Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **1**, 1–25; https://doi.org/10.14232/ejqtde.2022.1.1 www.math.u-szeged.hu/ejqtde/

New regularity coefficients

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Received 15 August 2021, appeared 11 January 2022
Communicated by Mihály Pituk

Abstract. We give two new characterizations of the notion of Lyapunov regularity in terms of the lower and upper exponential growth rates of the singular values. These characterizations motivate the introduction of new regularity coefficients. In particular, we establish relations between these regularity coefficients and the Lyapunov regularity coefficient. Moreover, we construct explicitly bounded sequences of matrices attaining specific values of the new regularity coefficients.

Keywords: Lyapunov regularity, regularity coefficients.

2020 Mathematics Subject Classification: 37D99.

1 Introduction

The purpose of this work is twofold: to introduce new regularity coefficients and to give new characterizations of Lyapunov regularity. The notion of regularity was introduced by Lyapunov and plays an important role in the stability theory of differential equations and dynamical systems. It is particularly ubiquitous in the context of ergodic theory. The new characterizations of Lyapunov regularity are expressed in terms of the lower and upper exponential growth rates of the singular values.

1.1 The notion of regularity

We start by describing the meaning and some of the implications of Lyapunov regularity. Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices with real entries. We assume that both sequences A_m and A_m^{-1} are bounded. For each $m \in \mathbb{N}$ let

$$\mathcal{A}_m = \begin{cases} A_{m-1}A_{m-2}\cdots A_1 & \text{if } m > 1, \\ \mathrm{Id} & \text{if } m = 1. \end{cases}$$

Given a basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$, any regularity coefficient measures how much

$$\lambda(v_i) := \limsup_{m\to\infty} \frac{1}{m} \log \|\mathcal{A}_m v_i\|, \tag{1.1}$$

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for $i = 1, \dots, q$, differs from being a limit, and how much

$$\alpha_{ij} := \limsup_{m\to\infty} \frac{1}{m} \log \angle(\mathcal{A}_m v_i, \mathcal{A}_m v_j), \tag{1.2}$$

for $i \neq j$, differs from zero. In particular, the *Lyapunov regularity coefficient* determined by the sequence $A = (A_m)_{m\in\mathbb{N}}$ is defined by

$$\sigma(A) = \min \sum_{i=1}^{q} \lambda(v_i) - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|, \tag{1.3}$$

where the minimum is taken over all bases $v_1, \dots, v_q$ for $\mathbb{R}^q$. One can show that $\sigma(A) \geq 0$ and that $\sigma(A) = 0$ if and only if each lim sup in (1.1) is a limit and each lim sup in (1.2) vanishes for any basis $v_1, \dots, v_q$ (see [2,3]). The sequence A is said to be *(Lyapunov) regular* if $\sigma(A) = 0$.

More generally, a *regularity coefficient* is a nonnegative function on the sequences of matrices $A = (A_m)_{m\in\mathbb{N}}$ vanishing only on the Lyapunov regular systems. Besides the Lyapunov regularity coefficient (see [11]), other regularity coefficients were introduced already at an early stage of the theory by Perron (see [13,14]) and Grobman (see [6]), although for a dynamics with continuous time obtained from a linear ordinary differential equation. We refer the reader to the books [2,3,6,10] for detailed accounts of various parts of the theory, both for discrete and continuous time.

1.2 Origins and relevance of regularity

The notion of regularity first appeared in the works of Lyapunov (see [11]) and Perron [13,14], in connection with the study of the stability of solutions of perturbations of linear ordinary differential equations. As already described above, one can introduce a similar notion of regularity and corresponding regularity coefficients for a dynamics with discrete time

$$x_{m+1} = A_m x_m \quad \text{for } m \in \mathbb{N}$$

on $\mathbb{R}^q$, obtained from a sequence $(A_m)_{m\in\mathbb{N}}$ of $q \times q$ matrices. Some works that consider the case of discrete time include [15] (see also [4]) with a study of the relation of regularity with the exponential growth rates of the singular values, [5,9] with descriptions of relations between regularity coefficients, and [8] with the introduction of a new regularity coefficient. For further references we refer the reader to [2] (see also [7]).

It turns out that Lyapunov regularity has various nontrivial applications to the stability theory of differential equations and dynamical systems. The reason for this is that any regularity coefficient measures how much the exponential stability or conditional stability of a given trajectory of a linear dynamics differs from being uniform on the initial time. For example, provided that a regularity coefficient is sufficiently small, one can construct stable and unstable invariant manifolds for any sufficiently small nonlinear perturbation when all Lyapunov exponents are nonzero (see [3] for details). This is particularly effective in the context of smooth ergodic theory, since a certain integrability assumption guarantees that the linearizations along almost all trajectories have zero regularity coefficient, as a consequence of Oseledets' Multiplicative ergodic theorem [12].

1.3 Characterizations of regularity

Now we describe briefly our results. In particular, we give new characterizations of Lyapunov regularity that are expressed in terms of the lower and upper exponential growth rates of the singular values. This also serves as a preparation for introducing new regularity coefficients.

Again let $A = (A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices with real entries. We assume that A_m and A_m^{-1} are bounded in m. Now let

$$\rho_1(m) \le \cdots \le \rho_q(m)$$

be the eigenvalues of the positive-semidefinite matrix $(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$. The *lower and upper exponential growth rates of the singular values* are defined by

$$a_i = \liminf_{m\to\infty} \frac{1}{m}\log\rho_i(m) \quad \text{and} \quad b_i = \limsup_{m\to\infty}\frac{1}{m}\log\rho_i(m).$$

In particular, we obtain new characterizations of Lyapunov regularity in terms of these numbers (see Theorem 2.1).

Theorem 1.1. *The following properties are equivalent:*

1. $(A_m)_{m\in\mathbb{N}}$ *is regular;*
2. $\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q b_i$ *when* $m\to\infty$*;*
3. $\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q a_i$ *when* $m\to\infty$*.*

Some arguments in the proof are inspired by work of Barabanov in [1] who considered a corresponding problem for the case of continuous time.

Now we consider the values $\lambda_1' \le \cdots \le \lambda_q'$ of the Lyapunov exponent determined by the sequence A, counted with their multiplicities. If $v_1, \ldots, v_q$ is a basis for $\mathbb{R}^q$ at which the minimum in (1.3) is attained, then $\lambda_i' = \lambda(v_i)$ for $i = 1, \ldots, q$ up to a reordering of the values. One can show that the sequence A is regular if and only if

$$\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q \lambda_i' \quad \text{when } m\to\infty$$

(see [3]). Incidentally, we have $a_i \le b_i \le \lambda_i'$ for $i = 1, \ldots, q$ and each of these inequalities can be strict (see [2,4]).

As an outcome of our approach, we also obtain a new proof of a characterization of regularity involving only the lower and upper exponential growth rates of the singular values: namely, A is regular if and only if

$$a_i = b_i \quad \text{for } i = 1, \ldots, q. \tag{1.4}$$

It follows from work of Ruelle in [15] that condition (1.4) yields the regularity of A. Barabanov [1] gave a new proof of this property and also obtained the other direction of the equivalence for a dynamics with continuous time. We consider the case of discrete time and we give a new proof of this equivalence (see [4] for a proof based on the existence of a structure of Oseledets type that is present even for a nonregular dynamics). Again, some arguments are inspired in [1].

1.4 New regularity coefficients

Finally, we introduce three new regularity coefficients motivated by Theorem 1.1 (see also Theorem 2.1). Then we establish some relations between these coefficients and the Lyapunov regularity coefficient. Given a sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices with real entries, we define

$$\alpha(A) = \max\{b_i - a_i : i = 1, \dots, q\},$$
$$\underline{\sigma}(A) = \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|,$$
$$\overline{\sigma}(A) = \limsup_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| - \sum_{i=1}^{q} a_i.$$

The advantage of having various regularity coefficients is that in each specific situation it is often easier to compute or at least to estimate one of them. We show in Theorem 3.1 that

$$0 \le \alpha(A) \le \underline{\sigma}(A) \le q\alpha(A) \quad \text{and} \quad 0 \le \alpha(A) \le \overline{\sigma}(A) \le q\alpha(A).$$

Finally, we construct bounded sequences of matrices attaining specific values of the regularity coefficients (see Theorem 3.3). The construction builds on former work in [5] although it required several nontrivial modifications.

