb{R}^+$, then $\lambda \in \mathcal{I}_k$. Similarly, if $\mathcal{D}_k^\nu$ meets $\mathcal{R}_\infty^+$ for some $\lambda \in \mathbb{R}^+$, then $\lambda \in \mathcal{I}_k$.*

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References

[1] Z. S. Aliev, On the global bifurcation of solutions of some nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207**(2016), No. 12, 3–29. https://doi.org/10.1070/SM8369; MR3588983; Zbl 1371.34038

[2] Z. S. Aliyev, Sh. M. Hasanova, Global bifurcation of positive solutions of semi-linear elliptic partial differential equations with indefinite weight, *Z. Anal. Anwend.* **38**(2019), No. 1, 1–15. https://doi.org/10.4171/ZAA/1625; Zbl 1412.35087

[3] Z. S. Aliyev, Sh. M. Hasanova, Global bifurcation of positive solutions from zero in nonlinearizable elliptic problems with indefinite weight, *J. Math. Anal. Appl.* **491**(2020), No. 1, 1–11. https://doi.org/10.1016/j.jmaa.2020.124252; Zbl 1451.35021

[4] Z. S. Aliyev, R.A. Huseynova, Bifurcation in nonlinearizable eigenvalue problems for ordinary differential equations of fourth order with indefinite weight, *Electron. J. Qual. Theory Differ. Equ.* **2017**, No. 92, 1–12. https://doi.org/10.14232/ejqtde.2017.1.92; Zbl 1413.34144

[5] Z. S. Aliyev, R.A. Huseynova, Global bifurcation from infinity in some nonlinearizable eigenvalue problems with indefinite weight, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **44**(2018), No. 1, 123–134. Zbl 1402.34021

[6] Z. S. Aliyev, N. A. Mustafayeva, Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, *Electron. J. Differential Equations* **2018**, No. 98, 1–19. MR3831844; Zbl 06866720

[7] Z. S. Aliyev, L. V. Ashurova (Nasirova), Bifurcation of positive and negative solutions of nonlinearizable Sturm–Liouville problems with indefinite weight, *Miskolc Math. Notes* **21**(2020), No. 1, 19–29. https://doi.org/10.18514/MMN.2020.2876; Zbl 07254879

[8] H. Berestycki, On some nonlinear Sturm–Liouville problems, *J. Diffential Equations* **26**(1977), No. 3, 375–390. https://doi.org/10.1016/0022-0396(77)90086-9; MR481230; Zbl 0331.34020

[9] K. J. Brown, Local and global bifurcation results for a semilinear boundary value problem, *J. Differential Equations* **239**(2007), No. 2, 296–310. https://doi.org/doi.org/10.1016/j.jde.2007.05.013; MR2344274; Zbl 1331.35129

[10] R. S. Cantrell, C. Cosner, *Spatial ecology via reaction-diffusion equations*, Wiley, Chichester, 2003. https://doi.org/10.1002/0470871296; MR2191264; Zbl 1087.92058

[11] G. Dai, Global bifurcation from intervals for Sturm–Liouville problems which are not linearizable, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 65, 1–7. https://doi.org/10.14232/ejqtde.2013.1.65; Zbl 1340.34110

[12] G. Dai, R. Ma, Bifurcation from intervals for Sturm–Liouville problems and its applications, *Electron. J. Differential Equations* **2014**, No. 3, 1–10. MR3159412; Zbl 1292.34027

[13] E. N. Dancer, On the structure of solutions of nonlinear eigenvalue problems, *Indiana Univ. Math. J.* **23**(1974), No. 3, 1069–1076. https://doi.org/10.1512/iumj.1974.23.23087; MR348567; Zbl 0276.47051

[14] W. H. Fleming, A selection-migration model in population genetics, *J. Math. Biol.* **2**(1975), No. 3, 219–233. https://doi.org/10.1007/BF00277151; MR403720; Zbl 0325.92009

[15] E. L. Ince, *Ordinary differential equations*, Dover, New York, 1926.

[16] B. Ko, K. J. Brown, The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problems, *Nonlinear Anal.* **3**(2000), No. 5, 587–597. https://doi.org/10.1016/S0362-546X(98)00223-5; MR 1727270; Zbl 0945.35036

[17] M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, The Macmillan Co., New York, 1964. MR0159197; Zbl 0111.30303

[18] Y. Lou, E. Yanagida, Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics, *Japan J. Ind. Appl. Math.* **23**(2006), No. 3, 275–292. https://doi.org/10.1007/BF03167595; MR2281509; Zbl 1185.35059

[19] A. P. Makhmudov, Z. S. Aliev, Global bifurcation of solutions of certain nonlinearizable eigenvalue problems, *Differential Equations* **25**(1989), No. 3, 71–76. MR986400; Zbl 0684.35014

[20] T. Nagylaki, Y. Lou, The dynamics of migration-selection models, in: A. Friedman (Ed.), *Tutorials in mathematical biosciences. IV*, Lecture Notes in Mathematics, Vol. 1922, Springer, Berlin, 2008, pp. 117–170. https://doi.org/10.1007/978-3-540-74331-6_4; MR2392286; Zbl 1300.92059

[21] L. V. Nasirova, Global bifurcation from intervals of solutions of nonlinear Sturm–Liouville problems with indefinite weight. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math.* **39**(2019), No. 4, 148–154. Zbl 1444.34040

[22] J. Przybycin, Bifurcation from infinity for the special class of nonlinear differential equations, *J. Differential Equations* **65**(1986), No. 2, 235–239. https://doi.org/10.1016/0022-0396(86)90035-5; Zbl 0599.34013

[23] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7**(1971), No. 3, 487–513. https://doi.org/10.1016/0022-1236(71)90030-9; MR0301587; Zbl 0212.16504

[24] P. H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14**(1973), No. 3, 462–475. https://doi.org/10.1016/0022-0396(73)90061-2; MR328705; Zbl 0272.35017

[25] B. P. Rynne, Bifurcation from zero or infinity in Sturm–Liouville problems which are not linearizable, *J. Math. Anal. Appl.* **228**(1998), No. 1, 141–156. https://doi.org/10.1006/jmaa.1998.6122; MR1659893; Zbl 0918.34028

[26] K. Schmitt, H. L. Smith, On eigenvalue problems for nondifferentiable mappings, *J. Differential Equations* **33**(1979), No. 3, 294–319. https://doi.org/10.1016/0022-0396(79)90067-6; MR543701; Zbl 0389.34019

[27] C. A. Stuart, Solutions of large norm for non-linear Sturm–Liouville problems, *Quart. J. Math. Oxford Ser. (2)* **24**(1973), No. 2, 129–139. https://doi.org/10.1093/qmath/24.1.129

[28] J. F. Toland, Asymptotic linearity and nonlinear eigenvalue problems, *Quart. J. Math. Oxford Ser. (2)* **24**(1973), No. 2, 241–250. https://doi.org/10.1093/qmath/24.1.241; Zbl 0256.47049

[29] S. Willard, *General topology*, Addison-Wesley, Reading, MA, 1970. MR0264581