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Regularity properties and blow-up of the solutions for improved Boussinesq equations

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Abstract. In this paper, we study the Cauchy problem for linear and nonlinear Boussinesq type equations that include the general differential operators. First, by virtue of the Fourier multipliers, embedding theorems in Sobolev and Besov spaces, the existence, uniqueness, and regularity properties of the solution of the Cauchy problem for the corresponding linear equation are established. Here, L^p-estimates for a solution with respect to space variables are obtained uniformly in time depending on the given data functions. Then, the estimates for the solution of linearized equation and perturbation of operators can be used to obtain the existence, uniqueness, regularity properties, and blow-up of solution at the finite time of the Cauchy for nonlinear for same classes of Boussinesq equations. Here, the existence, uniqueness, L^p-regularity, and blow-up properties of the solution of the Cauchy problem for Boussinesq equations with differential operators coefficients are handled associated with the growth nature of symbols of these differential operators and their interrelationships. We can obtain the existence, uniqueness, and qualitative properties of different classes of improved Boussinesq equations by choosing the given differential operators, which occur in a wide variety of physical systems.

Keywords: Boussinesq equations, hyperbolic equations, differential operators, blow-up, Fourier multipliers.

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1 Introduction

The aim of this paper is to investigate the existence, uniqueness, and quality properties of the solution of the Cauchy problem for the following improved Boussinesq equation

$$u_{tt} + L_0 u_{tt} + L_1 u = L_2 f(u), \quad x \in \mathbb{R}^n, \quad t \in (0, T), \tag{1.1}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \tag{1.2}$$

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where $u(x,t)$ is the complex-valued unknown function, $f(u) = f(x,t,u)$ is the given nonlinear function, L_i are differential operators with constant coefficients, $\varphi(x)$ and $\psi(x)$ are the given initial value functions.

Here, we find the sufficient conditions depending on the qualifications and mutual relevance of the elliptic operators included in the equation to ensure that there exists a unique solution of the problem, being L^p-regular and blow up infinite time. By choosing the operators L_i we obtain different classes of Boussinesq type equations which occur in a wide variety of physical systems, such as in the propagation of longitudinal deformation waves in an elastic rod, a hydro-dynamical process in plasma, in materials science which describe spinodal decomposition and in the absence of mechanical stresses (see [2,6,9,18,21,30–32]). We think this article is useful in the context of L^p-regularity theory of improved Boussinesg equations. For the first time here, the existence, uniqueness, L^p-regularity, and blow-up properties of solution (at the finite time) of the Cauchy problem for these type Boussinesq equations are established depending on the symbol of the differential operators and their orders, contained in the equation. We can obtain different classes of Boussinesq equations, by choosing these differential operators, which occur in a wide variety of physical systems. Moreover, in this paper, the method of proofs naturally differs from those used in previous works. Indeed, since the problem includes a general differential operator in the leading part, we need some extra mathematics tools for deriving considered conclusions.

For example, if we choose $L_0 = L_1 = L_2 = -\Delta$, where Δ is n-dimensional Laplace, we obtain the Cauchy problem for the Boussinesq equation

$$u_{tt} - \Delta u_{tt} - \Delta u = \Delta f(u), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{1.3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{1.4}$$

Let

$$L_0 = L_1 = L_2 = A_1 = \sum_{|\alpha|=2} a_\alpha D^\alpha,$$

where a_α are real numbers. Then the problem (1.1)–(1.2) is reduced to the Cauchy problem for the following Boussinesq equation

$$u_{tt} + A_1 u_{tt} + A_1 u = A_1 f(x,t,u), \quad x \in \mathbb{R}^2, \quad t \in (0,T), \tag{1.5}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$

here

$$\varphi, \psi \in W_p^s(\mathbb{R}^2), \quad s > \frac{2}{p}, \quad p \in (1,\infty).$$

Now let

$$L_0 = L_1 = L_2 = A_2 = \sum_{|\alpha|=4} a_\alpha D^\alpha,$$

where a_α are real numbers, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_k are natural numbers and $|\alpha| = \sum_{k=1}^{3} \alpha_k$.

Then we get the following Boussinesq equation

$$u_{tt} + A_2 u_{tt} + A_2 u = A_2 f(x,t,u), \quad x \in \mathbb{R}^3, \quad t \in (0,T), \tag{1.6}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

where

$$\varphi, \psi \in W_p^s(\mathbb{R}^3), \quad s > \frac{3}{p}, \quad p \in (1,\infty).$$

Finally, let

$$L_0 = \sum_{|\alpha|=4} a_{0\alpha} D^\alpha, \quad L_1 = \sum_{|\alpha|=2} a_{1\alpha} D^\alpha, \quad L_2 = \sum_{|\alpha|=4} a_{2\alpha} D^\alpha,$$

where $a_{\alpha i}$ are real numbers, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_k are natural numbers and $|\alpha| = \sum_{k=1}^{3} \alpha_k$.

The problem (1.1)–(1.2) reduced to Cauchy problem for the following Boussinesq equation

$$u_{tt} + L_0 u_{tt} + L_1 u = L_2 f(x,t,u), \quad x \in \mathbb{R}^3, \quad t \in (0,T), \tag{1.7}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$

where

$$\varphi,\ \psi \in W^{s,p}(\mathbb{R}^3), \quad s > \frac{3}{p}, \quad p \in (1,\infty).$$

By using the general result for (1.1)–(1.2), we obtain the existence, uniqueness, L^p-regularity, and blow-up properties of the solutions of the problems (1.5), (1.6) and (1.7).

The equation (1.3) arises in different situations (see [18, 30]). For example, for $n = 1$ it describes a limit of a one-dimensional nonlinear lattice [32], shallow-water waves [12, 31] and the propagation of longitudinal deformation waves in an elastic rod [4]. Rosenau [23] derived the equations governing dynamics of one, two and three-dimensional lattices. One of those equations is (1.3). Note that, the existence of solutions and regularity properties for different wave type equations are considered e.g. in [1, 7, 8, 14, 15, 17, 20, 22, 24, 29, 33]. In this respect we can show new results e.g. [1, 7, 8, 14, 15, 22, 29, 33]. In [27] and [28] the existence of the global classical solutions and the blow-up of the solutions of the initial value problem (1.3)–(1.4) are studied. In this paper, we obtain the existence, uniqueness of solution and regularity properties of the problem (1.1)–(1.2). The strategy is to express the Boussinesq equation as an integral equation. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on L^p norm of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate of the solutions of the linearized Boussinesq equation. The methods of harmonic analysis, operator theory, interpolation of Banach spaces and embedding theorems in Sobolev spaces are the main tools implemented to carry out the analysis.

2 Definitions and background

In order to state our results precisely, we introduce some notations and some function spaces. Let E be a Banach space. $L_p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{L_\infty(\Omega:E)} = \operatorname*{ess\,sup}_{x\in\Omega} \|f(x)\|_E.$$

Let $\mathbb{R}$, $\mathbb{C}$ denote the sets of all real and complex numbers, respectively. For $E = \mathbb{C}$ the $L_p(\Omega; E)$ denotes by $L_p(\Omega)$. Let m be a positive integer. $W_p^m(\Omega)$ denotes the Sobolev space, i.e. space of all functions $u \in L_p(\Omega)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L_p(\Omega)$, $1 \le p \le \infty$ with the norm

$$\|u\|_{W_p^m(\Omega)} = \|u\|_{L_p(\Omega)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L_p(\Omega)} < \infty.$$

Let F denotes the Fourier transform defined by

$$\hat{u}(\xi) = Fu = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)dx \quad \text{for } u \in S(\mathbb{R}^n; E) \text{ and } x, \xi \in \mathbb{R}^n.$$

Let $S(\mathbb{R}^n)$ denote the Schwartz class, i.e., the space of rapidly decreasing smooth functions on $\mathbb{R}^n$, equipped with its usual topology generated by seminorms. Let $S'(\mathbb{R}^n)$ denote the space of all continuous linear operators $L : S(\mathbb{R}^n) \to \mathbb{C}$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n)$ is norm dense in $L_p(\mathbb{R}^n)$ when $1 \leq p < \infty$. Let $1 \leq p \leq q < \infty$. A function $\Psi \in L_\infty(\mathbb{R}^n)$ is called a Fourier multiplier from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if the map $u \to F^{-1}\Psi(\xi)Fu$ for $u \in S(\mathbb{R}^n)$ is well defined and extends to a bounded linear operator

$$T : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n).$$

Let $L_p^s(\mathbb{R}^n)$, $-\infty < s < \infty$ denotes Liouville–Sobolev space of order s which is defined as:

$$L_p^s = L_p^s(\mathbb{R}^n) = (I - \Delta)^{-\frac{s}{2}} L_p(\mathbb{R}^n)$$

with the norm

$$\|u\|_{L_p^s} = \left\|(I-\Delta)^{\frac{s}{2}} u\right\|_{L_p(\mathbb{R}^n)} = \left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}\right\|_{L_p(\mathbb{R}^n)} < \infty.$$

It clear that $L_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. It is known that $L_p^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n)$ for the positive integer m (see e.g. [26, § 15].