Theorem 1.2. *Given numbers $p, g \ge 0$ such that $p \le g \le qp$, there exists a bounded sequence A of diagonal $q \times q$ matrices with $\alpha(A) = p$ and $\underline{\sigma}(A) = g$.*

In Section 4 we introduce two additional regularity coefficients. We also establish inequalities between these coefficients and the former ones.

1.5 Relevance of the results

Finally, we discuss the relevance of the results obtained in the paper. As noted above, the notion of regularity plays an important role in the stability theory of a dynamics with continuous or discrete time. In fact, a vanishing or sufficiently small regularity coefficient implies that the asymptotic stability of a trajectory a linear dynamics persists under sufficiently small nonlinear perturbations. This leads in particular to the construction of stable and unstable invariant manifolds, as well as to many other nontrivial properties. On the other hand, in each specific situation it may be easier to obtain bounds for a certain regularity coefficient. Thus, it is convenient to have additional coefficients. In particular, it may be easier in some specific situations to use instead the new regularity coefficients introduced in our paper.

In another direction, when a dynamics is regular (when some regularity coefficient vanishes, in which case all regularity coefficients vanish), there is a richer structure, such as for example the one illustrated by (1.1) and (1.2). Our work provides further additional properties caused by regularity that in fact also provide additional structure.

2 Characterizations of regularity

In this section we give new characterizations of Lyapunov regularity that are expressed in terms of the lower and upper exponential growth rates of the singular values. This serves as a preparation for introducing new regularity coefficients in Section 3, although the characterizations are also of interest by themselves.

Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices with real entries. We shall always assume that there exists $c \in \mathbb{R}$ such that

$$\|A_m\| \le c \quad \text{and} \quad \|A_m^{-1}\| \le c \tag{2.1}$$

for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, let

$$\mathcal{A}_m = \begin{cases} A_{m-1}A_{m-2}\cdots A_1 & \text{if } m > 1, \\ \mathrm{Id} & \text{if } m = 1. \end{cases}$$

The Lyapunov exponent $\lambda\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(A_m)_{m\in\mathbb{N}}$ is defined by

$$\lambda(v) = \limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v\|,$$

with the convention that $\log 0 = -\infty$. By the abstract theory of Lyapunov exponents, λ takes at most a number $s \le q$ of distinct values on $\mathbb{R}^q \setminus \{0\}$, say

$$\lambda_1 < \lambda_2 < \cdots < \lambda_s.$$

Moreover, for each $i = 1, \ldots, s$ the set

$$E_i = \{v \in \mathbb{R}^q : \lambda(v) \le \lambda_i\}$$

is a linear subspace of $\mathbb{R}^q$ and

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_s = \mathbb{R}^q.$$

We denote by $\lambda'_1 \le \lambda'_2 \le \cdots \le \lambda'_q$ the values of the Lyapunov exponent λ counted with their multiplicities. These are obtained repeating each value λ_i a number of times equal to $\dim E_i - \dim E_{i-1}$, with the convention that $E_0 = \{0\}$. The sequence $(A_m)_{m\in\mathbb{N}}$ is said to be *(Lyapunov) regular* if

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lambda'_i \tag{2.2}$$

(this includes the requirement that the limit on the left-hand side exists).

We also consider the singular values. The matrix

$$T_m = (\mathcal{A}_m^* \mathcal{A}_m)^{1/2} \tag{2.3}$$

is symmetric and positive-semidefinite. Hence, its eigenvalues

$$\rho_1(m) \le \cdots \le \rho_q(m)$$

(counted with their multiplicities) are real and nonnegative. They are called the *singular values* of the matrix $\mathcal{A}_m$. For $i = 1, \ldots, q$, we define *the lower and upper exponential growth rates of the singular values,* respectively, by

$$a_i = \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \quad \text{and} \quad b_i = \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m).$$

We note that

$$a_i \le b_i \le \lambda'_i \quad \text{for } i = 1, \ldots, q \tag{2.4}$$

(see for example Proposition 6.1.2 in [2]).

The following result gives two new characterizations of Lyapunov regularity (properties (ii) and (iii)). As an outcome of our approach, we also obtain a new proof of a characterization involving only the lower and upper exponential growth rates of the singular values (property (iv)).

Theorem 2.1. *Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q\times q$ matrices with real entries satisfying (2.1). Then the following properties are equivalent:*

(i) $(A_m)_{m\in\mathbb{N}}$ *is regular;*

(ii) $\frac{1}{m}\log|\det \mathcal{A}_m| \to \sum_{i=1}^q b_i$ *when* $m\to\infty$*;*

(iii) $\frac{1}{m}\log|\det \mathcal{A}_m| \to \sum_{i=1}^q a_i$ *when* $m\to\infty$*;*

(iv) $a_i = b_i$ *for* $i=1,\dots,q$*.*

Proof. $(i)\Rightarrow(ii)$. Since $|\det \mathcal{A}_m| = \det T_m$ (see (2.3)), we obtain

$$\frac{1}{m}\log|\det \mathcal{A}_m| = \frac{1}{m}\log\prod_{i=1}^q \rho_i(m) = \sum_{i=1}^q \frac{1}{m}\log\rho_i(m) \tag{2.5}$$

and so

$$\lim_{m\to\infty}\frac{1}{m}\log|\det \mathcal{A}_m| = \lim_{m\to\infty}\sum_{i=1}^q \frac{1}{m}\log\rho_i(m) \le \sum_{i=1}^q b_i.$$

Therefore, it follows from (2.2) that

$$\sum_{i=1}^q \lambda_i' \le \sum_{i=1}^q b_i.$$

By (2.4) we have $\sum_{i=1}^q \lambda_i' = \sum_{i=1}^q b_i$ and property (ii) follows readily from (2.2).

$(ii)\Rightarrow(iv)$. We proceed by contradiction. Assume that $a_k < b_k$ for some $k\in\{1,\dots,q\}$ and take a sequence $(m_l)_{l\in\mathbb{N}}$ such that

$$a_k = \liminf_{m\to\infty}\frac{1}{m}\log\rho_k(m) = \lim_{l\to\infty}\frac{1}{m_l}\log\rho_k(m_l).$$

By (ii) and (2.5), since $a_k < b_k$ we obtain

$$\begin{aligned}\sum_{i=1}^q b_i &= \lim_{m\to\infty}\frac{1}{m}\log|\det \mathcal{A}_m| = \lim_{m\to\infty}\sum_{i=1}^q\frac{1}{m}\log\rho_i(m)\\ &= \lim_{l\to\infty}\sum_{i=1}^q\frac{1}{m_l}\log\rho_i(m_l) = a_k + \lim_{l\to\infty}\sum_{i\ne k}\frac{1}{m_l}\log\rho_i(m_l)\\ &\le a_k + \sum_{i\ne k} b_i < \sum_{i=1}^q b_i.\end{aligned}$$

This contradiction implies that (iv) holds.

Now we obtain the former implications with (ii) replaced by (iii).

$(i)\Rightarrow(iii)$. Let $B_m = (A_m^*)^{-1}$ and define

$$\mathcal{B}_m = \begin{cases} B_{m-1}B_{m-2}\cdots B_1 & \text{if } m>1,\\ \mathrm{Id} & \text{if } m=1.\end{cases}$$

Note that $\mathcal{B}_m = (\mathcal{A}_m^*)^{-1}$. The Lyapunov exponent $\mu\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(B_m)_{m\in\mathbb{N}}$ is defined by

$$\mu(w) = \limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{B}_m w\|.$$

By the abstract theory of Lyapunov exponents, μ takes at most a number q of distinct values on $\mathbb{R}^q \setminus \{0\}$ and we denote by $\mu_1' \geq \mu_2' \geq \cdots \geq \mu_q'$ these values counted with their multiplicities. One can show that $(A_m)_{m\in\mathbb{N}}$ is regular if and only if $(B_m)_{m\in\mathbb{N}}$ is regular, in which case we have $\lambda_i' = -\mu_i'$ for $i = 1, \ldots, q$ (see for example Theorem 2.4.5 in [2]). Therefore,

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{B}_m| = \sum_{i=1}^q \mu_i'.$$

Proceeding as in the proof of the implication $(i) \Rightarrow (ii)$, one can show that

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{B}_m| = \sum_{i=1}^q \beta_i, \tag{2.6}$$

where

$$\beta_i = \limsup_{m\to\infty} \frac{1}{m} \log \sigma_i(m),$$

denoting by

$$\sigma_1(m) \leq \cdots \leq \sigma_q(m)$$

the eigenvalues of the matrix $S_m = (\mathcal{B}_m^* \mathcal{B}_m)^{1/2}$ (which is symmetric and positive-semidefinite). Note that

$$\sigma_i(m) = \frac{1}{\rho_{q-i+1}(m)} \quad \text{for } i = 1, \ldots, q$$

and so

$$\beta_i = -a_{q-i+1} \quad \text{and} \quad \alpha_i = -b_{q-i+1} \quad \text{for } i = 1, \ldots, q, \tag{2.7}$$

where

$$\alpha_i = \liminf_{m\to\infty} \frac{1}{m} \log \sigma_i(m).$$

In view of (2.6) and (2.7) we get

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = -\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{B}_m| = -\sum_{i=1}^q \beta_i = \sum_{i=1}^q a_i,$$

which establishes property (iii).

