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Bifurcation for a class of piecewise cubic systems with two centers

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Abstract. In this paper, a class of symmetric cubic planar piecewise polynomial systems are presented, which have two symmetric centers corresponding to two period annuli. By perturbation and considering piecewise first order Melnikov function, we show the existence of 18 limit cycles (not small-amplitude limit cycles) with the configuration $(9, 9)$ bifurcating from the two period annuli and 22 small-amplitude limit cycles with the configuration $(11, 11)$, respectively.

Keywords: piecewise cubic system, limit cycle, piecewise first order Melnikov function.

2020 Mathematics Subject Classification: 34C07, 34C23, 37G15.

1 Introduction

During the past sixty years, many problems arising from mechanics, electrical engineering and automatic control are described by non-smooth systems in [1,2,9]. Piecewise systems, a class of non-smooth systems which have different definitions for the vector fields in different regions divided by lines or curves, have attracted much attention due to their complex dynamic phenomena and wide applications. Usually, a planar piecewise system with two zones has the form

$$(\dot{x}, \dot{y}) = \begin{cases} Z^+(x, y), & h(x, y) > 0, \\ Z^-(x, y), & h(x, y) < 0, \end{cases}$$

where $Z^{\pm}(x, y)$ are analytic functions in $\{(x, y) : \pm h(x, y) \geq 0\}$ respectively, and $h(x, y)$ is a continuous function.

Similar to the planar smooth systems, a natural and important topic in the qualitative theory of planar piecewise systems is to find the number and configuration of limit cycles. Moreover, the piecewise systems can exhibit more complex dynamic behaviors than the classical smooth systems. For instance, in contrast to non-existence of the limit cycle in

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planar linear systems, the piecewise linear ones can possess limit cycles, one is referred to [4,10,14,15,17,18,22] for details.

Below we give a brief introduction to the piecewise quadratic and cubic systems. For piecewise quadratic system, da Cruz *et al.* in [8] recently constructed a system with at least 16 limit cycles. For more stories, one can see [5,21] and references therein. For piecewise cubic systems, only a small amount of work has been done recently, see [11,13,16,19,23] for example. In [16] Llibre *et al.* obtained 12 limit cycles bifurcating from a period annulus of some cubic system under piecewise cubic perturbation. Later, a piecewise cubic system with 15 limit cycles was constructed by Li *et al.* [19]. Recently, Guo *et al.* considered in [13] a class of $\mathbf{Z}_2$-equivariant piecewise cubic systems with two centers at $(-1,0)$ and $(1,0)$, and showed by computing the Lyapunov quantities that there exist 18 small-amplitude limit cycles with configuration $(9,9)$. Here by configuration $(9,9)$, we mean that 9 limit cycles surround $(1,0)$ and the remaining ones surround $(-1,0)$, simultaneously. Yu *et al.* in [23] also obtained the existence of 18 small-amplitude limit cycles by computing the Lyapunov quantities in a planar piecewise cubic polynomial system. Note that the difference between them is that the authors in [23] obtained 18 limit cycles bifurcating from the two symmetric foci, and each of them will present 9 limit cycles. Of course, the calculations in [13,23] have high techniques, as they involves nonlinear equations. In [11], Gouveia and Torregrosa got, also by computing the Lyapunov quantities through the parallelization algorithm, 24 crossing small-amplitude limit cycles emerging from a piecewise cubic polynomial center at the cost of quite complicated computations. Very recently, an improvement in the number of crossing limit cycles in the cubic family is obtained by Gouveia and Torregrosa in [12], where the calculations are also based on the parallelization algorithm.

Motivated by [13,23] in which two symmetrical nests are considered simultaneously, this paper is devoted to investigating limit cycles bifurcating from piecewise cubic polynomial system with two symmetric centers, each of which corresponds to a period annulus full of closed orbits. In detail, we focus on the following piecewise cubic polynomial system

$$\begin{cases} H^+(x,y) = \Psi^+(x) + \Phi^+(y), & x>0, \\ H^-(x,y) = \Psi^-(x) + \Phi^-(y), & x<0, \end{cases} \tag{1.1}$$

where $\Psi^{\pm}(x)$ and $\Phi^{\pm}(y)$ are quartic polynomials such that $\Phi^{\pm}(y) = \Phi^{\pm}(-y)$. Concretely, system (1.1) has form

$$(\dot{x},\dot{y}) = \begin{cases} \left(a_1 y + a_2 y^3,\, a_3 + a_4 x + a_5 x^2 + a_6 x^3\right), & x>0, \\ \left(b_1 y + b_2 y^3,\ b_3 + b_4 x + b_5 x^2 + b_6 x^3\right), & x<0, \end{cases} \tag{1.2}$$

where a_i and b_i, $i = 1,\dots,6$, are real coefficients. Without loss of generality, assume that system (1.2) has two symmetric centers at $(0,\pm 1)$ which yield

$$a_1 = -a_2, \quad b_1 = -b_2, \quad a_3 = b_3 = 0, \quad a_1 a_4 > 0, \quad b_1 b_4 > 0.$$