Let $L_q^*(E)$ denote the space of all E-valued function space such that

$$\|u\|_{L_q^*(E)} = \left(\int_0^\infty \|u(t)\|_E^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty, 1 \leq q < \infty, \ \|u\|_{L_\infty^*(E)} = \sup_{t \in (0,\infty)} \|u(t)\|_E.$$

Here, F denotes the Fourier transform. Fourier-analytic representation of Besov space on $\mathbb{R}^n$ are defined as:

$$B_{p,q}^s(\mathbb{R}^n) = \Big\{u \in S'(\mathbb{R}^n) : \|u\|_{B_{p,q}^s(\mathbb{R}^n)} = \left\|F^{-1} t^{\varkappa - s}\left(1+|\xi|^{\frac{\varkappa}{2}}\right) e^{-t|\xi|^2} Fu\right\|_{L_q^*\left(L_p(\mathbb{R}^n)\right)},$$
$$|\xi|^2 = \sum_{k=1}^n \xi_k^2, \ \xi = (\xi_1, \xi_2, \ldots, \xi_n), p \in (1,\infty), \ q \in [1,\infty], \ \varkappa > s\Big\}.$$

Here,

$$X_p = L^p(\mathbb{R}^n), \quad 1 \leq p \leq \infty, \quad Y^{s,p} = L^{s,p}(\mathbb{R}^n),$$

$$Y_1^{s,p} = L_p^s(\mathbb{R}^n) \cap L_1(\mathbb{R}^n), \quad Y_\infty^{s,p} = L^{s,p}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n),$$

It should be note that, the norm of Besov space does not depends on $\varkappa$ (see e.g. [25, § 2.3]. For $p = q$ the space $B_{p,q}^s(\mathbb{R}^n)$ will be denoted by $B_p^s(\mathbb{R}^n)$.

Definition 2.1. For any $T > 0$ the function $u \in C^2\left([0,T]; Y_\infty^{2,s,p}\right)$ satisfies the equation (1.1)–(1.2) a.e. in $\mathbb{R}_T^n = \mathbb{R}^n \times (0,T)$ is called the continuous solution or the strong solution of the problem (1.1)–(1.2). If $T < \infty$, then $u(x,t)$ is called the local strong solution of the problem (1.1)–(1.2). If $T = \infty$, then $u(x,t)$ is called the global strong solution of (1.1)–(1.2).

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of a unique solution and priory estimates for the solution of the linearized problem (1.1)–(1.2). In Section 3, we show the existence and uniqueness of the local strong solution of the problem (1.1)–(1.2). Section 4 is devoted to the existence of the global solution. In Section 5 the blow-up properties of the solution are derived. In Section 6 we show some applications of the problem (1.1)–(1.2).

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say h, we write C_h.

3 Estimates for linearized equation

In this section, we make the necessary estimates for solutions of the Cauchy problem for the following linear Boussinesq equation

$$u_{tt} + L_0 u_{tt} + L_1 u = L_2 g(x,t), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$

where

$$L_i u = \sum_{|\alpha|=2m_i} a_{i\alpha} D^\alpha u,\ a_{i\alpha} \in \mathbb{R}, \quad i = 0,1,2,$$

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_k are natural numbers, $|\alpha| = \sum_{k=1}^n \alpha_k$ and m_i are positive integers. Let

$$L_i(\xi) = \sum_{|\alpha|=2m_i} a_{i\alpha} (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \ldots (i\xi_n)^{\alpha_n}, \quad i = 0,1,2, \tag{3.2}$$

$$Q = Q(\xi) = L_1(\xi)[1 + L_0(\xi)]^{-1}, \quad L(\xi) = L_2(\xi)[1 + L_0(\xi)]^{-1}.$$

Condition 3.1. Assume that $L_1(\xi) \neq 0$, $L_0(\xi) \neq -1$ and there exist positive constants M_1 and M_2 depend only on $a_{i\alpha}$ such that

$$\left|Q^{\frac{1}{2}}(\xi)\right| \leq M_1 \left(1 + |\xi|^2\right)^{\frac{\nu}{2}}, \quad \left|L(\xi) Q^{\frac{1}{2}}(\xi)\right| \leq M_2 \left(1 + |\xi|^2\right)^{\frac{\nu}{2}} \tag{3.3}$$

for all $\xi \in \mathbb{R}^n$ and a real number ν.

Remark 3.2. It is not hard to see that if $\nu \geq m_1 - m_0$, then the first inequality verified. Moreover, if $\nu \geq m_1 + 2m_2 - (2m_0)^{\frac{3}{2}}$, then the second inequality holds.

First we need the following lemmas.

Lemma 3.3. *Suppose that $Q(\xi) \neq 0$ for each $\xi \in \mathbb{R}^n$. Then problem (3.1) has a strong solution.*

Proof. Since L_0, L_1 and L_2 are differential operators with constant coefficients, by using of Fourier transform and in view of (3.2), we get from (3.1):

$$\begin{aligned} &\hat{u}_{tt}(\xi,t) + Q(\xi)\hat{u}(\xi,t) = L(\xi)\hat{g}(\xi,t), \\ &\qquad \hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi), \quad \xi \in \mathbb{R}^n, \quad t \in (0,T), \end{aligned} \tag{3.4}$$

where $\hat{u}(\xi,t)$ is a Fourier transform of $u(x,t)$ with respect to x. By using the variation of constants we get that there exists a solution of the problem (3.4) that can be written as the following

$$\hat{u}(\xi,t) = C(\xi,t)\hat{\varphi}(\xi) + S(\xi,t)\hat{\psi}(\xi) + Og(\xi), \tag{3.5}$$

here,

$$C(\xi,t) = \cos\left(Q^{\frac{1}{2}}t\right), \quad S(\xi,t) = Q^{-\frac{1}{2}}\sin\left(Q^{\frac{1}{2}}t\right),$$

$$\hat{\Phi}(\xi,t) = L(\xi)Q^{-\frac{1}{2}}(\xi)\sin\left(Q^{\frac{1}{2}}t\right), \quad Og = Og(\xi) = \int_0^t \hat{\Phi}(\xi,t-\tau)\hat{g}(\xi,\tau)d\tau.$$

From (3.5) we get that the solution of the problem (3.1) can be expressed as

$$u(x,t) = S_1(t)\varphi(x) + S_2(t)\psi(x) + \int_0^t F^{-1}Og(\xi)d\xi, \quad t \in (0,T), \tag{3.6}$$

where F^{-1} denotes the inverse Fourier transformation, $S_1(t)$ and $S_2(t)$ are linear operators defined by

$$S_1(t)\varphi = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\xi}C(\xi,t)\hat{\varphi}(\xi)d\xi,$$
$$S_2(t)\psi = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\xi}S(\xi,t)\hat{\psi}(\xi)d\xi. \qquad \square$$

Theorem 3.4. *Assume that the Condition 3.1 holds and*

$$s > n\left(\frac{2}{q}+\frac{1}{p}\right)+\nu \quad \text{if } \nu \geq 0, \quad s > n\left(\frac{2}{q}+\frac{1}{p}\right) \quad \text{if } \nu < 0 \tag{3.7}$$

for $p \in [1,\infty]$ and for a $q \in [1,2]$. Then for φ, ψ, $g(\cdot,t) \in Y_1^{s,p}$ for $t \in (0,T)$ and $g(x,\cdot) \in L^1\left(0,T;Y_1^{s,p}\right)$ for $x \in \mathbb{R}^n$ problem (3.1) has a unique solution $u(x,t)$ satisfies the following estimate

$$\begin{aligned}\|u\|_{X_\infty} &+ \|u_t\|_{X_\infty} \\ &\leq C\left[\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1} + \int_0^t \left(\|g(\cdot,\tau)\|_{Y^{s,p}} + \|g(\cdot,\tau)\|_{X_1}\right)d\tau\right]\end{aligned} \tag{3.8}$$

uniformly with respect to $t \in [0,T]$.