$(iii) \Rightarrow (iv)$**.** The proof is identical to the proof of the implication $(ii) \Rightarrow (iv)$ using the identities in (2.7).

$(iv) \Rightarrow (i)$. We assume that the upper exponential growth rates of the singular values take r distinct values

$$c_1 < \cdots < c_r. \tag{2.8}$$

Let $n_1, \ldots, n_r$ be their multiplicities. Moreover, let $v_1(m), \ldots, v_q(m)$ be an orthonormal basis for $\mathbb{R}^q$ formed by eigenvectors of the matrix T_m associated, respectively, with the eigenvalues $\rho_1(m), \ldots, \rho_q(m)$. For $i = 1, \ldots, q$ and $m \in \mathbb{N}$, let

$$E_i(m) = \operatorname{span}\{v_{n_1+\cdots+n_{i-1}+1}(m), \ldots, v_{n_1+\cdots+n_i}(m)\}$$

and

$$F_i(m) = \operatorname{span}\{v_1(m), \ldots, v_{n_1+\cdots+n_i}(m)\}.$$

Since the basis $v_1(m), \ldots, v_q(m)$ is orthonormal, we have

$$F_i(m)^\perp = \operatorname{span}\{v_{n_1+\cdots+n_i+1}(m), \ldots, v_q(m)\}. \tag{2.9}$$

Considering the 2-norm for $\mathbb{R}^q$, we obtain

$$\|\mathcal{A}_m v\|^2 = v^* \mathcal{A}_m^* \mathcal{A}_m v = v^* T_m^2 v = v^* T_m^* T_m v = \|T_m v\|^2$$

and so

$$\|\mathcal{A}_m|_{F_i(m)}\| = \sup_{v \in F_i(m) \setminus \{0\}} \frac{\|\mathcal{A}_m v\|}{\|v\|} = \sup_{v \in F_i(m) \setminus \{0\}} \frac{\|T_m v\|}{\|v\|} = \rho_{l_i}(m), \tag{2.10}$$

where $l_i = n_1 + \cdots + n_i$. Moreover, for any subspace $L \subset F_i(m)^\perp$, it follows from (2.9) that

$$\rho_{l_{i+1}}(m) \le \|\mathcal{A}_m|_L\| \le \rho_q(m). \tag{2.11}$$

Properties (2.10) and (2.11) are crucial in the remainder of the proof.

Before proceeding, we recall some notions and results concerning the distance between two linear spaces. Let $\angle(v, w)$ be the angle between two vectors v and w. Given linear subspaces $E, F \subset \mathbb{R}^q$, we define

$$d(E, F) = \sin \angle(E, F),$$

where

$$\angle(E, F) = \max\{\theta(E, F), \theta(F, E)\}$$

and

$$\theta(E, F) = \max_{v \in E \setminus \{0\}} \min_{w \in F \setminus \{0\}} \angle(v, w).$$

Note that

$$\theta(E, F) = \max_{v \in E \setminus \{0\}} \angle(v, \operatorname{proj}_F v),$$

where $\operatorname{proj}_F v$ is the orthogonal projection of v onto F.

The following result is proved in [1].

Lemma 2.2. *The following properties hold:*

1. *If* $\dim E = \dim F$*, then* $\theta(E, F) = \theta(F, E)$ *and* $d(E, F) = d(E^\perp, F^\perp)$.
2. *If a sequence* $(E_k)_{k \in \mathbb{N}}$ *of linear spaces of equal dimensions is a Cauchy sequence, then it converges to a linear space* E *of the same dimension.*

3. *Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of linear spaces converging to a linear space E such that $E_k = F_k \oplus G_k$, where G_k is the orthogonal complement of F_k in E_k. If the sequence $(F_k)_{k\in\mathbb{N}}$ converges to a linear space F, then $F \subset E$ and the sequence $(G_k)_{k\in\mathbb{N}}$ converges to a linear space G that is the orthogonal complement of F in E.*

Now let

$$\alpha_i(m) = \angle(F_i(m), F_i(m+1)) \quad \text{for } i = 1,\dots,r-1.$$

Lemma 2.3. *We have*

$$\limsup_{m\to\infty} \frac{1}{m} \log \alpha_i(m) \le c_i - c_{i+1} \quad \textit{for } i = 1,\dots,r-1.$$

Proof of the lemma. We proceed by contradiction. Assume the contrary. Then for some $i \in \{1,\dots,r-1\}$ there exist $\varepsilon > 0$ and a sequence $(m_l)_{l\in\mathbb{N}}$ such that

$$\frac{1}{m_l} \log \alpha_i(m_l) > c_i - c_{i+1} + \varepsilon \tag{2.12}$$

for $l \in \mathbb{N}$. Take $v \in F_i(m+1)$ with $\|v\| = 1$ such that

$$\angle(v, F_i(m)) = \alpha_i(m)$$

and write it in the form $v = v_1 + v_2$ with $v_1 \in F_i(m)$ and $v_2 \in F_i(m)^\perp$. By (2.12) we have $v_2 \neq 0$. It follows from (2.10) that

$$\|\mathcal{A}_m v_1\| \le \rho_{l_i}(m)\|v_1\|$$

and taking $L = \operatorname{span}\{v_2\}$ in (2.11) we obtain

$$\|\mathcal{A}_m v_2\| \ge \rho_{l_{i+1}}(m)\|v_2\|.$$

On the other hand, we have

$$\|v_1\| = \|v\| \cos \alpha_i(m) \le \|v\|$$

and

$$\|v_2\| = \|v\| \sin \alpha_i(m) \ge \frac{2}{\pi}\alpha_i(m)\|v\|$$

because $\sin x \ge \frac{2}{\pi}x$ for $x \in [0, \pi/2]$. Therefore,

$$\begin{aligned}
\|A_m v\| &\ge \|\mathcal{A}_m v_2\| - \|\mathcal{A}_m v_1\| \\
&\ge \rho_{l_{i+1}}(m)\|v_2\| - \rho_{l_i}(m)\|v_1\| \\
&\ge \left(\rho_{l_{i+1}}(m)\frac{2}{\pi}\alpha_{l_i}(m) - \rho_{l_i}(m)\right)\|v\|.
\end{aligned}$$

Note that given $\delta > 0$, by (2.8) there exists $m = m(\delta)$ such that

$$e^{(a_j-\delta)m} \le \rho_j(m) \le e^{(a_j+\delta)m}$$

for all $j = 1,\dots,q$ and $m \ge m(\delta)$. Hence, for $m = m_l$ we obtain

$$\begin{aligned}
\|\mathcal{A}_{m_l} v\| &\ge \left(e^{(c_{i+1}-\delta)m_l}\frac{2}{\pi}e^{(c_i-c_{i+1}+\varepsilon)m_l} - e^{(c_i+\delta)m_l}\right)\|v\| \\
&= \left(\frac{2}{\pi}e^{(c_i-\delta+\varepsilon)m_l} - e^{(c_i+\delta)m_l}\right)\|v\|.
\end{aligned} \tag{2.13}$$