Next, by the transformations $(x,y,t) \to \left(\sqrt{\frac{a_1}{2a_4}}x, y, \frac{1}{\sqrt{2a_1a_4}}t\right)$ and $(x,y,t) \to \left(\sqrt{\frac{b_1}{2b_4}}x, y, \frac{1}{\sqrt{2b_1b_4}}t\right)$ for $x>0$ and $x<0$, respectively, the system (1.2) becomes

$$(\dot{x},\dot{y}) = \begin{cases} \left(y(1-y^2),\, \dfrac{1}{2}x + \bar{a}x^2 + \bar{c}x^3\right), & x>0, \\ \left(y(1-y^2),\, \dfrac{1}{2}x + \bar{b}x^2 + \bar{d}x^3\right), & x<0, \end{cases} \tag{1.3}$$

where $\bar{a} = \frac{a_5}{2a_4}\sqrt{\frac{a_1}{2a_4}}$, $\bar{c} = \frac{a_1 a_6}{4a_4^2}$, $\bar{b} = \frac{b_5}{2b_4}\sqrt{\frac{b_1}{2b_4}}$, and $\bar{d} = \frac{b_1 b_6}{4b_4^2}$. Under piecewise cubic polynomial perturbation, we consider

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y(1-y^2) + \epsilon f^+(x,y),\ \dfrac{1}{2}x + \bar{a}x^2 + \bar{c}x^3 + \epsilon g^+(x,y) \right), & x > 0, \\ \left(y(1-y^2) + \epsilon f^-(x,y),\ \dfrac{1}{2}x + \bar{b}x^2 + \bar{d}x^3 + \epsilon g^-(x,y) \right), & x < 0, \end{cases} \tag{1.4}$$

where $f^+(x,y) = \sum_{i+j=0}^{3} a_{ij}x^i y^j$, $f^-(x,y) = \sum_{i+j=0}^{3} c_{ij}x^i y^j$, $g^+(x,y) = \sum_{i+j=0}^{3} b_{ij}x^i y^j$, $g^-(x,y) = \sum_{i+j=0}^{3} d_{ij}x^i y^j$.

To investigate the number of the limit cycles bifurcating from the two period annuli, we will apply the first order Melnikov function, also known as the Abelian integral, rather than the Lyapunov quantities to reduce the computation. One of our two main results is stated as follows.

Theorem 1.1. *For sufficiently small $|\epsilon| > 0$, there exists a system of the form (1.4) possessing at least 18 limit cycles with configuration $(9, 9)$.*

Note that 18 limit cycles in Theorem 1.1 obtained by the first order Melnikov function and Lemma 2.2 are no longer small-amplitude, which differs from the conclusions of 18 small-amplitude limit cycles in [13, 23].

The following result states the existence of 22 small-amplitude limit cycles (near the centers) with configuration $(11, 11)$, which improves the results in [13, 23]. Although 22 small-amplitude limit cycles is not as good as previous results in [11, 12], this is new and good from the point of view of simultaneity.

Theorem 1.2. *For sufficiently small $|\epsilon| > 0$, there exists a system of the form (1.4) possessing at least 22 small-amplitude limit cycles with configuration $(11, 11)$.*

The rest of this paper is organized as follows. In section 2, we will introduce the piecewise first order Melnikov function firstly. Meanwhile, some lemmas which will be applied to prove our main theorems are presented. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2 Preliminary results

In this section, we introduce the piecewise first order Melnikov function. For this we need the following result from the work of Liu and Han [20] in which the authors studied system

$$(\dot{x}, \dot{y}) = \begin{cases} (H_y^+(x,y) + \epsilon f^+(x,y), -H_x^+(x,y) + \epsilon g^+(x,y)) & x > 0, \\ (H_y^-(x,y) + \epsilon f^-(x,y), -H_x^-(x,y) + \epsilon g^-(x,y)) & x < 0, \end{cases} \tag{2.1}$$

where $f^{\pm}(x,y), g^{\pm}(x,y), H^{\pm}(x,y)$ are analytic functions and suppose the following two assumptions **H1** and **H2** hold.

H1. There exists an open interval (α, β), and two points $A(h) = (0, r(h)), C(h) = (0, \tilde{r}(h))$, where $r(h) \neq \tilde{r}(h)$. For $h \in (\alpha, \beta)$, we have $H^+(A(h)) = H^+(C(h)) = h$, $H^-(A(h)) = H^-(C(h))$.

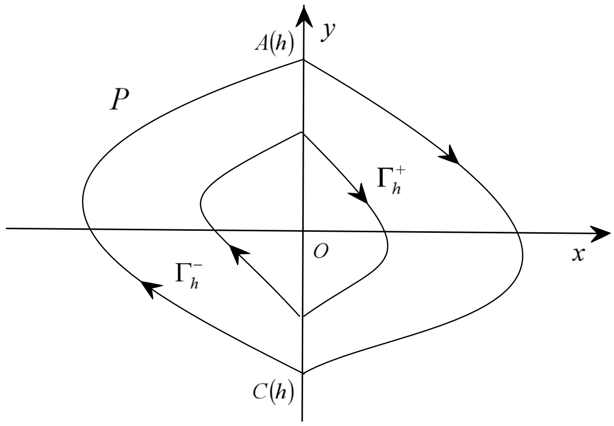

Figure 2.1: The graph shows the structure of **H1** and **H2**.