Proof. Let $N \in \mathbb{N}$ and

$$\Pi_N = \{\xi : \xi \in \mathbb{R}^n,\ |\xi| \leq N\}, \quad \Pi_N' = \{\xi : \xi \in \mathbb{R}^n,\ |\xi| \geq N\}.$$

It is clear to see that

$$\begin{aligned}\|u\|_{X_\infty} &\leq \left\|F^{-1}C(\xi,t)\hat{\varphi}(\xi)\right\|_{X_\infty} + \left\|F^{-1}S(\xi)\hat{\psi}(\xi,t)\right\|_{X_\infty} \\ &\leq \left\|\int_{\mathbb{R}^n} e^{ix\xi}C(\xi,t)\varphi(x)dx\right\|_{L_\infty(\Pi_N)} + \left\|\int_{\mathbb{R}^n} e^{ix\xi}S(\xi,t)\psi(x)dx\right\|_{L_\infty(\Pi_N)} \\ &\quad + \left\|F^{-1}C(\xi,t)\hat{\varphi}(\xi)\right\|_{L_\infty(\Pi_N')} + \left\|F^{-1}S(\xi,t)\hat{\psi}(\xi)\right\|_{L\infty(\Pi_N')} \\ &\quad + \left\|F^{-1}C(\xi,t)Og(\xi)\right\|_{L_\infty(\Pi_N')} + \left\|F^{-1}Og(\xi)\right\|_{L\infty(\Pi_N')}\end{aligned} \tag{3.9}$$

Using Minkowski's inequality for integrals and uniformly boundedness of $C(\xi,t)$, $S(\xi,t)$ on Π_N we have

$$\left\| \int_{\mathbb{R}^n} e^{ix\xi} C(\xi,t)\varphi(x)dx \right\|_{L_\infty(\Pi_N)} + \left\| \int_{\mathbb{R}^n} e^{ix\xi} S(\xi,t)\psi(x)dx \right\|_{L_\infty(\Pi_N)} \leq C\left[\|\varphi\|_{X_1} + \|\psi\|_{X_1}\right]. \quad (3.10)$$

It is clear to see that

$$\begin{aligned} &\left\|F^{-1}C(\xi,t)\hat{\varphi}(\xi)\right\|_{L^\infty(\Pi_N')} + \left\|F^{-1}S(\xi,t)\hat{\psi}(\xi)\right\|_{L^\infty(\Pi_N')} \\ &= \left\|F^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}} C(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{\varphi}(\xi)\right\|_{L^\infty(\Pi_N')} \\ &+ \left\|F^{-1}\left(1+|\xi|^2\right)^{-s} S(\xi,t)(1+|\xi|)^{\frac{s}{2}}\hat{\psi}(\xi)\right\|_{L^\infty(\Pi_N')}. \end{aligned} \quad (3.11)$$

By using (3.5) and (3.3) we get the estimates

$$\begin{aligned} &\sup_{\xi\in\mathbb{R}^n, t\in[0,T]} |\xi|^{|\alpha|+\frac{n}{p}} \left| D^\alpha\left[\left(1+|\xi|^2\right)^{-\frac{s}{2}} C(\xi,t)\right]\right| \leq C_2, \\ &\sup_{\xi\in\mathbb{R}^n, t\in[0,T]} |\xi|^{|\alpha|+\frac{n}{p}} \left| D^\alpha\left[\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)\right]\right| \leq C_2, \end{aligned} \quad (3.12)$$

uniformly in $t \in [0,T]$ for $s > \frac{n}{p}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_k \in \{0,1\}$, $\xi \in \mathbb{R}^n$ and $\xi \neq 0$.

Let we show that $G(\cdot,t)$, $V(\cdot,t) \in B_{q,1}^{\frac{n}{q}+\frac{1}{p}}(\mathbb{R}^n;E)$ for some $q \in (1,2)$ and for all $t \in [0,T]$, where

$$G(\xi,t) = \left(1+|\xi|^2\right)^{-\frac{s}{2}} Q^{\frac{1}{2}}(\xi)C(\xi,t),\ V(\cdot,t) = \left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t).$$

By embedding properties of Sobolev and Besov spaces it sufficient to derive that G, $V \in W_q^{n(\frac{1}{q}+\frac{1}{p})+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Indeed by construction of solution, by Condition 3.1 and by (3.3) we get $G \in L_q(\mathbb{R}^n)$. Let $\sigma > n(\frac{1}{q}+\frac{1}{p})$. For deriving the embedding relating $G \in W_q^{\sigma+\varepsilon}(\mathbb{R}^n)$, it sufficient to show

$$\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} G(\cdot,t) \in L_\sigma(\mathbb{R}^n) \quad \text{for all } t \in [0,T].$$

Indeed, in view of (3.3), (3.12) the function $\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} G(\xi,t)$ is uniformly bounded for $\xi \in \mathbb{R}^n$ and $s > \sigma$. By virtue of (3.3), (3.12) and by assumption (3.7) we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{\sigma}{2}q} |G(\xi,t)|^q d\xi &\lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{(s-\sigma)}{2}q} |C(\xi,t)|^q d\xi \\ &\lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\left(\frac{s-\sigma}{2}\right)q} d\xi < \infty. \end{aligned}$$

In a similar way we obtain the following

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{\sigma}{2}q} |V(\xi,t)|^q d\xi \lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\left(\frac{s-\sigma}{2}\right)q} |S(\xi,t)|^q d\xi < \infty.$$

By the Fourier multiplier theorem [10, Theorem 4.3], from (3.12) we get that the functions $\left(1+|\xi|^2\right)^{-\frac{s}{2}} Q^{\frac{1}{2}}(\xi) C(\xi,t)$, $\left(1+|\xi|^2\right)^{-\frac{s}{2}} L(\xi) Q^{\frac{1}{2}}(\xi) S(\xi,t)$ are $L_p(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$ Fourier multipliers. Then by Minkowski's inequality for integrals from (3.10) and (3.11) we obtain

$$\left\| F^{-1} C(\xi,t) \hat{\varphi}(\xi) \right\|_{L_\infty(\Pi'_N)} + \left\| F^{-1} S(\xi,t) \hat{\psi}(\xi) \right\|_{L_\infty(\Pi'_N)} \leq C[\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}}]. \tag{3.13}$$

Moreover, by using the representation of $\hat{\Phi}(\xi,t)$ in (3.5) and the estimate (3.3) we get the uniform estimate

$$\sup_{\xi \in \mathbb{R}^n, t \in [0,T]} |\xi| \left|^{|\alpha| + \frac{n}{p}} D^\alpha \left[\left(1+|\xi|^2\right)^{-\frac{s}{2}} \hat{\Phi}(\xi,t) \right] \right| \leq C_3. \tag{3.14}$$

By reasoning as the above and in view of (3.3) we get that the function $\left(1+|\xi|^2\right)^{-\frac{s}{2}} Og(\xi)$ is a $L_p(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$ Fourier multiplier, i.e. we have the following uniform estimate

$$\left\| F^{-1} \int_0^t \hat{\Phi}(\xi, t-\tau) \hat{g}(\xi,\tau) d\tau \right\|_{X_\infty} \leq C \int_0^t \left(\|g(\cdot,\tau)\|_{Y^s} + \|g(\cdot,\tau)\|_{X_1} \right) d\tau.$$