Taking $\delta < \varepsilon/2$, we have $c_i - \delta + \varepsilon > c_i + \delta$. Since $v \in F_i(m_l + 1)$ and $\|A_m^{-1}\| \le c$, it follows from (2.13) that

$$c_i + \delta < \limsup_{l\to\infty} \frac{1}{m_l} \log\|\mathcal{A}_{m_l} v\| \le \lim_{l\to\infty} \frac{1}{m_l+1} \log \rho_{l_i}(m_l+1) = c_i,$$

which is impossible. This contradiction yields the desired result. $\square$

We proceed with the proof of the theorem. We first show that $(F_i(m))_{m\in\mathbb{N}}$ is a Cauchy sequence. By Lemma 2.3, taking $\varepsilon > 0$ such that $c_i - c_{i+1} + \varepsilon < 0$ we have

$$\begin{aligned} d(F_i(m), F_i(k)) &\le \sum_{j=m}^{k-1} d(F_i(j), F_i(j+1)) \\ &\le \sum_{j=m}^{\infty} d(F_i(j), F_i(j+1)) = \sum_{j=m}^{\infty} \sin\alpha_i(j) \\ &\le \sum_{j=m}^{\infty} e^{(c_i - c_{i+1}+\varepsilon)j} = \frac{e^{(c_i-c_{i+1}+\varepsilon)m}}{1-e^{c_i-c_{i+1}+\varepsilon}} \end{aligned}$$

for all sufficiently large m and all $k > m$. This shows that $(F_i(m))_{m\in\mathbb{N}}$ is a Cauchy sequence. In view of Lemma 2.2 (second item), we conclude that $(F_i(m))_{m\in\mathbb{N}}$ converges to some linear space F satisfying

$$d(F_i(m), F_i) \le \frac{e^{(c_i-c_{i+1}+\varepsilon)m}}{1-e^{c_i-c_{i+1}+\varepsilon}}. \tag{2.14}$$

Moreover, also in view of Lemma 2.2 (third item), the sequence $(E_i(m))_{m\in\mathbb{N}}$ also converges to some linear space E. Indeed, since $F_{i+1}(m) \to F_{i+1}$ when $m \to \infty$ and

$$F_{i+1}(m) = F_i(m) \oplus E_{i+1}(m)$$

with

$$E_{i+1}(m) = F_i(m)^\perp \cap F_{i+1}(m),$$

we conclude that $(E_i(m))_{m\in\mathbb{N}}$ converges to some linear space E_i.

Lemma 2.4. *For $k = 1, \dots, r$ we have*

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| = c_k \quad \text{for } w \in E_k \setminus \{0\}$$

and

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{k-1} \quad \text{for } w \in F_{k-1} \setminus \{0\}.$$

Proof of the lemma. We proceed by backwards induction on k. Take $j \in \{0, 1, \dots, r-2\}$ and given $w \in \mathbb{R}^q$, write it in the form $w = w_1 + w_2$ with $w_1 \in F_{r-j-1}(m)^\perp$ and $w_2 \in F_{r-j-1}(m)$.

First take $j = 0$ and $w \in E_r \setminus \{0\}$. Then $w = w_1 + w_2$ with $w_1 \in F_{r-1}(m)^\perp$ and $w_2 \in F_{r-1}(m)$. Note that $w_1 \ne 0$ for any sufficiently large m, since $\angle(E_r, E_r(m)) \to 0$ when $m \to \infty$ and $E_r(m) = F_{r-1}(m)^\perp$. In view of (2.11) we have

$$\rho_{q-n_r+1}(m)\|w_1\| \le \|\mathcal{A}_m w_1\| \le \rho_q(m)\|w_1\|,$$

which implies that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w_1\| = c_r.$$

Moreover, in view of (2.10) we also have

$$\|\mathcal{A}_m w_2\| \le \rho_{q-r_n}(m)\|w_2\|,$$

which implies that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w_2\| \le c_{r-1}.$$

Since $\mathcal{A}_m w = \mathcal{A}_m w_1 + \mathcal{A}_m w_2$ and $c_{r-1} < c_r$, we conclude that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| = c_r.$$

Now take $w \in F_{r-1} \setminus \{0\}$ and recall that $w = w_1 + w_2$ with $w_1 \in F_{r-1}(m)^\perp$ and $w_2 \in F_{r-1}(m)$. Since $\angle(F_{r-1}, F_{r-1}(m)) \to 0$ when $m \to \infty$, we have $w_2 \neq 0$ for any sufficiently large m. Note that

$$\|w_1\| = \|w\| \sin\angle(w, w_2)$$

and

$$\|w_2\| = \|w\| \cos\angle(w, w_2) \le \|w\|.$$

Since $\sin\angle(w, w_2) \le d(F_{r-1}, F_{r-1}(m))$, it follows from (2.14) that

$$\begin{aligned}
\|\mathcal{A}_m w\| &\le \|\mathcal{A}_m w_1\| + \|\mathcal{A}_m w_2\| \\
&\le \rho_q(m)\|w_1\| + \rho_{q-n_r}(m)\|w_2\| \\
&\le \frac{e^{(c_{r-1}-c_r+\varepsilon)m}}{1-e^{c_{r-1}-c_r+\varepsilon}} \rho_q(m)\|w\| + \rho_{q-n_r}(m)\|w\|.
\end{aligned}$$

We have

$$\lim_{m\to\infty} \frac{1}{m} \log\left(\frac{e^{(c_{r-1}-c_r+\varepsilon)m}}{1-e^{c_{r-1}-c_r+\varepsilon}} \rho_q(m)\right) = c_{r-1} - c_r + \varepsilon + c_r = c_{r-1} + \varepsilon$$

and

$$\limsup_{m\to\infty} \frac{1}{m} \log \rho_{q-n_r}(m) = c_{r-1}.$$

Therefore,

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-1} + \varepsilon$$

and since ε is arbitrary, we obtain

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-1}.$$

This establishes the induction hypothesis for $j = 0$.

Now assume that the statement in Lemma 2.4 holds for $k = r, \dots, r-j+1$ and some $j \ge 1$. We want to show that it also holds for $k = r - j$. Take $w \in E_{r-j} \setminus \{0\}$. Since $E_{r-j} \setminus \{0\} \subset F_{r-j} \setminus \{0\}$, it follows from the induction hypothesis that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-j}. \tag{2.15}$$

We first show that

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge c_{r-j} \tag{2.16}$$

for $w \in E_{r-j} \setminus \{0\}$. Then it follows from (2.15) and (2.16) that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| = c_{r-j},$$

which establishes the first statement in the lemma.

Since $E_{r-j} \setminus \{0\} \subset F_{r-j-1}^{\perp} \setminus \{0\}$, we take $w \in F_{r-j-1}^{\perp} \setminus \{0\}$ and write it in the form $w = w_1 + w_2$ with $w_1 \in F_{r-j-1}(m)^{\perp}$ and $w_2 \in F_{r-j-1}(m)$. Since

$$d(F_{r-j-1}^{\perp}, F_{r-j-1}(m)^{\perp}) \to 0 \quad \text{when } m \to \infty,$$

we have $w_1 \neq 0$ for any sufficiently large m. Moreover,

$$\|w_1\| = \|w\| \cos\angle(w, w_1), \quad \|w_2\| = \|w\| \sin\angle(w, w_2). \tag{2.17}$$

In view of (2.14) we have

$$\begin{aligned} \sin\angle(w, w_1) &\le d(F_{r-j-1}^{\perp}, F_{r-j-1}(m)^{\perp}) \\ &= d(F_{r-j-1}, F_{r-j-1}(m)) \\ &\le \frac{e^{(c_{r-j-1}-c_{r-j}+\varepsilon)m}}{1 - e^{c_{r-j-1}-c_{r-j}+\varepsilon}} =: \alpha_j(m). \end{aligned} \tag{2.18}$$

Hence, by (2.10) and (2.11) together with (2.17) and (2.18), we obtain

$$\begin{aligned} \|\mathcal{A}_m w\| &\ge \|\mathcal{A}_m w_1\| - \|\mathcal{A}_w w_2\| \\ &\ge \rho_{l_{r-j}}(m)\|w_1\| - \rho_{l_{r-j-1}}(m)\|w_2\| \\ &\ge \rho_{l_{r-j}}(m)\sqrt{1-\alpha_j(m)^2}\|w\| - \rho_{l_{r-j-1}}(m)\|w\|. \end{aligned}$$

Therefore, since $\alpha_j(m) \to 0$ when $m \to \infty$, we have

$$\lim_{m\to\infty} \frac{1}{m} \log\Big(\rho_{l_{r-j}}(m)\sqrt{1-\alpha_j(m)^2}\Big) = c_{r-j}$$

and

$$\lim_{m\to\infty} \frac{1}{m} \log \rho_{l_{r-j-1}}(m) = c_{r-j-1}.$$

Finally, since $c_{r-j} > c_{r-j-1}$, we conclude that

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \ge c_{r-j},$$

which establishes (2.16).