H2. When $x > 0$, system (2.1)$|_{\epsilon=0}$ has an orbital arc Γ_h^+ starting from $A(h)$ and ending at $C(h)$ defined by $H^+(x,y) = h$. When $x \leq 0$, system (2.1)$|_{\epsilon=0}$ has an orbital arc Γ_h^- starting from $C(h)$ and ending at $A(h)$ defined by $H^-(x,y) = H^-(A(h))$, as illustrated in Figure 2.1.

Then Liu and Han give the piecewise first order Melnikov function, also known as Abelian integral, for (2.1) in [20] as follows.

Lemma 2.1. *Under assumptions* **H1** *and* **H2***, for sufficiently small* $|\epsilon| > 0$*, then*

(1) *the Abelian integral of system* (2.1) *can be expressed as*

$$I(h) = \frac{H_y^+(A(h))}{H_y^-(A(h))}\left(\frac{H_y^-(C(h))}{H_y^+(C(h))}\int_{\Gamma_h^+} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y + \int_{\Gamma_h^-} g^-(x,y)\mathrm{d}x - f^-(x,y)\mathrm{d}y\right); \quad (2.2)$$

(2) *system* (2.1) *has a limit cycle near* Γ_{h^*}*, if* $I(h)$ *has a simple root* h^* $(I(h^*) = 0, I'(h^*) \neq 0)$*;*

(3) *system* (2.1) *has at least* k *limit cycles, if* $I(h)$ *has* k *roots.*

Applying Lemma 2.1 to consider the problem of limit cycles, a difficult and necessary work is to estimate the number of roots of (2.2). For this purpose, a well-known result, see Lemma 4.5 in [7], is presented as follows.

Lemma 2.2. *Consider* $p+1$ *linearly independent analytical functions* $f_i : U \subset R \to R, i = 0, 1, \ldots, p$*.*

(1) *Given* p *arbitrary values* $x_i \in U, i = 1, 2, \ldots, p$*, there exist* $p+1$ *constants* $C_i, i = 0, 1, \ldots, p$*, such that*

$$f(x) := \sum_{i=0}^{p} C_i f_i(x), \quad (2.3)$$

is not the zero function and $f(x_i) = 0$ *for* $i = 1, 2, \ldots, p$*.*

(2) *Furthermore, if there exists* $j \in \{0, 1, \ldots, p\}$ *such that* $f_j\,|_U$ *has constant sign, it is possible to get* $f(x)$ *in* (2.3) *such that it has at least* p *simple roots in* U*.*

If we consider small-amplitude limit cycles, the following lemma gives sufficient conditions, see Lemma 3.2 in [5] and Theorem 2.1 in [6].

Lemma 2.3. *Suppose* $c = (c_1, c_2, \dots, c_N)$, $I(h) = \sum_{i=0}^{\infty} A_i(c)h^i$ *where* $A_i(c^*) = 0, i = 0, 1, 2, \dots, N-1, A_N(c^*) \neq 0$, *and*

$$\operatorname{rank}\left(\frac{\partial(A_0(c), A_1(c), \dots, A_{N-1}(c))}{\partial(c_1, c_2, \dots, c_N)}\Big|_{c^*}\right) = N,$$

then there exists $(c_1, c_2, \dots, c_N)$ *such that* $I(h)$ *can have N simple real positive roots near* $h = 0$.

3 Proof of the main results

In this section, we will prove Theorem 1.1 and Theorem 1.2 by Lemma 2.2 and Lemma 2.3 respectively. For system $(1.4)_{\epsilon=0}$, there exists a first integral

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{1}{4}(y^2-1)^2 + \frac{1}{4}x^2(1 + \frac{4}{3}\bar{a}x + \bar{c}x^2), \\ H^-(x,y) = \frac{1}{4}(y^2-1)^2 + \frac{1}{4}x^2(1 + \frac{4}{3}\bar{b}x + \bar{d}x^2). \end{cases}$$

Define $\Gamma_{hi}^{\pm} = \{(x,y) : H^{\pm}(x,y) = h,\ 0 < h < \frac{1}{4}\}, i = 1, 2$, which form two annuli corresponding to two centers $(0,1)$ and $(0,-1)$, respectively (see Figure 3.1). More precisely, $\Gamma_{h1}^+ \cup \Gamma_{h1}^-$ and $\Gamma_{h2}^+ \cup \Gamma_{h2}^-$ are the closed orbits surrounding $(0,1)$ and $(0,-1)$, respectively.