Hence, from (3.9)–(3.11), we deduced the following

$$\|u\|_{X_\infty} \leq C \left[\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1} + \int_0^t \left(\|g(\cdot,\tau)\|_{Y^{s,p}} + \|g(\cdot,\tau)\|_{X_1} \right) d\tau \right]. \tag{3.15}$$

By differentiating from (3.5) we get

$$\begin{aligned} \hat{u}_t(\xi,t) = & -Q^{\frac{1}{2}}(\xi) \sin\left(Q^{\frac{1}{2}} t\right) \hat{\varphi}(\xi) + \cos\left(Q^{\frac{1}{2}} t\right) \hat{\psi}(\xi) \\ & + \int_0^t Q^{\frac{1}{2}}(\xi) L(\xi) \sin\left(Q^{\frac{1}{2}}(\xi, t-\tau)\right) \hat{g}(\xi,\tau) d\tau, \quad t \in (0,T). \end{aligned} \tag{3.16}$$

By using (3.3) and (3.16) in a similar way, we get

$$\|u_t\|_{X_\infty} \leq C \left[\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1} + \int_0^t \left(\|g(\cdot,\tau)\|_{Y^{s,p}} + \|g(\cdot,\tau)\|_{X_1} \right) d\tau \right] \tag{3.17}$$

Then from (3.15) and (3.17), we obtain the estimate (3.8). Let us now show that problem (3.1) has a unique solution $u \in C^{(1)}([0,T]; Y^{s,p})$. Let us admit it is the opposite. So let us assume that the problem (3.1) has two solutions u_1, $u_2 \in C^{(1)}([0,T]; Y^{s,p})$. Then by linearity of (3.1), we get that $v = u_1 - u_2$ is also a solution of the corresponding homogenous equation

$$v_{tt} + L_0 v_{tt} + L_1 v = 0, \quad v(x,0) = 0, \quad v_t(x,0) = 0, \quad x \in \mathbb{R}^n, \quad t \in (0,T).$$

Moreover, by (3.8) we have the following estimate

$$\|v\|_{X_\infty} \leq 0.$$

The above estimate implies that $v = 0$. □

Remark 3.5. In view of Remark 3.2 we see that the assumption (3.7) is satisfied if $\nu \geq 0$ and

$$s > n\left(\frac{2}{q} + \frac{1}{p}\right) + \max\left\{ m_1 - m_0,\, m_1 + 2m_2 - (2m_0)^{\frac{3}{2}} \right\}.$$

By reasoning as in Theorem 3.4 we obtain

Theorem 3.6. *Let the Condition 3.1 hold. Then for φ, ψ, $g(\cdot,t) \in Y^{s,p}$ for $t \in (0,T)$, $g(x,\cdot) \in L^1\left(0,T;Y_1^{s,p}\right)$ for $x \in \mathbb{R}^n$ problem (3.1) has a unique solution $u(x,t)$ and the following uniform estimate holds*

$$\|u\|_{Y^{s,p}} + \|u_t\|_{Y^{s,p}} \leq C\left[\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|g(.,\tau)\|_{Y^{s,p}} d\tau\right]. \tag{3.18}$$

Proof. From (3.5) we have the following uniform estimate

$$\begin{aligned}
&\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}\right\|_{X_p} + \left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}_t\right\|_{X_p} \\
&\quad \leq C\left\{\left\|F^{-1}(1+|\xi|)^{\frac{s}{2}}C(\xi,t)\hat{\varphi}\right\|_{X_p} + \left\|F^{-1}(1+|\xi|)^{\frac{s}{2}}S(\xi,t)\hat{\psi}\right\|_{X_p}\right. \\
&\qquad \left.+ \int_0^t \left\|(1+|\xi|)^{\frac{s}{2}}\hat{\Phi}(\xi,t-\tau)\hat{g}(\cdot,\tau)\right\|_{X_p} d\tau\right\}.
\end{aligned} \tag{3.19}$$

By Condition 3.1 and by virtue of Fourier multiplier theorems (see e.g. [10, Theorem 4.3], we get that $C(\xi,t)$, $S(\xi,t)$ and $\hat{\Phi}(\xi,t)$ are Fourier multipliers in $L_p(\mathbb{R}^n)$ uniformly with respect to $t \in [0,T]$. So, the estimate (3.19) by using Minkowski's inequality for integrals implies (3.18).

The uniqueness of (3.3) is obtained by reasoning as in Theorem 3.4. □

4 Initial value problem for nonlinear equation

In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1)–(1.2).

For the study of the nonlinear problem (1.1)–(1.2) we need the following lemmas

Lemma 4.1 (Nirenberg's inequality [19]). *Assume that $u \in L_p(\Omega)$, $D^m u \in L_q(\Omega)$, $p,q \in (1,\infty)$. Then for i with $0 \leq i \leq m$, $m > \frac{n}{q}$ we have*

$$\left\|D^i u\right\|_r \leq C\|u\|_p^{1-\mu} \sum_{k=1}^n \|D_k^m u\|_q^\mu, \tag{4.1}$$

where

$$\frac{1}{r} = \frac{i}{m} + \mu\left(\frac{1}{q} - \frac{m}{n}\right) + (1-\mu)\frac{1}{p}, \quad \frac{i}{m} \leq \mu \leq 1.$$

Lemma 4.2 ([19]). *Assume that $u \in W_p^m(\Omega) \cap L_\infty(\Omega)$ and $f(u)$ possesses continuous derivatives up to order $m \geq 1$. Then $f(u) - f(0) \in W_p^m(\Omega)$ and*

$$\|f(u) - f(0)\|_p \leq \left\|f^{(1)}(u)\right\|_\infty \|u\|_p,$$

$$\left\|D^k f(u)\right\|_p \leq C_0 \sum_{j=1}^k \left\|f^{(j)}(u)\right\|_\infty \|u\|_\infty^{j-1} \left\|D^k u\right\|_p, \quad 1 \leq k \leq m, \tag{4.2}$$

where $C_0 \geq 1$ is a constant.

Let

$$E_0 = \left(Y^{s,p}, X_p\right)_{\frac{1}{2p},p} = B_p^{s\left(1-\frac{1}{2p}\right)}(\mathbb{R}^n).$$

Remark 4.3. By using a result by J. Lions and I. Petree (see e.g. [25, § 1.8]) we obtain that the map $u \to u(t_0)$, $t_0 \in [0,T]$ is continuous and surjective from $W_p^2(0,T;Y^{s,p},X_p)$ onto E_0 and there is a constant C_1 such that

$$\|u(t_0)\|_{E_0} \leq C_1 \|u\|_{W_p^2(0,T;Y^{s,p},X_p)}, \quad 1 \leq p \leq \infty.$$

Let

$$C^{(m)}(p) = C^{(m)}\left([0,T];Y_\infty^{s,p}\right).$$

First all of, we define the space $Y(T) = C\left([0,T];Y_\infty^{s,p}\right)$ equipped with the norm defined by

$$\|u\|_{Y(T)} = \max_{t\in[0,T]} \|u\|_{Y^{s,p}} + \max_{t\in[0,T]} \|u\|_{X_\infty},\ u \in Y(T).$$

It is easy to see that $Y(T)$ is a Banach space. For φ, $\psi \in Y^{s,p}$, let

$$M = \|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_\infty} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_\infty}.$$

Condition 4.4. Assume:

(1) The Condition 3.1 holds, φ, $\psi \in Y_1^{s,p}$ and $s > n\left(\frac{2}{q}+\frac{1}{p}\right)+\nu$ if $\nu \geq 0$, $s > n\left(\frac{2}{q}+\frac{1}{p}\right)$ if $\nu < 0$ for $p \in [1,\infty]$ and for a $q \in [1,2]$;

(2) (2) the function $u \to \hat{f}(\xi,t,u)$: $\mathbb{R}^n \times [0,T] \times E_0 \to \mathbb{C}$ is a measurable in $(\xi,t) \in \mathbb{R}^n \times [0,T]$ for $u \in E_0$; moreover, $\hat{f}(\xi,t,u)$ is continuous in $u \in E_0$ and $\hat{f}(\xi,t,u) \in C^{([s]+1)}(E_0;\mathbb{C})$ uniformly for $\xi \in \mathbb{R}^n$ and $t \in [0,T]$.