Now we prove the second statement in the lemma. Take $w \in F_{r-j-1} \setminus \{0\}$ and write it in the form $w = w_1 + w_2$ with $w_1 \in F_{r-j-1}^{\perp}(m)$ and $w_2 \in F_{r-j-1}(m)$. Since

$$\angle(F_{r-j-1}, F_{r-j-1}(m)) \to 0 \quad \text{when } m \to \infty,$$

we have $w_2 \neq 0$ for any sufficiently large m. Note that

$$\|w_1\| = \|w\| \sin\angle(w, w_2)$$

and

$$\|w_2\| = \|w\| \cos\angle(w, w_2) \le \|w\|.$$

Since $\sin\angle(w, w_2) \le d(F_{r-j-1}, F_{r-j-1}(m))$ and

$$d(F_{r-j-1}, F_{r-j-1}(m)) \le \alpha_j(m),$$

using (2.10) and (2.11), we conclude as above that

$$\begin{aligned}\|\mathcal{A}_m w\| &\le \|\mathcal{A}_m w_1\| + \|\mathcal{A}_m w_2\| \\ &\le \alpha_j(m)\rho_{l_{r-j}}(m)\|w\| + \rho_{l_{r-j-1}}(m)\|w\|.\end{aligned}$$

We have

$$\lim_{m\to\infty} \frac{1}{m} \log\big(\alpha_j(m)\rho_{l_{r-j}}(m)\big) = c_{r-j-1} - c_{r-j} + \varepsilon + c_{r-j} = c_{r-j-1} + \varepsilon$$

and

$$\lim_{m\to\infty} \frac{1}{m} \log \rho_{l_{r-j-1}}(m) = c_{r-j-1}.$$

Therefore,

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-j-1} + \varepsilon$$

and since ε is arbitrary, we obtain

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-j-1}.$$

This establishes the induction hypothesis for $k = r - j$. □

We proceed with the proof of the theorem. Note that

$$\det T_m = \prod_{i=1}^{q} \rho_i(m).$$

Therefore, using (2.5) we have

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lim_{m\to\infty} \frac{1}{m} \log \rho_i(m).$$

Finally, in view of Lemma 2.4 we obtain

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lambda_i',$$

which shows that $(iv) \Rightarrow (i)$. This completes the proof of the theorem. □

3 New regularity coefficients

In this section we introduce three new regularity coefficients motivated by the properties (ii), (iii) and (iv) in Theorem 2.1. We also establish some relations between these coefficients and the Lyapunov regularity coefficient.

3.1 Regularity coefficients

Given a sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices with real entries, we define

$$\begin{aligned}
\alpha(A) &= \max\{b_i - a_i : i = 1, \dots, q\},\\
\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|,\\
\overline{\sigma}(A) &= \limsup_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| - \sum_{i=1}^{q} a_i.
\end{aligned}$$

The following result gives some relations between these functions. In particular, together with Theorem 2.1, it shows that the three are indeed regularity coefficients.

Theorem 3.1. *For any bounded sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices, we have*

$$0 \le \alpha(A) \le \underline{\sigma}(A) \le q\alpha(A) \tag{3.1}$$

and

$$0 \le \alpha(A) \le \overline{\sigma}(A) \le q\alpha(A). \tag{3.2}$$

Proof. Clearly, $\alpha(A) \ge 0$. Note that

$$\begin{aligned}
\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m)\\
&\ge \sum_{i=1}^{q} b_i - \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i\ne j} \rho_i(m) - \liminf_{m\to\infty} \frac{1}{m} \log \rho_j(m)\\
&\ge \sum_{i=1}^{q} b_i - \sum_{i\ne j} b_i - a_j = b_j - a_j
\end{aligned}$$

and so $\underline{\sigma}(A) \ge \alpha(A)$. Moreover,

$$\begin{aligned}
\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m)\\
&\le \sum_{i=1}^{q} b_i - \sum_{i=1}^{q} a_i = \sum_{i=1}^{q} (b_i - a_i) \le q\alpha(A),
\end{aligned}$$

which establishes (3.1).

On the other hand, we have

$$\begin{aligned}
\overline{\sigma}(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m) - \sum_{i=1}^{q} a_i\\
&\ge \limsup_{m\to\infty} \frac{1}{m} \log \rho_j(m) + \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i\ne j} \rho_i(m) - \sum_{i=1}^{q} a_i\\
&\ge b_j + \sum_{i\ne j} a_i - \sum_{i=1}^{q} a_i = b_j - a_j
\end{aligned}$$

and so $\overline{\sigma}(A) \geq \alpha(A)$. Finally,

$$\begin{aligned}\overline{\sigma}(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m) - \sum_{i=1}^{q} a_i \\ &\leq \sum_{i=1}^{q} b_i - \sum_{i=1}^{q} a_i = \sum_{i=1}^{q} (b_i - a_i) \leq q\alpha(A),\end{aligned}$$

which establishes (3.2). ☐

It follows readily from Theorem 3.1 that

$$q^{-1}\overline{\sigma}(A) \leq \underline{\sigma}(A) \leq q\overline{\sigma}(A)$$

and

$$q^{-1}\underline{\sigma}(A) \leq \overline{\sigma}(A) \leq q\underline{\sigma}(A).$$

We also establish some relations between these three coefficients and the Lyapunov regularity coefficient. We recall that the *Lyapunov regularity coefficient* of a sequence $A = (A_m)_{m\in\mathbb{N}}$ is defined by

$$\sigma(A) = \sum_{i=1}^{q} \lambda_i' - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|.$$

Theorem 3.2. *For any bounded sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices, we have*

$$\underline{\sigma}(A) \leq \sigma(A) \quad \text{and} \quad \sigma(A) \leq q^2\alpha(A).$$

Proof. The first inequality follows readily from the fact that $b_i \leq \lambda_i'$ for $i = 1, \ldots, q$ and the definitions of $\underline{\sigma}(A)$ and $\sigma(A)$.

For the second identity, we first observe that it suffices to consider upper-triangular matrices. Indeed, given a sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices, there exists a sequence $(U_m)_{m\in\mathbb{N}}$ of orthogonal $q \times q$ matrices with $U_1 = \mathrm{Id}$ such that

$$C_m = U_{m+1}^* A_m U_m$$

is upper-triangular for each $m \in \mathbb{N}$ (see Theorem 3.2.1 in [2]). Clearly, the sequence $C = (C_m)_{m\in\mathbb{N}}$ is also bounded and one can easily verify that

$$\alpha(C) = \alpha(A), \quad \underline{\sigma}(C) = \underline{\sigma}(A), \quad \overline{\sigma}(C) = \overline{\sigma}(A) \quad \text{and} \quad \sigma(C) = \sigma(A).$$

Without loss of generality, we assume from now on that all matrices are upper-triangular. We also consider the *Grobman regularity coefficient* $\gamma(A)$ that is defined by

$$\gamma(A) = \min \max\{\lambda(v_i) + \mu(w_i) : 1 \leq i \leq q\},$$

where the minimum is taken over all dual bases $v_1, \ldots, v_q$ and $w_1, \ldots, w_q$. Denoting by $a_{ij}(l)$ the entries of A_l, we have

$$\gamma(A) \leq \sum_{i=1}^{q} \left(\limsup_{m\to\infty} \frac{1}{m} \log \prod_{l=1}^{m} |a_{ii}(l)| - \liminf_{m\to\infty} \frac{1}{m} \log \prod_{l=1}^{m} |a_{ii}(l)| \right)$$

(see Theorem 3.1.3 in [2]). Since the matrices A_l are upper-triangular, we obtain

$$\frac{1}{m} \log \prod_{l=1}^{m} |a_{ii}(l)| = \frac{1}{m} \log \rho_{k_i}(m)$$

for some integer k_i and so

$$\begin{aligned}\gamma(A) &\leq \sum_{i=1}^{q}\left(\limsup_{m\to\infty}\frac{1}{m}\log\rho_{k_i}(m)-\liminf_{m\to\infty}\frac{1}{m}\log\rho_{k_i}(m)\right)\\&=\sum_{i=1}^{q}(b_{k_i}-a_{k_i})\leq q\alpha(A).\end{aligned}$$

Moreover, we have

$$\frac{\sigma(A)}{q}\leq\gamma(A)$$

(see Theorem 7.3.2 in [2]) and so $\sigma(A)\leq q^2\alpha(A)$. This completes the proof of the theorem. □

3.2 Realization problem I

In this section we construct bounded sequences of matrices $A=(A_m)_{m\in\mathbb{N}}$ attaining specific values of the regularity coefficients $\alpha(A)$ and $\underline{\sigma}(A)$.