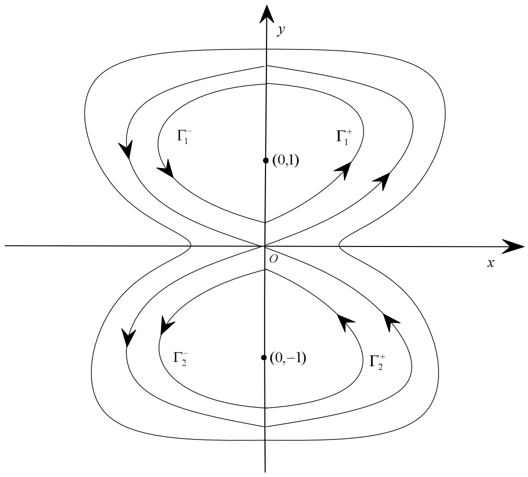

Figure 3.1: The phase graph of system $(1.4)_{\epsilon=0}$.

Since $H_y^+(x,y) = H_y^-(x,y)$, the Abelian integral (2.2) of the system (1.4) corresponding to the two annuli can be written as

$$I_i(h) = \int_{\Gamma_{hi}^+} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y + \int_{\Gamma_{hi}^-} g^-(x,y)\mathrm{d}x - f^-(x,y)\mathrm{d}y, \qquad i = 1, 2, \tag{3.1}$$

respectively. A direct computation for (3.1) yields the following result.

Lemma 3.1.

$$\begin{aligned} I_i(h) = &-(a_{00} - c_{00})\int_{\Gamma_{hi}^+}\mathrm{d}y - (a_{01} - c_{01})\int_{\Gamma_{hi}^+} y\mathrm{d}y - (a_{02} - c_{02})\int_{\Gamma_{hi}^+} y^2\mathrm{d}y - (a_{03} - c_{03})\int_{\Gamma_{hi}^+} y^3\mathrm{d}y \\ &- \int_{\Gamma_{hi}^+}\Big[(a_{10} + b_{01})x + (a_{11} + 2b_{02})xy + \left(a_{20} + \frac{1}{2}b_{11}\right)x^2 + \left(a_{30} + \frac{1}{3}b_{21}\right)x^3 \\ &+ (a_{21} + b_{12})x^2y + \left(a_{12} + 3b_{03}\right)xy^2\Big]\mathrm{d}y - \int_{\Gamma_{hi}^-}\Big[\left(c_{10} + d_{01}\right)x + (c_{11} + 2d_{02})xy \\ &+ \left(c_{20} + \frac{1}{2}d_{11}\right)x^2 + \left(c_{30} + \frac{1}{3}d_{21}\right)x^3 + (c_{21} + d_{12})x^2y + (c_{12} + 3d_{03})xy^2\Big]\mathrm{d}y, \quad i = 1, 2. \end{aligned}$$

In the proof of Lemma 3.1, we will impose

$$\int_{\Gamma_{hi}^{+}} \mathrm{d}y = -\int_{\Gamma_{hi}^{-}} \mathrm{d}y, \quad \int_{\Gamma_{hi}^{+}} x^m y^n \mathrm{d}x = -\int_{\Gamma_{hi}^{+}} \frac{n}{m+1} x^{m+1} y^{n-1} \mathrm{d}y.$$

Since the proof is direct, we omit it. Furthermore, if we take the following hypothesis

$$\text{(H)} \qquad a_{01}-c_{01}=a_{03}-c_{03}=a_{11}+2b_{02}=a_{21}+b_{12}=c_{11}+2d_{02}=c_{21}+d_{12}=0,$$

the Abelian integrals $I_1(h)$ and $I_2(h)$ will have the same expression defined as $I(h)$. Define

$$\begin{aligned} &J_1(h)=\int_{\Gamma_{h1}^{+}} dy, \quad J_2(h)=\int_{\Gamma_{h1}^{+}} y^2 dy, \quad J_3(h)=\int_{\Gamma_{h1}^{+}} x dy, \quad J_4(h)=\int_{\Gamma_{h1}^{+}} x^2 dy, \quad J_5(h)=\int_{\Gamma_{h1}^{+}} x^3 dy, \\ &J_6(h)=\int_{\Gamma_{h1}^{+}} xy^2 dy, \quad J_7(h)=\int_{\Gamma_{h1}^{-}} x dy, \quad J_8(h)=\int_{\Gamma_{h1}^{-}} x^2 dy, \quad J_9(h)=\int_{\Gamma_{h1}^{-}} x^3 dy, \quad J_{10}(h)=\int_{\Gamma_{h1}^{-}} xy^2 dy. \end{aligned}$$

Then we show the expression of $I(h)$ as the following result.

Lemma 3.2. *When the hypothesis* (H) *holds, then*

$$\begin{aligned} I(h) = & -(a_{00}-c_{00})J_1(h)-(a_{02}-c_{02})J_2(h)-(a_{10}+b_{01})J_3(h)-\left(a_{20}+\frac{1}{2}b_{11}\right)J_4(h) \\ & -\left(a_{30}+\frac{1}{3}b_{21}\right)J_5(h)-(a_{12}+3b_{03})J_6(h)-(c_{10}+d_{01})J_7(h)-\left(c_{20}+\frac{1}{2}d_{11}\right)J_8(h) \\ & -\left(c_{30}+\frac{1}{3}d_{21}\right)J_9(h)-(c_{12}+3d_{03})J_{10}(h). \end{aligned}$$