The main aim of this section is to prove the following result.

Theorem 4.5. *Let the Condition 4.4 hold. Then problem (1.1)–(1.2) has a unique strong solution $u \in C^{(2)}(p)$, where T_0 is a maximal time that is appropriately small relative to M. Moreover, if*

$$\sup_{t\in[0,\,T_0)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty \tag{4.3}$$

then $T_0 = \infty$.

Proof. First, we are going to prove the existence and the uniqueness of the local strong solution of (1.1)–(1.2) by contraction mapping principle. Consider a map G on $Y(T)$ such that $G(u)$ is the operator defined by

$$G(u) = G(u)(x,t) = S_1(t)\varphi(x) + S_2(t)\psi(x) + O(u), \tag{4.4}$$

where

$$O(u) = \int_0^t F^{-1}\left[S(\xi,t-\tau)L(\xi)\hat{f}(u)(\xi,\tau)\right]d\tau, \quad t \in (0,T). \tag{4.5}$$

From Lemma 4.2 we know that $f(u) \in L_p\left(0,T;Y_\infty^{s,p}\right)$ for any $T > 0$. From Lemma 4.2 it is easy to see that the map G is well defined for $f \in C^{(2)}(X_0;\mathbb{C})$. We put

$$Q(M;T) = \left\{u \mid u \in Y(T),\ \|u\|_{Y(T)} \leq M+1\right\}.$$

First, by reasoning as in [12] let us prove that the map G has a unique fixed point in $Q(M;T)$. From Lemma 4.2 it is easy to see that the map G is well defined for $f \in C^{(2)}(X_0;\mathbb{C})$. Let

$$W(u) = F^{-1}[S(\xi,t-\tau)L(\xi)f(u)](x,\tau).$$

By assumption (2) of Condition 4.4 and by virtue [10, Theorem 4.3], the function $U(\xi, t-\tau)L(\xi)$ is a Fourier multiplier theorem in X_p, i.e. if $f(u) \in X_p$, then $W(u) \in X_p$.

First, by reasoning as in [12] let us prove that the map G has a unique fixed point in $Q(M;T)$. For this aim, it is sufficient to show that the operator G maps $Q(M;T)$ into $Q(M;T)$ and $G : Q(M;T) \to Q(M;T)$ is strictly contractive if T is appropriately small relative to M. Consider the function $\overline{W}(\xi)$: $[0, \ \infty) \to [0, \ \infty)$ defined by

$$\overline{W}(\sigma) = \max_{|\xi|\le\sigma}\left\{\left|\overline{W}^{(1)}(\xi)\right|, \left|\overline{W}^{(2)}(\xi)\right|, \ldots, \left|\overline{W}^{([s])}(\xi)\right|\right\}, \quad \sigma \ge 0.$$

It is clear to see that the function $\overline{W}(\sigma)$ is continuous and nondecreasing on $[0,\infty)$. From Lemma 4.2 we have

$$\begin{aligned}
\|W(u)\|_{Y^{2,p}} \le & \left\|W^{(1)}(u)\right\|_{X_\infty}\|u\|_{X_p} + \left\|W^{(1)}(u)\right\|_{X_\infty}\|Du\|_{X_p} \\
& + C_0\left[\left\|W^{(1)}(u)\right\|_{X_\infty}\|u\|_{X_p} + \cdots + \left\|W^{([s])}(u)\right\|_{X_\infty}\|u\|_{X_\infty}\left\|D^{[s]}u\right\|_{X_p}\right] \\
& \le 2C_0\overline{W}(M+1)(M+1)\|u\|_{Y^{s,p}}.
\end{aligned} \tag{4.6}$$

By using Theorem 3.4 we obtain from (4.5):

$$\|G(u)\|_{X_\infty} \le \|\varphi\|_{X_\infty} + \|\psi\|_{X_\infty} + \int_0^t \|W(x,\tau,u(\tau))\|_{X_\infty}, \tag{4.7}$$

$$\|G(u)\|_{Y^{2sp}} \le \|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|W(x,\tau,u(\tau))\|_{Y^{2,p}} d\tau. \tag{4.8}$$

Thus, from (4.6)–(4.8) and Lemma 4.2 we get

$$\|G(u)\|_{Y(T)} \le M + T(M+1)\left[1+2C_0(M+1)\bar{f}(M+1)\right].$$

If T satisfies

$$T \le \left\{(M+1)\left[1+2C_0(M+1)\bar{f}(M+1)\right]\right\}^{-1}, \tag{4.9}$$

then

$$\|Gu\|_{Y(T)} \le M+1.$$

Therefore, if (4.9) holds, then G maps $Q(M;T)$ into $Q(M;T)$. Now, we are going to prove that the map G is strictly contractive. Assume $T > 0$ and u_1, $u_2 \in Q(M;T)$ given. We get

$$G(u_1) - G(u_2) = \int_0^t [W(u_1)(x,\tau) - W(u_2)(x,\tau)]d\tau, \quad t \in (0,T).$$

By using the assumption (3) and the mean value theorem, we obtain

$$\begin{aligned}
W(u_1) - W(u_2) &= W^{(1)}(u_2 + \eta_1(u_1-u_2))(u_1-u_2), \\
D[W(u_1) - W(u_2)] &= W^{(2)}(u_2 + \eta_2(u_1-u_2))(u_1-u_2)D_\xi u_1 + W^{(1)}(u_2)(Du_1 - D_\xi u_2), \\
D^2\left[\hat{f}(u_1) - \hat{f}(u_2)\right] &= W^{(3)}(u_2 + \eta_3(u_1-u_2))(u_1-u_2)(Du_1)^2 \\
&\quad + W^{(2)}(u_2)(Du_1 - Du_2)(Du_1 + Du_2) \\
&\quad + W^{(2)}(u_2 + \eta_4(u_1-u_2))(u_1-u_2)D^2u_1 + W^{(1)}(u_2)\left(D^2u_1 - D^2u_2\right),
\end{aligned}$$

where $0 < \eta_i < 1$. Thus, using Hormander's and Nirenberg's inequality, we have

$$\|W(u_1) - W(u_2)\|_{X_\infty} \le \overline{W}(M+1)\|u_1 - u_2\|_{X_\infty}, \tag{4.10}$$

$$\|(u_1) - W(u_2)\|_{X_p} \le \overline{W}(M+1)\|u_1 - u_2\|_{X_p},$$

$$\begin{aligned}\|D[W(u_1) - W(u_2)]\|_{X_p} \le{}& (M+1)\overline{W}(M+1)\|u_1 - u_2\|_{X_\infty} \\ &+ \overline{W}(M+1)\|W(u_1) - W(u_2)\|_{X_p},\end{aligned} \tag{4.11}$$

$$\begin{aligned}&\left\|D^2[W(u_1) - W(u_2)]\right\|_{X_p} \\ &\quad\le (M+1)\overline{W}(M+1)\|u_1 - u_2\|_{X_\infty}\left\|D^2 u_1\right\|^2_{Y^{2,p}} \\ &\qquad + \overline{W}(M+1)\|D(u_1 - u_2)\|_{Y^{2,p}}\|D(u_1 + u_2)\|_{Y^{2,p}} \\ &\qquad + \overline{W}(M+1)\|u_1 - u_2\|_{X_\infty}\left\|D^2 u_1\right\|_{X_p} + \overline{W}(M+1)\|D(u_1 - u_2)\|_{X_p} \\ &\quad\le C^2\overline{W}(M+1)\|u_1 - u_2\|_{X_\infty}\|u_1\|_{X_\infty}\left\|D^2 u_1\right\|_{X_p} \\ &\qquad + C^2\overline{W}(M+1)\|u_1 - u_2\|_{X_\infty}\left\|D^2(u_1 - u_2)\right\|_{X_p}\|u_1 + u_2\|_{X_\infty}\left\|D^2(u_1 + u_2)\right\|_{X_p} \\ &\qquad + (M+1)\overline{W}(M+1)\|u_1 - u_2\|_{X_\infty} + \overline{W}(M+1)\left\|D^2(u_1 - u_2)\right\|_{X_p} \\ &\quad\le 3C^2(M+1)^2\overline{W}(M+1)\|u_1 - u_2\|_{X_\infty} + 2C^2(M+1)\overline{W}(M+1)\left\|D^2(u_1 - u_2)\right\|_{X_p}.\end{aligned} \tag{4.12}$$