Theorem 3.3. *Given numbers $p,g\geq 0$ such that*

$$p\leq g\leq qp,$$

there exists a bounded sequence $A=(A_m)_{m\in\mathbb{N}}$ of diagonal $q\times q$ matrices with $\alpha(A)=p$ and $\underline{\sigma}(A)=g$.

Proof. Note that $\alpha(A)=\underline{\sigma}(A)=0$ for any regular sequence A. So it suffices to take $p>0$. We divide the proof into steps.

Step 1. Construction of sequences of numbers. Given $r,c,d\in\mathbb{R}$ with $r>1$ and $c\geq d$, for each $m\in\mathbb{N}$ let

$$a(m)=\begin{cases}e^d & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^c & \text{if } m\in T_k \text{ for } k\in\mathbb{N},\end{cases}$$

where

$$S_k=\{m\in\mathbb{N}: r^{2k-2}\leq m<r^{2k-1}\}$$

and

$$T_k=\{m\in\mathbb{N}: r^{2k-1}\leq m<r^{2k}\}.$$

The following result is taken from [5].

Lemma 3.4. *For $\rho(m)=\prod_{j=1}^{m-1}a(j)$ we have*

$$\rho(m)=\begin{cases}e^{\frac{d+cr}{r+1}(r^{2k-2}-1)+d(m-r^{2k-2})} & \textit{if } m\in S_k \textit{ for } k\in\mathbb{N},\\ e^{\frac{d+cr}{r+1}(r^{2k-2}-1)+d(r^{2k-1}-r^{2k-2})+c(m-r^{2k-1})} & \textit{if } m\in T_k \textit{ for } k\in\mathbb{N}.\end{cases}$$

Moreover,

$$\limsup_{m\to\infty}\frac{1}{m}\log\rho(m)=\frac{d+cr}{r+1}$$

and

$$\liminf_{m\to\infty}\frac{1}{m}\log\rho(m)=\frac{c+dr}{r+1}.$$

Step 2. Construction of sequences of matrices. We say that the sequence of matrices $(A_m)_{m\in\mathbb{N}}$, where $A_m = \operatorname{diag}(a_1(m),\dots,a_q(m))$ for $m\in\mathbb{N}$ with

$$a_i(m) = \begin{cases} e^{d_i} & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^{c_i} & \text{if } m\in T_k \text{ for } k\in\mathbb{N}, \end{cases} \tag{3.3}$$

is *r-regular* if the following conditions hold:

1. $c_i \ge d_i$ for $i=1,\dots,q$;
2. $c_i \le c_{i+1}$ and $d_i\le d_{i+1}$ for $i=1,\dots,q-1$.

Lemma 3.5. *For any r-regular sequence $A=(A_m)_{m\in\mathbb{N}}$, we have*

$$\alpha(A) = \frac{r-1}{r+1}\max_{1\le i\le q}(c_i-d_i)$$

and

$$\underline{\sigma}(A) = \frac{r-1}{r+1}\sum_{i=1}^q (c_i-d_i).$$

Proof of the lemma. It follows from Lemma 3.4 that

$$\rho_i(m) = \begin{cases} e^{\frac{d_i+c_ir}{r+1}(r^{2k-2}-1)+d_i(m-r^{2k-2})} & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^{\frac{d_i+c_ir}{r+1}(r^{2k-2}-1)+d_i(r^{2k-1}-r^{2k-2})+c_i(m-r^{2k-1})} & \text{if } m\in T_k \text{ for } k\in\mathbb{N}, \end{cases}$$

for $i=1,\dots,q$. Indeed, if $m\in S_k$, then

$$\begin{aligned}\rho_i(m) &= e^{\frac{d_i+c_ir}{r+1}(r^{2k-2}-1)+d_i(m-r^{2k-2})}\\ &\le e^{\frac{d_{i+1}+c_{i+1}r}{r+1}(r^{2k-2}-1)+d_{i+1}(m-r^{2k-2})} = \rho_{i+1}(m)\end{aligned}$$

and if $m\in T_k$, then

$$\begin{aligned}\rho_i(m) &= e^{\frac{d_ir}{r+1}(r^{2k-2}-1)+d_i(r^{2k-1}-r^{2k-2})+c_i(m-r^{2k-1})}\\ &\le e^{\frac{d_{i+1}r}{r+1}(r^{2k-2}-1)+d_{i+1}(r^{2k-1}-r^{2k-2})+c_{i+1}(m-r^{2k-1})} = \rho_{i+1}(m).\end{aligned}$$

It also follows from Lemma 3.4 that

$$\alpha(A) = \frac{r-1}{r+1}\max_{1\le i\le q}(c_i-d_i)$$

and

$$\begin{aligned}\underline{\sigma}(A) &= \sum_{i=1}^q b_i - \liminf_{m\to\infty}\frac{1}{m}\log\det\mathcal{A}_m\\ &= \sum_{i=1}^q \frac{d_i+c_ir}{r+1} - \liminf_{m\to\infty}\frac{1}{m}\log\det\mathcal{A}_m.\end{aligned} \tag{3.4}$$

We have

$$\liminf_{m\to\infty}\frac{1}{m}\log\det\mathcal{A}_m = \liminf_{m\to\infty}\frac{1}{m}\log\prod_{j=1}^{m-1}\prod_{i=1}^q a_i(j),$$

where

$$\prod_{i=1}^{q} a_i(m) = \begin{cases} e^{d_1+\cdots+d_q} & \text{if } m \in S_k \text{ for } k \in \mathbb{N}, \\ e^{c_1+\cdots+c_q} & \text{if } m \in T_k \text{ for } k \in \mathbb{N}. \end{cases}$$

Applying Lemma 3.4 with $c = c_1 + \cdots + c_q$ and $d = d_1 + \cdots + d_q$ we obtain

$$\begin{aligned} \liminf_{m\to\infty} \frac{1}{m} \log \prod_{j=1}^{m-1} \prod_{i=1}^{q} a_i(j) &= \frac{c_1 + \cdots + c_q + (d_1 + \cdots + d_q)r}{r+1} \\ &= \sum_{i=1}^{q} \frac{c_i + d_i r}{r+1} \end{aligned}$$

and it follows from (3.4) that

$$\underline{\sigma}(A) = \sum_{i=1}^{q} \frac{d_i + c_i r}{r+1} - \sum_{i=1}^{q} \frac{c_i + d_i r}{r+1} = \frac{r-1}{r+1} \sum_{i=1}^{q} (c_i - d_i).$$

This completes the proof of the lemma. □

Step 3. Conclusion of the argument. We first construct auxiliary sequences.