Proof. Using symmetry, we have

$$\int_{\Gamma_{h1}^{+}} x^i y^j \mathrm{d}y = (-1)^j \int_{\Gamma_{h2}^{+}} x^i y^j \mathrm{d}y, \qquad \int_{\Gamma_{h1}^{-}} x^i y^j \mathrm{d}y = (-1)^j \int_{\Gamma_{h2}^{-}} x^i y^j \mathrm{d}y,$$

where $0 \le i+j \le 3$. Combining Lemma 3.1 and hypothesis (H), we directly obtain that $I_1(h)$ and $I_2(h)$ have the same expression $I(h)$. □

Proof of Theorem 1.1. For simplicity, we may take $\bar{a}=\frac{3}{4}, \bar{b}=\frac{6}{4}$, and $\bar{c}=\bar{d}=1$, then

$$\Gamma_{h1}^{+} = \left\{(x,y) : \frac{1}{4}(y^2-1)^2 + \frac{1}{4}x^2(1+x+x^2) = h\right\},$$

and

$$\Gamma_{h1}^{-} = \left\{(x,y) : \frac{1}{4}(y^2-1)^2 + \frac{1}{4}x^2(1+2x+x^2) = h\right\},$$

where $0 < h < \frac{1}{4}$.

From the proof of Lemma 3.2, it is easy to check that the coefficients of $J_i(h), i=1,2,\ldots,10$, are arbitrary and $J_1(h) > 0$ for all $h \in (0, \frac{1}{4})$. By Lemma 2.2 and Theorem 2.1, we only need to prove that $J_i(h), i=1,2,\ldots,10$, are linearly independent functions.

Let $h = \frac{r^2}{4}$. When $0 < h \ll 1$, on Γ_{1h}^{+} we apply the transformations $x^2(1+x+x^2) = u^2$, $(y^2-1)^2 = v^2$ with $x>0, y>0$, where $u = r\cos\theta,\ v = r\sin\theta,\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. With the aid of algebra-system Maple [3], we obtain

$$\begin{aligned} x &= u - \frac{1}{2}u^2 + \frac{1}{8}u^3 + \frac{1}{2}u^4 - \frac{161}{128}u^5 + \frac{3}{2}u^6 + \frac{33}{1024}u^7 - \frac{9}{2}u^8 + \frac{350779}{32768}u^9 - \frac{23}{2}u^{10} + O(u^{11}), \\ y &= 1 + \frac{1}{2}v - \frac{1}{8}v^2 + \frac{1}{16}v^3 - \frac{5}{128}v^4 + \frac{7}{256}v^5 - \frac{21}{1024}v^6 + \frac{33}{2048}v^7 - \frac{429}{32768}v^8 + \frac{715}{65536}v^9 \\ &\quad - \frac{2431}{262144}v^{10} + O(v^{11}). \end{aligned}$$

Furthermore,

$$
\begin{aligned}
J_1(h) &= \int_{\Gamma_{h1}^+} \mathrm{d}y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta \frac{\mathrm{d}y}{\mathrm{d}v}\,|_{v=r\sin\theta}\,\mathrm{d}\theta = r + \frac{1}{8}r^3 + \frac{7}{128}r^5 + \frac{33}{1024}r^7 + \frac{715}{32768}r^9 + O(r^{11}),\\
J_2(h) &= r - \frac{1}{24}r^3 - \frac{1}{128}r^5 - \frac{3}{1024}r^7 - \frac{143}{98304}r^9 + O(r^{11}),\\
J_3(h) &= \frac{1}{4}\pi r^2 - \frac{1}{3}r^3 + \frac{3}{64}\pi r^4 + \frac{29}{120}r^5 - \frac{191}{1024}\pi r^6 + \frac{443}{640}r^7 - \frac{82949}{46080}r^9 + \frac{1382241}{1048576}\pi r^{10} + O(r^{11}),\\
J_4(h) &= \frac{2}{3}r^3 - \frac{3}{16}\pi r^4 + \frac{19}{60}r^5 + \frac{1}{8}\pi r^6 - \frac{601}{448}r^7 + \frac{4935}{8192}\pi r^8 - \frac{3023}{4608}r^9 - \frac{78477}{65536}\pi r^{10} + O(r^{11}),\\
J_5(h) &= \frac{3}{16}\pi r^4 - \frac{4}{5}r^5 + \frac{3}{16}\pi r^6 + \frac{29}{70}r^7 - \frac{5655}{8192}\pi r^8 + \frac{12317}{3360}r^9 - \frac{47691}{65536}\pi r^{10} + O(r^{11}),\\
J_6(h) &= \frac{1}{4}\pi r^2 - \frac{1}{3}r^3 + \frac{1}{64}\pi r^4 + \frac{11}{40}r^5 - \frac{203}{1024}\pi r^6 + \frac{9167}{13440}r^7 + \frac{115}{16384}\pi r^8 - \frac{592951}{322560}r^9\\
&\quad + \frac{1381233}{1048576}\pi r^{10} + O(r^{11}).
\end{aligned}
$$