In a similar way, we have

$$\left\|D^{[s]}[W(u_1) - W(u_2)]\right\|_{X_p} \le C_1\|u_1 - u_2\|_{X_\infty} + C_2\left\|D^{[s]}(u_1 - u_2)\right\|_{X_p}. \tag{4.13}$$

From (4.10)–(4.13), using Minkowski's inequality for integrals, Fourier multiplier theorem in X_p spaces and Young's inequality, we obtain

$$\begin{aligned}\|G(u_1) - G(u_2)\|_{Y(T)} \le{}& \int_0^t \|u_1 - u_2\|_{X_\infty} d\tau + \int_0^t \|u_1 - u_2\|_{Y^{s,p}} d\tau \\ &+ \int_0^t \|W(u_1) - W(u_2)\|_{X_\infty} d\tau + \int_0^t \|W(u_1) - W(u_2)\|_{Y^{s,p}} d\tau \\ \le{}& T\left[1 + C_1(M+1)^2\overline{W}(M+1)\right]\|u_1 - u_2\|_{Y(T)},\end{aligned}$$

where C_1 is a constant. If T satisfies (4.9) and the following inequality

$$T \le \frac{1}{2}\left[1 + C_1(M+1)^2\overline{W}(M+1)\right]^{-1}, \tag{4.14}$$

then

$$\|Gu_1 - Gu_2\|_{Y(T)} \le \frac{1}{2}\|u_1 - u_2\|_{Y(T)}.$$

That is, G is a contractive map. By contraction mapping principle we know that $G(u)$ has a fixed point $u(x,t) \in Q(M;T)$ that is a solution of (1.1)–(1.2). From (3.6) we get that u is a solution of the following integral equation

$$u(t,x) = S_1(t)\varphi(x) + S_2(t)\psi(x) + + \int_0^t W(u)(x,\tau)d\tau, \quad t \in (0,T).$$

Let us show that this solution is a unique in $Y(T)$. Let u_1, $u_2 \in Y(T)$ be two solutions of the problem (1.1)–(1.2). Then

$$u_1 - u_2 = \int_0^t [W(u_1)(x,\tau) - W(u_2)(x,\tau)]d\tau. \tag{4.15}$$

By the definition of the space $Y(T)$, we can assume that

$$\|u_1\|_{X_\infty} \leq C_1(T), \quad \|u_1\|_{X_\infty} \leq C_1(T).$$

Hence, by Minkowski's inequality for integrals and Theorem 3.6 we obtain from (4.15)

$$\|u_1 - u_2\|_{Y^{s,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{2,p}} d\tau. \tag{4.16}$$

From (4.16) and Gronwall's inequality, we have $\|u_1 - u_2\|_{Y^{s,p}} = 0$, i.e. problem (1.1)–(1.2) has a unique solution which belongs to $Y(T)$. That is, we obtain the first part of the assertion. Now, let $[0,\ T_0)$ be the maximal time interval of existence for $u \in Y(T_0)$. It remains only to show that if (4.3) is satisfied, then $T_0 = \infty$. Assume contrary that, (4.3) holds and $T_0 < \infty$. For $T \in [0, T_0)$, we consider the following integral equation

$$v(x,t) = S_1(t)u(x,T) + S_2(t)u_t(x,T) + \int_0^t W(v)(x,\tau)d\tau, \quad t \in (0,T). \tag{4.17}$$

By virtue of (4.3), for $T' > T$ we have

$$\sup_{t\in[0,T)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty.$$

By reasoning as in the first part of the theorem and by the contraction mapping principle, there is a $T^* \in (0, T_0)$ such that for each $T \in [0,\ T_0)$ the equation (4.17) has a unique solution $v \in Y(T^*)$. The estimates (4.9) and (4.14) imply that T^* can be selected independently of $T \in [0,\ T_0)$. Set $T = T_0 - \frac{T^*}{2}$ and define

$$\tilde{u}(x,t) = \begin{cases} u(x,t), \ t \in [0,T], \\ v(x,t-T), t \in \left[T, T_0 + \frac{T^*}{2}\right] \end{cases}$$

By construction $\tilde{u}(x,t)$ is a solution of the problem (1.1)–(1.2) on $\left[T, T_0 + \frac{T^*}{2}\right]$ and in view of local uniqueness, $\tilde{u}(x,t)$ extends u. This is against to the maximality of $[0,\ T_0)$, i.e. we obtain $T_0 = \infty$. □

From [27], we have

Lemma 4.6. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$ with $f(0) = 0$. Then for any $u \in Y^{s,p} \cap L^\infty$, we have $f(u) \in Y^{s,p} \cap X_\infty$. Moreover, there is some constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^\infty$ with $\|u\|_{X_\infty} \leq M$,*

$$\|f(u)\|_{Y^{s,p}} \leq C(M)\|u)\|_{Y^{s,p}}.$$

By using Lemma 4.1 and properties of convolution operators we obtain

Corollary 4.7. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$ with $f(0) = 0$. Moreover, assume $\Phi \in L^\infty(\mathbb{R}^n)$. Then for any $u \in Y^{s,p} \cap L^\infty$, we have $f(u) \in Y^{s,p} \cap X_\infty$. Moreover, there is some constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^\infty$ with $\|u\|_{X_\infty} \leq M$,*

$$\|\Phi * f(u)\|_{Y^{s,p}} \leq C(M)\|u)\|_{Y^{s,p}}.$$

Lemma 4.8. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$. Then for for any M there is some constant $K(M)$ depending on M such that for all $u, v \in Y^{s,p} \cap X_\infty$ with $\|u\|_{X_\infty} \leq M$, $\|v\|_{X_\infty} \leq M$, $\|u\|_{Y^{s,p}} \leq M$, $\|v\|_{Y^{s,p}} \leq M$,*

$$\|f(u) - f(v\|_{Y^{s,p}} \leq K(M)\|u - v\|_{Y^{s,p}},\ \|f(u) - f(v\|_{X_\infty} \leq K(M)\|u - v\|_{X_\infty}.$$

By reasoning as in [27, Lemma 3.4] and [5, Lemma X 4] we have, respectively

Corollary 4.9. *Let $s > \frac{n}{p}$, $f \in C^{[s]+1}(\mathbb{R})$. Then for any M there is a constant $K(M)$ depending on M such that for all $u, v \in Y^{s,p}$ with $\|u\|_{Y^{s,p}} \leq M$, $\|v\|_{Y^{s,p}} \leq M$,*

$$\|f(u) - f(v\|_{Y^{s,p}} \leq K(M)\|u - v\|_{Y^{s,p}}.$$

Lemma 4.10. *If $s > 0$, then $Y_\infty^{s,p}$ is an algebra. Moreover, for $f, g \in Y_\infty^{s,p}$,*

$$\|fg\|_{Y^{s,p}} \leq C\Big[\|f\|_{X_\infty} + \|g\|_{Y^{s,p}} + \|f\|_{Y^{s,p}} + \|g\|_{X_\infty}\Big].$$

By using Corollary 4.7 and Lemma 4.10 we obtain

Lemma 4.11. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$ and $f(u) = O\big(|u|^{\gamma+1}\big)$ for $u \to 0$, $\gamma \geq 1$ be a positive integer. If $u \in Y_\infty^{s,p}$ and $\|u\|_{X_\infty} \leq M$, then*

$$\|f(u)\|_{Y^{s,p}} \leq C(M)\Big[\|u\|_{Y^{s,p}}\|u\|_{X_\infty}^{\gamma}\Big],$$
$$\|f(u)\|_{X_1} \leq C(M)\|u\|_{X_p}^{p}\|u\|_{X_\infty}^{\gamma-1}.$$

The solution in Theorems 4.2–4.4 can be extended to a maximal interval $[0, T_{\max})$, where finite $T_{\max}$ is characterized by the blow-up condition

$$\limsup_{T\to T_{\max}}\|u\|_{Y^{s,p}(A;E)} = \infty.$$

Lemma 4.12. *Let the Condition 4.4 hold and u be a solution of (1.1)–(1.2). Then there is a global solution if for any $T < \infty$, we have*

$$\sup_{t\in[0,T]}\Big(\|u\|_{Y_\infty^{s,p}} + \|u_t\|_{Y_\infty^{s,p}}\Big) < \infty. \tag{4.18}$$

Proof. Indeed, by reasoning as in the second part of the proof of Theorem 4.5, by using a continuation of local solution of (1.1)–(1.2) and assuming contrary that, (4.18) holds and $T_0 < \infty$, then we obtain contradiction, i.e. we get $T_0 = T_{\max} = \infty$. □

5 Conservation of energy and global existence.

Consider the problem for $p = 2$. Let us denote $Y^{s,2}$ by W^s. We prove the following results.