Lemma 3.6. *Given $c_q > d_q > 0$, there exist an r-regular sequence $C = (C_m)_{m\in\mathbb{N}}$ with*

$$\underline{\sigma}(C) = \alpha(C) = \frac{(r-1)(c_q - d_q)}{r+1}$$

and an r-regular sequence $D = (D_m)_{m\in\mathbb{N}}$ with

$$\underline{\sigma}(D) = q\alpha(D) = q\frac{(r-1)(c_q - d_q)}{r+1}. \tag{3.5}$$

Proof of the lemma. Let

$$c_1 = \cdots = c_{q-1} = d_1 = \cdots = d_{q-1} = 0$$

and denote the corresponding matrices A_m (see (3.3)) by C_m. Then the sequence C is r-regular and by Lemma 3.5 we have

$$\underline{\sigma}(C) = \alpha(C) = \frac{(r-1)(c_q - d_q)}{r+1}.$$

Now let

$$c_1 = \cdots = c_q, \quad d_1 = \cdots = d_q$$

and denote the corresponding matrices A_m (see (3.3)) by D_m. Then the sequence D is r-regular and by Lemma 3.5 we have

$$\underline{\sigma}(D) = q\frac{(r-1)(c_q - d_q)}{r+1} \quad \text{and} \quad \alpha(D) = \frac{(r-1)(c_q - d_q)}{r+1}$$

which gives identity (3.5). □

We use the sequences C and D to show that for each $\mu \in [1, q]$ there exists an r-regular sequence $E = (E_m)_{m\in\mathbb{N}}$ with

$$\underline{\sigma}(E) = \mu\alpha(E).$$

First observe that replacing C by the sequence $C' = (C'_m)_{m\in\mathbb{N}}$ with $C'_m = C^\kappa_m$ for some $\kappa > 0$ corresponds to replace the numbers c_i and d_i, respectively, by κc_i and κd_i for each i. Therefore,

$$\alpha(C') = \kappa\alpha(C) \quad \text{and} \quad \underline{\sigma}(C') = \kappa\underline{\sigma}(C).$$

Moreover, for each $\nu \in [0, 1]$, the sequence of matrices $E = (E_m)_{m\in\mathbb{N}}$ with $E_m = C^\nu_m D^{1-\nu}_m$ for $m \in \mathbb{N}$ is r-regular, with

$$\alpha(E) = \nu\alpha(C) + (1-\nu)\alpha(D) \quad \text{and} \quad \underline{\sigma}(E) = \nu\underline{\sigma}(C) + (1-\nu)\underline{\sigma}(D). \tag{3.6}$$

Indeed, let $e^{c_i(m)}$ and $e^{d_i(m)}$ be, respectively, the entries on the diagonals of C_m and D_m. Then the entries on the diagonal of E_m are $e^{\nu c_i(m)+(1-\nu)d_i(m)}$ and one can easily verify that the two properties in the notion of r-regularity hold as well as (3.6). By Lemma 3.5 and (3.6) we obtain

$$\begin{aligned}\alpha(E) &= \nu\frac{(c_q - d_q)(r-1)}{r+1} + (1-\nu)\frac{(c_q - d_q)(r-1)}{r+1} \\ &= \frac{(c_q - d_q)(r-1)}{r+1}\end{aligned}$$

and

$$\begin{aligned}\underline{\sigma}(E) &= \nu\frac{(c_q - d_q)(r-1)}{r+1} + (1-\nu)\frac{q(c_q - d_q)(r-1)}{r+1} \\ &= \frac{(\nu + (1-\nu)q)(c_q - d_q)(r-1)}{r+1}.\end{aligned}$$

In particular,

$$\underline{\sigma}(E)/\alpha(E) = \nu + (1-\nu)q.$$

Note that when ν goes from 0 to 1, this expression goes from q to 1 and so it takes any value $\mu \in [1, q]$. Moreover, $\alpha(E)$ can take any prescribed positive value by choosing c_q and d_q. This completes the proof of the theorem. □

3.3 Realization problem II

In this section we construct specific sequences of matrices attaining each possible value of the regularity coefficients $\underline{\sigma}(A)$ and $\overline{\sigma}(A)$.

Theorem 3.7. *Given $s \geq 0$, there exists:*

1. *a bounded sequence of matrices A with $\underline{\sigma}(A) = s$;*
2. *a bounded sequence of matrices A with $\overline{\sigma}(A) = s$.*

Proof. Note that $\underline{\sigma}(A) = \overline{\sigma}(A) = 0$ for any regular sequence A. So it suffices to take $s > 0$.

We first show that given $s > 0$, there exists a bounded sequence of matrices A with $\underline{\sigma}(A) = s$. Consider the sequence of diagonal matrices

$$A_m = \text{diag}(a_1(m), \ldots, a_q(m)),$$

where

$$a_i(m) = \begin{cases} e^{\beta_i} & \text{if } k! \le m < (k+1)! \text{ with } k \text{ odd}, \\ 1 & \text{otherwise} \end{cases}$$

for some nonnegative numbers

$$\beta_1 \le \beta_2 \le \cdots \le \beta_q$$

such that $\sum_{i=1}^q \beta_i = s$. Then $\rho_i(m) \le e^{\beta_i m}$ and so

$$b_i = \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m) \le \beta_i.$$

On the other hand, for k odd we have

$$\rho_i((k+1)!) \ge e^{\beta_i((k+1)!-k!)}$$

and so

$$\begin{aligned} b_i &\ge \limsup_{k\to\infty} \frac{1}{(k+1)!} \log \rho_i((k+1)!) \\ &\ge \limsup_{k\to\infty} \frac{1}{(k+1)!} \beta_i((k+1)! - k!) = \beta_i. \end{aligned}$$

This shows that $b_i = \beta_i$ for $i = 1, \ldots, q$. Moreover, since $a_i(j) \ge 1$ we have

$$\liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \ge 0. \tag{3.7}$$

On the other hand, denoting by $e_1, \ldots, e_q$ the canonical basis for $\mathbb{R}^q$ and writing $r_n = (2n+1)!$, we obtain

$$\begin{aligned} \liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m &= \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^q \|\mathcal{A}_m e_i\| \\ &\le \liminf_{n\to\infty} \frac{1}{r_n} \log \prod_{i=1}^q \|\mathcal{A}_{r_n} e_i\|. \end{aligned}$$

We have

$$\prod_{i=1}^q \|\mathcal{A}_{r_n} e_i\| \le \prod_{i=1}^q e^{\beta_i (2n)!} = e^{s(2n)!}$$

and so

$$\liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \le \liminf_{n\to\infty} \frac{s(2n)!}{r_n} = 0.$$

Together with (3.7), this implies that

$$\liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m = 0$$

and so

$$\underline{\sigma}(A) = \sum_{i=1}^q \beta_i - \liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m = \sum_{i=1}^q \beta_i = s.$$

Now we show that given $s > 0$, there exists a bounded sequence of matrices A with $\overline{\sigma}(A) = s$. Consider the sequence of diagonal matrices

$$A_m = \operatorname{diag}(a_1(m), \ldots, a_q(m)),$$

where

$$a_i(m) = \begin{cases} e^s & \text{if } k! \le m < (k+1)! \text{ and } k \equiv q \bmod (q+1), \\ 1 & \text{otherwise.} \end{cases}$$

Since $a_i(m) \ge 1$, we have $\rho_i(m) \ge 1$ and so

$$a_i = \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \ge 0.$$

Moreover, letting $r_{n,j} = (q + n(q+1) + j)!$ we have

$$a_i \le \liminf_{n\to\infty} \frac{1}{r_{n,2}} \max_{1\le i\le q} \log \prod_{i=1}^{q} \|\mathcal{A}_{r_{n,2}} e_i\| \le \liminf_{n\to\infty} \frac{r_{n,1}}{r_{n,2}} = 0.$$

Therefore, $a_i = 0$ for $i = 1, \ldots, q$. Moreover, $\det \mathcal{A}_m \le e^{ms}$ and so

$$\limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \le s. \tag{3.8}$$

Finally, we have

$$\det \mathcal{A}_{(k+1)!} \ge \prod_{i=1}^{q} \prod_{j=k!}^{(k+1)!-1} a_i(j) = e^{s((k+1)!-k!)},$$

for $k \equiv q \bmod (q+1)$, which gives

$$\limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \ge \limsup_{n\to\infty} \frac{s(r_{n,1} - r_{n,0})}{r_{n,1}} = s.$$

Together with (3.8), this implies that

$$\limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m = s$$

and so

$$\overline{\sigma}(A) = \limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m - \sum_{i=1}^{q} a_i = s.$$

This completes the proof of the theorem. $\square$

4 Further regularity coefficients

In this section we introduce two additional regularity coefficients based on the matrices $\mathcal{A}_m$. We also establish inequalities between these coefficients and the former ones.