On Γ_{h1}^-, let $x^2(1+2x+x^2) = u^2,\ (y^2-1)^2 = v^2$ with $x<0$ and $y>0$, where $u = r\cos\theta,\ v = r\sin\theta,\ \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Similarly, we have

$$
\begin{aligned}
x &= u - u^2 + 2u^3 - 5u^4 + 14u^5 - 42u^6 + 132u^7 - 429u^8 + 1430u^9 - 4862u^{10} + O(u^{11}),\\
y &= 1 + \frac{1}{2}v - \frac{1}{8}v^2 + \frac{1}{16}v^3 - \frac{5}{128}v^4 + \frac{7}{256}v^5 - \frac{21}{1024}v^6 + \frac{33}{2048}v^7 - \frac{429}{32768}v^8 + \frac{715}{65536}v^9\\
&\quad - \frac{2431}{262144}v^{10} + O(v^{11}).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
J_7(h) &= \frac{1}{4}\pi r^2 + \frac{2}{3}r^3 + \frac{51}{128}\pi r^4 + \frac{163}{60}r^5 + \frac{9091}{4096}\pi r^6 + \frac{43363}{2240}r^7 + \frac{4760595}{262144}\pi r^8 + \frac{28250723}{161280}r^9\\
&\quad + \frac{2963888949}{16777216}\pi r^{10} + O(r^{11}),\\
J_8(h) &= -\frac{2}{3}r^3 - \frac{3}{8}\pi r^4 - \frac{163}{60}r^5 - \frac{283}{128}\pi r^6 - \frac{43363}{2240}r^7 - \frac{297465}{16384}\pi r^8 - \frac{28250723}{161280}r^9\\
&\quad - \frac{46310061}{262144}\pi r^{10} + O(r^{11}),\\
J_9(h) &= \frac{3}{16}\pi r^4 + \frac{8}{5}r^5 + \frac{363}{256}\pi r^6 + \frac{451}{35}r^7 + \frac{405465}{32768}\pi r^8 + \frac{203683}{1680}r^9 + \frac{64826685}{524288}\pi r^{10} + O(r^{11}),\\
J_{10}(h) &= \frac{1}{4}\pi r^2 + \frac{2}{3}r^3 + \frac{47}{128}\pi r^4 + \frac{53}{20}r^5 + \frac{8923}{4096}\pi r^6 + \frac{128689}{6720}r^7 + \frac{4721575}{262144}\pi r^8 + \frac{28071031}{161280}r^9\\
&\quad + \frac{2948223621}{16777216}\pi r^{10} + O(r^{11}).
\end{aligned}
$$

Define $J_i(h) = \sum_{j=1}^{10} C_{i,j} r^j + O(r^{10})$, $i = 1, 2, \ldots, 10$, and $C = (C_{i,j})_{10\times 10}$, we obtain $\mathrm{rank}(C) = 10$ which means $J_i(h)$, $i = 1, 2, \ldots, 10$, are linearly independent functions.

The proof is completed. □

Next, we will apply Lemma 2.3 to prove that there exists a system of (1.4) with 22 small-amplitude limit cycles. Here we will take $\bar{a} = \frac{3}{4}a$, $\bar{b} = \frac{3}{4}b$, and $\bar{c} = \bar{d} = 1$ for simplicity.

Proof of Theorem 1.2. Let

$$
I(h) = \sum_{j=1}^{10} k_j J_j(h),
$$

where k_j, $j = 1, 2, \ldots, 10$, are arbitrary real constants.

By similar calculations used in Theorem 1.1, with the relation $h = \frac{r^2}{4}$, we get the Taylor expansions of $J_i(h)$, $i = 1, 2, \ldots, 10$, with 12th-order r which yield

$$I(h) = \sum_{i=0}^{12} F_i r^i + O(r^{13}) = \sum_{i=0}^{12} 2F_i h^{\frac{i}{2}} + O(h^{\frac{13}{2}}),$$