Condition 5.1. Assume the Condition 4.4 holds for $p = 2$. Let $L_0 = L_1 = L_2 = -L$ and L be a negative symmetric operator in $L_2(\mathbb{R}^n)$. Suppose $(I - L)^{-1}$, $A = L(I - L)^{-1}$ are bounded in $L_2(\mathbb{R}^n)$ and assume

$$\psi \in L_2(\mathbb{R}^n),\quad (Au, u) \in L^2(\mathbb{R}^n),\quad \Phi(\cdot) \in L^1(\mathbb{R}^n),$$

where (u, v) denotes the inner product in $L_2(\mathbb{R}^n)$.

Let

$$F(u) = A[f(u) - u], \Phi(\eta) = \int_0^\eta F(\sigma)d\sigma.$$

Remark 5.2. Note that if $-L$ is self-adjoint positive operator in $L_2(\mathbb{R}^n)$, then the operators $(I - L)^{-1}$, A are bounded in $L_2(\mathbb{R}^n)$.

Lemma 5.3. *Let the Condition 4.4 hold and let $u \in C^{(2)}([0, T]; W^s)$ be solution of (1.1)–(1.2) for any $t \in [0, T)$. Then the energy*

$$E(t) = \|u_t\|^2 + 2\int_{\mathbb{R}^n} \Phi(u)dx \tag{5.1}$$

is constant.

Proof. By use of (1.1) and in view of Condition 4.4, it follows from straightforward calculation that

$$\frac{d}{dt}E(t) = 2(u_{tt}, u_t) + 2\int_{\mathbb{R}^n} \Phi_u(u)u_t dx = 2(u_{tt} + Au - Af(u), u_t) = 0.$$

Hence, we obtain the assertion. □

By using the above lemmas we obtain the following results.

Theorem 5.4. *Assume the Condition 5.1 is satisfied and φ, $\psi \in Y_\infty^{s,2}$. Moreover, there is some $k > 0$ so that*

$$\Phi(s) \geq -k|s|^2, \quad \textit{for all } s \in \mathbb{R}. \tag{5.2}$$

Then there is some $T > 0$ such that problem (1.1)–(1.2) has a global solution

$$u \in C^{(2)}([0, T]; W^s).$$

Proof. Since $r > 2 + \frac{n}{2}$, by Theorem 4.5 we get local existence in $u \in C^{(2)}([0, T]; W^s)$ for some $T > 0$. Assume that u exists on $[0, T)$. By assumption (5.2), we obtain

$$E(t) = \|u_t\|^2 + 2\int_{\mathbb{R}^n} \Phi(u)dx \leq E(0) + 2k\|u(\cdot, t)\|^2. \tag{5.3}$$

for all $t \in [0, T)$. By properties of norms in Hilbert spaces and by the Cauchy–Schwarz inequality, from (5.3) we get

$$\begin{aligned}\frac{d}{dt}\|u(\cdot, t)\|_{W^s}^2 &\leq 2\|u_t(\cdot, t)\|_{W^s}\|u(\cdot, t)\|_{W^s} \\ &\leq \|u_t(\cdot, t)\|_{W^s}^2 + \|u(\cdot, t)\|_{W^s}^2 \leq CE(0) + (2Ck + 1)\|u(t)\|_{W^s}^2.\end{aligned}$$

Gronwall's lemma implies that $\|u(\cdot, t)\|_{W^s}$ is bounded in $[0, T)$. But, since $s > \frac{n}{2}$, we conclude that $\|u(t)\|_{L^\infty}$ also is bounded in $[0, T)$. By Lemma 4.12 this implies a global solution. □

6 Blow up in finite time

We will use the following lemma to prove blow up in finite time.

Lemma 6.1 ([11]). *Suppose $H(t)$, $t \geq 0$ is a positive, twice differentiable function satisfying*

$$H^{(2)}H - (1+\nu)\left(H^{(1)}\right)^2 \geq 0 \quad \text{for } \nu > 0.$$

If $H(0) > 0$ and $H^{(1)}(0) > 0$, then $H(t) \to \infty$ when $t \to t_1$ for some

$$t_1 \leq H(0)\left[\nu H^{(1)}(0)\right]^{-1}.$$

We rewrite the energy identity as

$$E(t) = \|u_t\|^2 + 2\int_{\mathbb{R}^n} \Phi(u)dx = E(0),$$

where

$$\Phi(\eta) = \int_0^\eta F(\sigma)d\sigma, \quad A = L(I-L)^{-1}. \tag{6.1}$$

We prove here the following result.

Theorem 6.2. *Assume the Condition 5.1 is satisfied and φ, $\psi \in Y_\infty^{s,2}$. Let $u \in C^{(2)}([0,T]; W^s)$ be solution of (1.1)–(1.2) for any $t \in [0, T)$. Suppose there are some positive numbers ν, t_0 and b such that*

$$\sigma F(\sigma) \leq 2(1+2\nu)\Phi(\sigma) \quad \text{for all } \sigma \in \mathbb{R} \tag{6.2}$$

and

$$E(0) = \|u_t\|^2 + 2\int_{\mathbb{R}^n} \Phi(u)dx < 0. \tag{6.3}$$

Then the solution u of the problem (1.1)–(1.2) blows up in finite time.

Proof. Assume that there is a global solution. Let

$$H(t) = \|u\|^2 + b(t+t_0)^2.$$

for some positive b and t_0 that will be determined later. We have

$$\begin{aligned} H^{(1)}(t) &= 2(u, u_t) + 2b(t+t_0), \\ H^{(2)}(t) &= 2\|u_t\|^2 + 2(u, u_{tt}) + 2b. \end{aligned} \tag{6.4}$$

Hence, from (1.1)we get

$$(u, u_{tt}) = -(u, AF(u)) = -\int_{\mathbb{R}^n} uAF(u)dx. \tag{6.5}$$

From (6.2)–(6.3) and (6.5) we deduced

$$(u, u_{tt}) \geq -2(1+\nu)\int_{\mathbb{R}^n} \Phi(u)dx = 2(1+\nu)\left[\|u_t\|^2 - E(0)\right]. \tag{6.6}$$

From (6.4) and (6.6), we obtain

$$H^{(2)}(t) \geq 2\|u_t\|^2 + 2(1+\nu)\left[E(0) - \|u_t\|^2\right] + 2b. \tag{6.7}$$

On the other hand, in view of the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left(H^{(1)}(t)\right)^2 &= [2(u, u_t) + 2b(t+t_0)]^2 \\ &\leq 4\left[\|u\|^2\|u_t\|^2 + b^2(t+t_0)^2\left(\|u\|^2 + \|u_t\|^2\right)\right] + 4b^2(t+t_0)^2. \end{aligned} \tag{6.8}$$

Hence, combining (6.4), (6.7) and (6.8) we obtain

$$
\begin{aligned}
&H^{(2)}H-(1+\nu)\left(H^{(1)}\right)^{2}\\
&\quad\geq\left[2\|u_t\|^2+4(1+\nu)\|u_t\|^2-2(1+2\nu)E(0)+2b\right]\left[\|u\|^2+b(t+t_0)^2\right]\\
&\qquad-4(1+\nu)\left[\|u\|^2\|u_t\|^2+b^2(t+t_0)^2\left(\|u\|^2+\|u_t\|^2\right)\right]+4b^2(t+t_0)^2\\
&\quad=\ -2(1+2\nu)[b+E(0)]H(t).
\end{aligned}
$$

Hence, if we choose $b\leq -E(0)$, this gives

$$H^{(2)}H-(1+\nu)\left(H^{(1)}\right)^{2}\geq 0.$$

Moreover,

$$H^{(1)}(0)=2(\varphi,\psi)+2b(t_0)\geq 0$$

for sufficiently large t_0. According to Lemma 6.1, this implies that $H(t)$, and thus $\|u(t)\|^2$ blows up in finite time contradicting the assumption that the global solution exists. □

7 Applications

In this section we give some application of Theorem 4.5.