For each $k = 1, \ldots, q$, let $(\mathbb{R}^q)^{\wedge k}$ be the set of alternating k-linear forms on $\mathbb{R}^q$. We define an inner product on $(\mathbb{R}^q)^{\wedge k}$ by requiring that

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det B,$$

where B is the $k \times k$ matrix with entries $b_{ij} = \langle v_i, w_j \rangle$ for $i, j = 1, \ldots, k$ and where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^q$. In particular, for $k = 1$ we recover the standard inner product and so the 2-norm on $\mathbb{R}^q$.

Now let $A = (A_m)_{m\in\mathbb{N}}$ be a sequence of $q \times q$ matrices with real entries. For each $k = 1, \ldots, q$, we define

$$c_k(A) = \liminf_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge k}\|$$

and

$$d_k(A) = \limsup_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge k}\|,$$

where

$$(\mathcal{A}_m)^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = \mathcal{A}_m v_1 \wedge \cdots \wedge \mathcal{A}_m v_k.$$

Finally, let

$$\varepsilon(A) = \max\{d_k(A) - c_k(A) : k = 1, \ldots, q\}.$$

Theorem 4.1. *The function $\varepsilon(A)$ is a regularity coefficient. Moreover, for each bounded sequence of matrices $A = (A_m)_{m\in\mathbb{N}}$ we have*

$$\frac{1}{2}\alpha(A) \le \varepsilon(A) \le q\alpha(A). \tag{4.1}$$

Proof. Note that it suffices to establish (4.1) since then $\varepsilon(A) \ge \alpha(A)/2 \ge 0$ and $\varepsilon(A) = 0$ if and only if $\alpha(A) = 0$, that is, if and only if A is regular.

Recall that for any $q \times q$ matrix B we have

$$\|B^{\wedge k}\| = \prod_{i=1}^{k} \rho_{q-i+1},$$

where $\rho_1 \le \cdots \le \rho_q$ are the (real nonnegative) eigenvalues of the matrix $(B^*B)^{1/2}$. Taking $B = \mathcal{A}_m$ we obtain

$$\|(\mathcal{A}_m)^{\wedge k}\| = \prod_{i=1}^{k} \rho_{q-i+1}(m). \tag{4.2}$$

Therefore,

$$\begin{aligned} c_k(A) &= \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{k} \rho_{q-i+1}(m) \\ &\ge \sum_{i=1}^{k} \liminf_{m\to\infty} \frac{1}{m} \log \rho_{q-i+1}(m) = \sum_{i=1}^{k} a_{q-i+1}(A) \end{aligned}$$

and

$$\begin{aligned} d_k(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{k} \rho_{q-i+1}(m) \\ &\le \sum_{i=1}^{k} \limsup_{m\to\infty} \frac{1}{m} \log \rho_{q-i+1}(m) = \sum_{i=1}^{k} b_{q-i+1}(A). \end{aligned}$$

This readily implies that

$$d_k(A) - c_k(A) \le \sum_{i=1}^{k} (b_{q-i+1}(A) - a_{q-i+1}(A)) \le k\alpha(A)$$

and so $\varepsilon(A) \le q\alpha(A)$. On the other hand, by (4.2) we have

$$\rho_i(m) = \frac{\|(\mathcal{A}_m)^{\wedge(q-i+1)}\|}{\|(\mathcal{A}_m)^{\wedge(q-i)}\|}$$

for $i = 1, \ldots, q-1$ and

$$\rho_q(m) = \|(\mathcal{A}_m)^{\wedge 1}\| = \|\mathcal{A}_m\|.$$

Therefore,

$$\begin{aligned} a_i(A) &= \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \\ &\ge \liminf_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i+1)}\| + \liminf_{m\to\infty} -\frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i)}\| \\ &= c_{q-i+1}(A) - d_{q-i}(A) \end{aligned}$$

and

$$\begin{aligned} b_i(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m) \\ &\le \limsup_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i+1)}\| + \limsup_{m\to\infty} -\frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i)}\| \\ &= d_{q-i+1}(A) - c_{q-i}(A) \end{aligned}$$

for $i = 1, \ldots, q-1$. These inequalities also hold for $i = q$, with the convention that $c_0(A) = d_0(A) = 0$. This implies that

$$b_i(A) - a_i(A) \le d_{q-i+1}(A) - c_{q-i+1}(A) + d_{q-i}(A) - c_{q-i}(A)$$

and so

$$\begin{aligned} \alpha(A) &\le \max\{d_{q-i+1}(A) - c_{q-i+1}(A) : i = 1, \ldots, q\} + \max\{d_{q-i}(A) - c_{q-i}(A) : i = 1, \ldots, q\} \\ &\le 2\varepsilon(A). \end{aligned}$$

This completes the proof of the theorem. $\square$

We also introduce a second regularity coefficient. First recall that a bounded sequence of matrices $A = (A_m)_{m\in\mathbb{N}}$ is regular if and only if the sequence of matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}$ converges entry by entry when $m \to \infty$ (see [4]). Therefore, the function

$$\mu(A) = \sum_{i=1}^{q}\sum_{j=1}^{q}\left(\limsup_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)} - \liminf_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)}\right),$$

where B_{ij} denotes the ij entry of a matrix B, is a regularity coefficient. Moreover, we have the following result.

Theorem 4.2. *There exists a constant $C_q > 0$ depending only on q such that for each bounded sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices such that $A^{-1} = (A_m^{-1})_{m\in\mathbb{N}}$ is also bounded we have*

$$C_q^{-1}\|A^{-1}\|_\infty^{-1}\alpha(A) \le \mu(A) \le C_q\|A\|_\infty\alpha(A).$$

Proof. Since the matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$ are symmetric and positive definite, there exist orthogonal matrices S_m such that

$$S_m^{-1}(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}S_m = \operatorname{diag}(\rho_1(m)^{1/m}, \ldots, \rho_q(m)^{1/m}). \tag{4.3}$$

Moreover, it is shown in [4] that one can always choose the matrices S_m so that they converge entry by entry to some orthogonal matrix S when $m \to \infty$. Hence, it follows from (4.3) that there exists a constant $C_q > 0$ depending only on q such that

$$\begin{aligned}
&\max_{1\le i\le q}\left(\limsup_{m\to\infty}\rho_i(m)^{1/m} - \liminf_{m\to\infty}\rho_i(m)^{1/m}\right) \\
&\quad\le C_q\sum_{i=1}^q\sum_{j=1}^q\left(\limsup_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)} - \liminf_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)}\right).
\end{aligned} \tag{4.4}$$

Again by (4.3) we have

$$(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)} = S_m\operatorname{diag}(\rho_1(m)^{1/m}, \ldots, \rho_q(m)^{1/m})S_m^{-1}$$

and so we also obtain

$$\begin{aligned}
&\sum_{i=1}^q\sum_{j=1}^q\left(\limsup_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)} - \liminf_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)}\right) \\
&\quad\le C_q\max_{1\le i\le q}\left(\limsup_{m\to\infty}\rho_i(m)^{1/m} - \liminf_{m\to\infty}\rho_i(m)^{1/m}\right),
\end{aligned} \tag{4.5}$$

taking the same constant C_q without loss of generality.

Now observe that by (4.2) with $k = 1$, we have

$$\|\mathcal{A}_m^{-1}\|^{-1} \le \rho_i(m) \le \|\mathcal{A}_m\|$$

for each $i = 1, \ldots, q$. Therefore,

$$\|A^{-1}\|_\infty^{-1} \le \rho_i(m)^{1/m} \le \|A\|_\infty$$

and it follows from the mean value theorem that

$$\begin{aligned}
b_i(A) - a_i(A) &= \log\limsup_{m\to\infty}\rho_i(m)^{1/m} - \log\liminf_{m\to\infty}\rho_i(m)^{1/m} \\
&\le \|A^{-1}\|_\infty\left(\limsup_{m\to\infty}\rho_i(m)^{1/m} - \liminf_{m\to\infty}\rho_i(m)^{1/m}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\limsup_{m\to\infty}\rho_i(m)^{1/m} - \liminf_{m\to\infty}\rho_i(m)^{1/m} &= \exp\limsup_{m\to\infty}\frac{1}{m}\log\rho_i(m) - \exp\liminf_{m\to\infty}\frac{1}{m}\log\rho_i(m) \\
&\le \|A\|_\infty(b_i(A) - a_i(A)).
\end{aligned}$$

Together with (4.4) and (4.5) this yields the desired result. □

Acknowledgment

This research was partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

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