where

$$\begin{aligned}
F_0 &= 0,\\
F_1 &= k_1 + k_2,\\
F_2 &= \frac{1}{4}\pi k_3 + \frac{1}{4}\pi k_6 + \frac{1}{4}\pi k_7 + \frac{1}{4}\pi k_{10},\\
F_3 &= \frac{1}{8}k_1 - \frac{1}{24}k_2 - \frac{1}{3}ak_3 + \frac{2}{3}k_4 - \frac{1}{3}ak_6 + \frac{1}{3}bk_7 - \frac{2}{3}k_8 + \frac{1}{3}bk_{10},\\
F_4 &= \left(-\frac{9}{128}\pi + \frac{15}{128}a^2\pi\right)k_3 - \frac{3}{16}\pi ak_4 + \frac{3}{16}\pi k_5 + \left(-\frac{13}{128}\pi + \frac{15}{128}a^2\pi\right)k_6\\
&\quad + \left(-\frac{9}{128}\pi + \frac{15}{128}b^2\pi\right)k_7 - \frac{3}{16}\pi bk_8 + \frac{3}{16}\pi k_9 + \left(-\frac{13}{128}\pi + \frac{15}{128}b^2\pi\right)k_{10},\\
F_5 &= \frac{7}{128}k_1 - \frac{1}{128}k_2 + \left(-\frac{8}{15}a^3 + \frac{31}{40}a\right)k_3 + \left(-\frac{29}{60} + \frac{4}{5}a^2\right)k_4 - \frac{4}{5}ak_5 + \left(-\frac{8}{15}a^3 + \frac{97}{120}a\right)k_6\\
&\quad + \left(\frac{8}{15}b^3 - \frac{31}{40}b\right)k_7 + \left(-\frac{4}{5}b^2 + \frac{29}{60}\right)k_8 + \frac{4}{5}bk_9 + \left(\frac{8}{15}b^3 - \frac{97}{120}b\right)k_{10},\\
F_6 &= \left(\frac{571}{4096}\pi - \frac{1245}{2048}a^2\pi + \frac{1155}{4096}a^4\pi\right)k_3 + \left(-\frac{105}{256}\pi a^3 + \frac{137}{256}\pi a\right)k_4\\
&\quad + \left(-\frac{57}{256}\pi + \frac{105}{256}a^2\pi\right)k_5 + \left(\frac{563}{4096}\pi - \frac{1265}{2048}a^2\pi + \frac{1155}{4096}a^4\pi\right)k_6\\
&\quad + \left(\frac{571}{4096}\pi - \frac{1245}{2048}b^2\pi + \frac{1155}{4096}b^4\pi\right)k_7 + \left(-\frac{105}{256}\pi b^3 + \frac{137}{256}\pi b\right)k_8\\
&\quad + \left(-\frac{57}{256}\pi + \frac{105}{256}b^2\pi\right)k_9 + \left(\frac{563}{4096}\pi - \frac{1265}{2048}b^2\pi + \frac{1155}{4096}b^4\pi\right)k_{10},
\end{aligned}$$

and F_i, $i = 7, 8, 9, 10, 11, 12$, are polynomials of a, b, and k_j, $j = 1, 2, \ldots, 10$, which are omitted here because of the large scale.

Solving $F_i = 0$, $i = 1, \ldots, 9$, we obtain $k_1, k_3, k_2, k_4, k_5, k_6, k_7, k_8, k_9$ as follows.

$$\begin{aligned}
k_1 &= -k_2,\\
k_3 &= -k_6 - k_7 - k_{10},\\
k_2 &= 4k_4 + (2a + 2b)k_7 - 4k_8 + (2a + 2b)k_{10},\\
k_4 &= -\frac{1}{24a}\Big(-24k_5 + 4k_6 + (15a^2 - 15b^2)k_7 + 24bk_8 - 24k_9 + (15a^2 - 15b^2 + 4)k_{10}\Big),\\
k_5 &= \left(-\frac{3}{22}a^2 + \frac{1}{6}\right)k_6 + \left(\frac{1}{22}a^4 + \frac{15}{22}a^2b^2 + \frac{8}{11}b^3a - \frac{53}{88}a^2 - \frac{27}{22}ba - \frac{5}{8}b^2\right)k_7\\
&\quad + \left(-\frac{12}{11}a^2b - \frac{12}{11}b^2a + a + b\right)k_8 + \left(\frac{12}{11}a^2 + \frac{12}{11}ba - 1\right)k_9\\
&\quad + \left(\frac{1}{22}a^4 + \frac{15}{22}a^2b^2 + \frac{8}{11}b^3a - \frac{69}{88}a^2 - \frac{14}{11}ba - \frac{5}{8}b^2 + \frac{1}{6}\right)k_{10},
\end{aligned}$$

$$\begin{aligned}k_6 = {} & \frac{1}{16(9a^2-22)}\Big((103a^4+390a^2b^2-2048ab^3-2541b^4-768a^2+3456ab+4224b^2)k_7\\ & +(-624a^2b+3072ab^2+3696b^3-2816a-2816b)k_8+(624a^2-3072ab-3696b^2)k_9\\ & +(103a^4+390a^2b^2-2048ab^3-2541b^4-872a^2+3584ab+4312b^2+352)k_{10}\Big),\\ k_7 = {} & -\frac{N_7(a,b,k_8,k_9,k_{10})}{3\,M_7(a,b)},\qquad k_8=-\frac{N_8(a,b,k_9,k_{10})}{6\,M_8(a,b)},\qquad k_9=-\frac{k_{10}\,N_9(a,b)}{24\,(a+b)^2\,M(a,b)},\end{aligned}$$

where M_7, N_7, M_8, N_8, N_9, and M are polynomials of degree 7, 6, 10, 10, 14, and 10, respectively. Substituting the above results into F_{10}, F_{11}, and F_{12}, we have

$$F_{10}=-\frac{21\pi k_{10}}{1048576M}(a^2-b^2)P_{10}(a,b),\qquad F_{11}=\frac{k_{10}}{3465M}(a+b)P_{11}(a,b),$$
$$F_{12}=-\frac{21\pi k_{10}}{33554432M}(a^2-b^2)P_{12}(a,b),$$

where $P_{10}(a,b)$, $P_{11}(a,b)$, and $P_{12}(a,b)$ are polynomials of degree 12, 14, and 14 respectively.