1. Let

$$L_0=L_1=L_2=A_1=\sum_{|\alpha|=2}a_\alpha D^\alpha,$$

where a_α are real numbers.

Then the problem (1.1)–(1.2) is reduced to the Cauchy problem for the following Boussinesq equation

$$
\begin{gathered}
u_{tt}+A_1u_{tt}+A_1u=A_1f(x,t,u),\quad x\in\mathbb{R}^2,\quad t\in(0,T),\\
u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),
\end{gathered}
\tag{7.1}
$$

Let

$$X_p=L_p\left(\mathbb{R}^2\right),\quad 1\leq p\leq\infty,\quad Y^{s,p}=L_p^s\left(\mathbb{R}^2\right).$$

Assumption 7.1. Assume that $A_2(\xi)\neq 0$, $A_2(\xi)\neq -1$ for all $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$. Let φ, $\psi\in Y^{s,p}\cap X_1$ and

$$M=\|\varphi\|_{Y^{s,p}}+\|\varphi\|_{X_1}+\|\psi\|_{Y^{s,p}}+\|\psi\|_{X_1}.$$

It is not hard to see that Assumtion 7.1 implies Condition 3.1. Hence, from Theorem 4.5 we obtain:

Theorem 7.2. *Suppose that the Assumption 7.1 holds. Let $s>2\left(\frac{2}{q}+\frac{1}{p}\right)$ for $p\in[1,\infty]$ and for a $q\in[1,2]$. Assume that the function $u\rightarrow f(x,t,u)$: $\mathbb{R}^2\times[0,T]\times B_p^{s(1-\frac{1}{2p})}\left(\mathbb{R}^2\right)\rightarrow L_p\left(\mathbb{R}^2\right)$ is measurable in $(x,t)\in\mathbb{R}^2\times[0,T]$ for $u\in B_p^{s(1-\frac{1}{2p})}\left(\mathbb{R}^2\right)$. Moreover, $f(x,t,u)$ is continuous in $u\in B_p^{s(1-\frac{1}{2p})}\left(\mathbb{R}^2\right)$ and*

$$f(x,t,u)\in C^{(3)}\left(B_p^{s(1-\frac{1}{2p})}\left(\mathbb{R}^2\right)\right)$$

uniformly with respect to $(x,t) \in \mathbb{R}^2 \times [0,T]$. *Then for* $\varphi, \psi \in Y^{s,p} \cap X_1$ *problem* (7.1) *has a unique local strong solution* $u \in C^{(2)}([0,\ T_0); Y_\infty^{s,p})$, *where* T_0 *is a maximal time interval that is appropriately small relative to M. Moreover, if*

$$\sup_{t\in[0,T_0)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

2. Let

$$L_0 = L_1 = L_2 = A_2 = \sum_{|\alpha|=4} a_\alpha D^\alpha,$$

where a_α are real numbers, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_k are natural numbers and

$$|\alpha| = \sum_{k=1}^{3} \alpha_k.$$

Then the problem (1.1)–(1.2) is reduced to the Cauchy problem for the following Boussinesq equation

$$u_{tt} + A_2 u_{tt} + A_2 u = A_2 f(x,t,u), \quad x \in \mathbb{R}^3, \quad t \in (0,T), \tag{7.2}$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

where

$$\varphi, \quad \psi \in L_p^s(\mathbb{R}^3), \quad s > \frac{3}{p}, \quad p \in [1,\infty].$$

Assumption 7.3. Assume that $A_2(\xi) \neq 0$, $A_2(\xi) \neq -1$ for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Let φ, $\psi \in Y^{s,p} \cap X_1$ and

$$M = \|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1}.$$

It is clear to see that if Assumption 7.1 holds, then Condition 3.1 is satisfied.

Let

$$X_p = L_p(\mathbb{R}^3), 1 \leq p \leq \infty, \quad Y^{s,p} = L_p^s(\mathbb{R}^3).$$

Hence, from Theorem 4.5 we obtain:

Theorem 7.4. *Suppose that the Assumption* 7.3 *holds. Let* $s > 3\left(\frac{2}{q} + \frac{1}{p}\right)$ *for* $p \in [1,\infty]$ *and for a* $q \in [1,2]$. *Suppose that the function* $u \to f(x,t,u)$: $\mathbb{R}^3 \times [0,T] \times B_p^{s(1-\frac{1}{2p})}(\mathbb{R}^3) \to L_p(\mathbb{R}^3)$ *is measurable in* $(x,t) \in \mathbb{R}^3 \times [0,T]$ *for* $u \in B_p^{s\left(1-\frac{1}{2p}\right)}(\mathbb{R}^3)$. *Moreover,* $f(x,t,u)$ *is continuous in* $u \in B_p^{s\left(1-\frac{1}{2p}\right)}(\mathbb{R}^3)$ *and*

$$f(x,t,u) \in C^{(3)}\left(B_p^{s(1-\frac{1}{2p})}(\mathbb{R}^3)\right)$$

uniformly with respect to $(x,t) \in \mathbb{R}^3 \times [0,T]$. *Then problem* (7.2) *has a unique local strong solution*

$$u \in C^{(2)}([0,\ T_0); Y_\infty^{s,p}),$$

where T_0 *is a maximal time interval that is appropriately small relative to M. Moreover, if*

$$\sup_{t\in[0,T_0)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

3. Let

$$L_0 = \sum_{|\alpha|=4} a_{0\alpha} D^\alpha, \quad L_1 = \sum_{|\alpha|=2} a_{1\alpha} D^\alpha, \quad L_2 = \sum_{|\alpha|=4} a_{2\alpha} D^\alpha,$$

where $a_{\alpha i}$ are real numbers, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_k are natural numbers and

$$|\alpha| = \sum_{k=1}^{3} \alpha_k.$$

Then the problem (1.1)–(1.2) is reduced to Cauchy problem for the following Boussinesq equation

$$\begin{gathered} u_{tt} + L_0 u_{tt} + L_1 u = L_2 f(x,t,u), \quad x \in \mathbb{R}^3, \quad t \in (0,T), \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \end{gathered} \tag{7.3}$$

where

$$\varphi,\ \psi \in L^{s,p}(\mathbb{R}^3), \quad p \in [1,\infty].$$

Hence, from Theorem 4.5 we obtain:

Theorem 7.5. *Assume that the Condition 3.1 is satisfied. Let φ, $\psi \in Y^{s,p} \cap X_1$ and*

$$M = \|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1}$$

for $s > 3(\frac{2}{q} + \frac{1}{p}) + \nu$, $p \in [1,\infty]$ and for a $q \in [1,2]$. Suppose that the function $u \to f(x,t,u)$: $\mathbb{R}^3 \times [0,T] \times B_p^{s(1-\frac{1}{2p})}(\mathbb{R}^3) \to L_p(\mathbb{R}^3)$ is measurable in $(x,t) \in \mathbb{R}^3 \times [0,T]$ for $u \in B_p^{s(1-\frac{1}{2p})}(\mathbb{R}^3)$. Moreover, $f(x,t,u)$ is continuous in $u \in B_p^{s(1-\frac{1}{2p})}(\mathbb{R}^3)$ and

$$f(x,t,u) \in C^{(3)}\left(B_p^{s(1-\frac{1}{2p})}(\mathbb{R}^3)\right)$$

uniformly with respect to $(x,t) \in \mathbb{R}^3 \times [0,T]$. Then problem (7.3) has a unique strong solution

$$u \in C^{(2)}([0,\ T_0); Y_\infty^{s,p}),$$

where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if

$$\sup_{t \in [0,T_0)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

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