Define

$$P=\det\left[\frac{\partial(F_1,F_2,F_3,F_4,F_5,F_6,F_7,F_8,F_9,F_{10},F_{11})}{\partial(k_1,k_3,k_2,k_4,k_5,k_6,k_7,k_8,k_9,a,b)}\right].$$

When $F_i=0$, $i=1,\ldots,9$, we take $k_1,k_3,k_2,k_4,k_5,k_6,k_7,k_8$, and k_9 into function P one by one. Then we obtain

$$P=\frac{\pi^5k_{10}^2(a+b)^6}{8192610672140353536\,00}\frac{\bar{P}(a,b)}{M},$$

where $\bar{P}(a,b)$ is a polynomial of degree 36.

Next, we prove the existence of a, b such that $P_{10}(a,b)=P_{11}(a,b)=0$ and $P_{12}(a,b)\cdot\bar{P}(a,b)\neq 0$ in three steps.

Firstly, we determine the common roots of $P_{10}(a,b)$ and $P_{11}(a,b)$. By the Maple built-in command 'RealRootIsolate' where the width of the interval is less than or equal $\frac{1}{2^{15}}$, we have

$$\begin{aligned}R_1\triangleq\{ & [[1.678741455,1.678771973],[-0.9492201089,-0.9492201089]],\\ & [[0.9492034912,0.9492263794],[-1.678745722,-1.678745722]],\\ & [[-0.9492263794,-0.9492034912],[1.678745722,1.678745722]],\\ & [[-1.678771973,-1.678741455],[0.9492201089,0.9492201089]],\\ & [[-1.901992798,-1.901976585],[-0.03034826851,-0.03034826851]],\\ & [[-2.100128174,-2.100101471],[-0.8025713876,-0.8025713876]],\\ & [[-2.040435791,-2.040405273],[-1.611435396,-1.611435396]],\\ & [[-0.03036117554,-0.03033447266],[-1.901976879,-1.901976879]],\\ & [[-1.611436844,-1.611415863],[-2.040434847,-2.040434847]],\\ & [[-0.8025817871,-0.8025512695],[-2.100106852,-2.100106852]],\\ & [[0.8025512695,0.8025817871],[2.100106852,2.100106852]],\\ & [[1.611415863,1.611436844],[2.040434847,2.040434847]],\\ & [[0.03033447266,0.03036117554],[1.901976879,1.901976879]],\\ & [[2.040405273,2.040435791],[1.611435396,1.611435396]],\\ & [[2.100101471,2.100128174],[0.8025713876,0.8025713876]],\\ & [[1.901976585,1.901992798],[0.03034826851,0.03034826851]]\},\end{aligned}$$

where the common roots are located.

Secondly, we estimate the common roots of $P_{10}(a,b)$, $P_{11}(a,b)$, and $P_{12}(a,b)$, and the common roots of $P_{10}(a,b)$, $P_{11}(a,b)$, and $\bar{P}(a,b)$. By Groebner Basis and the Maple built-in command 'Basis', we get

$$\mathrm{Basis}([P_{10}(a,b),P_{11}(a,b),P_{12}(a,b)],\mathrm{plex}(a,b))=\mathrm{Basis}([P_{10}(a,b),P_{11}(a,b),\bar{P}(a,b)],\mathrm{plex}(a,b)),$$

and two polynomials $P_1(b)$ and $P_2(a,b)$ with degrees 44 and 43, respectively, which mean the common roots of P_{10}, P_{11}, and P_{12} are the same as the common roots of P_{10}, P_{11}, and $\bar{P}(a,b)$ and they are determined by the common roots of $P_1(b)$ and $P_2(a,b)$. Furthermore, we find the intervals where the roots of $P_1(b)$ are located as follows

$$\begin{aligned}R_2 \triangleq &\{[-1.678746223,-1.678745270],[-0.9492206573,-0.9492197037],\\&[0.9492197037,0.9492206573],[1.678745270,1.678746223]\}.\end{aligned}$$

Thirdly, we take

$$(a^*,b^*)\in[[-1.901992798,-1.901976585],[-0.03034826851,-0.03034826851]],$$

which means $a^*\in[-1.901992798,-1.901976585]$, $b^*\in[-0.03034826851,-0.03034826851]$ with $b^*\notin R_2$. Then $P_{10}(a^*,b^*)=P_{11}(a^*,b^*)=0$ and $P_{12}(a^*,b^*)\cdot\bar{P}(a^*,b^*)\neq 0$. By the same method, we can prove (a^*,b^*) such that $M\neq 0$ and $M_i\neq 0, i=7,8$. Furthermore, these properties imply $F_{10}=F_{11}=0$ and $F_{12}\cdot P\neq 0$.

Finally, we can solve $k_9^*,\dots,k_1^*$ one by one, which combined with a^* and b^* imply $F_i=0$, $i=1,2,\dots,11$, and $F_{12}\cdot P\neq 0$. According to Lemma 2.3, $I(h)$ has 11 simple positive roots near $h=0$. By symmetry, the proof is completed. □

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