Irreducible Cyclic Presentations of the Trivial Group

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CONTENTS

We produce families of irreducible cyclic presentations of the trivial group. These families comprehensively answer questions about such presentations asked by Dunwoody and by Edjvet, Hammond, and Thomas. Our theorems are purely theoretical, but their derivation is based on practical computations. We explain how we chose the computations and how we deduced the theorems.

1. INTRODUCTION

Edjvet, Hammond, and Thomas [Edjvet et al. 01] made the following definitions: Let F_n denote the free group on n (free) generators $x_0, \ldots, x_{n-1}$ and let $\theta : F_n \to F_n$ be the automorphism for which $x_i\theta = x_{i+1}$ (where subscripts are taken modulo n). Following [Johnson 80], for (cyclically reduced) $w \in F_n$ define $G_n(w) = F_n/N$ where N is the normal closure in F_n of the set $\{w, w\theta, \ldots, w\theta^{n-1}\}$. A group G is said to have a *cyclic presentation* or to be *cyclically presented* if $G \cong G_n(w)$ for some w and for some n.

The *polynomial associated with* the cyclic presentation for $G_n(w)$ is defined to be $f_w(t) = \sum_{i=0}^{n-1} a_i t^i$ where a_i is the exponent sum of x_i in w. Set $A_n(w) = G_n(w)^{ab}$. It is shown in [Johnson 80] that the order of $A_n(w)$ is equal to the absolute value of the product $\prod_{i=0}^{n-1} f_w(\xi_i)$ where ξ_i ranges over the set of complex n-th roots of unity (with the convention that $A_n(w)$ is infinite whenever the product vanishes). Furthermore, $A_n(w)$ is trivial if and only if $f_w(t)$ is a unit in the ring $\mathbb{Z}[t]/(t^n - 1)$.

A presentation for $G_n(w)$ is *irreducible* if whenever w involves only $x_{i_1}, \ldots, x_{i_k}$, where $i_j < i_{j+1}$, then $\gcd(n, i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}) = 1$. Otherwise, when this greatest common divisor is not equal to 1, the group $G_n(w)$ decomposes into a free product of copies of $G_m(\hat{w})$ where m divides n.

Edjvet, Hammond, and Thomas posed the following two questions:

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Question 1.1. Is there an irreducible cyclic presentation of the trivial group with more than five generators?

Question 1.2. Is there an example w where $G_5(w)$ is trivial and $f_w(t) = \pm t^i$?

One motivation for Edjvet, Hammond, and Thomas was that any irreducible cyclic presentation of the trivial group could be a possible counterexample to the well-known Andrews-Curtis conjecture (see [Burns and Macedońska 93] for a good group-theoretic survey of this and see [Miasnikov 99], [Havas and Ramsay 03a], and [Havas and Ramsay 03b] for computational approaches). Edjvet, Hammond, and Thomas conducted a computer search and as a result found the following examples of irreducible cyclic presentations of the trivial group:

$$\begin{array}{ll}
G_4(x_2x_1x_0x_1^{-1}x_0^{-1}) & f_w(t) = t^2 \\
G_4(x_2x_1^{-1}x_3^{-1}x_1x_3) & f_w(t) = t^2 \\
G_4(x_3x_2x_1x_0^{-1}x_1^{-1}x_2^{-1}x_0) & f_w(t) = t^3 \\
G_4(x_3x_0^{-1}x_1x_2x_1^{-1}x_2^{-1}x_0) & f_w(t) = t^3 \\
G_4(x_3x_2x_1x_0^{-1}x_1^{-1}x_0x_2^{-1}) & f_w(t) = t^3 \\
G_4(x_3x_0^{-1}x_1x_2x_1^{-1}x_0x_2^{-1}) & f_w(t) = t^3 \\
G_5(x_0^{-1}x_1^{-1}x_3x_2x_1) & f_w(t) = -1 + t^2 + t^3 \\
G_5(x_0^{-1}x_2x_3^{-1}x_0x_4) & f_w(t) = t^2 - t^3 + t^4 \\
G_5(x_0^{-1}x_2x_1^{-1}x_3x_1) & f_w(t) = -1 + t^2 + t^3 \\
G_5(x_0^{-1}x_2^{-1}x_0x_3x_1) & f_w(t) = t - t^2 + t^3.
\end{array}$$

Dunwoody ([Johnson 76, page 191], [Dunwoody 95]) conjectured that if $G_n(w)$ is trivial then $f_w(t) = \pm t^i$. If $n = 2$, 3, 4, or 6, then the only units in $\mathbb{Z}[t]/(t^n - 1)$ are cosets containing elements of the form $\pm t^i$. It follows that the conjecture is true for these values of n. Edjvet, Hammond, and Thomas, however, showed that the conjecture is false for $n = 5$, but their investigation led them to pose Questions 1.1 and 1.2 above. This already answered the query in [Dunwoody 95] as to whether "there are any non-trivial cyclic presentations of the trivial group for $n > 3$."

We assert the following theorems which comprehensively answer the questions of Dunwoody and of Edjvet, Hammond, and Thomas.

Theorem 1.3. *For each $n \geq 2$ there exist infinitely many irreducible cyclic presentations of the trivial group with n generators.*

Theorem 1.4. *For each $n \geq 2$ there exist infinitely many irreducible cyclic presentations $G_n(w)$ of the trivial group with n generators and $f_w(t) = \pm t^i$.*

Theorem 1.5. *For each $n = 5k$ there exist infinitely many irreducible cyclic presentations $G_n(w)$ of the trivial group with n generators and $f_w(t) \neq \pm t^i$.*

The proofs of these theorems are simple corollaries of our constructions in the next section.

2. FAMILIES OF IRREDUCIBLE PRESENTATIONS

Theorem 2.1. *Each of the balanced presentations derived from the following word w on km generators (where $k > 1$ and $m \geq 1$) is irreducible and defines the trivial group:*

$$w = x_1^{-1}(x_0x_mx_{2m}\ldots x_{(k-1)m})^{-1}x_mx_{2m}\ldots x_{(k-1)m}x_0.$$

Proof:

$$\begin{aligned}
x_0 &= (x_{km-1}x_{m-1}x_{2m-1}\ldots x_{(k-1)m-1})^{-1} \\
&\quad \times x_{m-1}x_{2m-1}\ldots x_{(k-1)m-1}x_{km-1}; \\
x_m &= (x_{m-1}x_{2m-1}\ldots x_{(k-1)m-1}x_{km-1})^{-1} \\
&\quad \times x_{2m-1}\ldots x_{(k-1)m-1}x_{km-1}x_{m-1}; \\
&\ \vdots \\
x_{(k-1)m} &= (x_{(k-1)m-1}x_{km-1}x_{m-1}\ldots x_{(k-2)m-1})^{-1} \\
&\quad \times x_{km-1}x_{m-1}x_{2m-1}\ldots x_{(k-1)m-1}.
\end{aligned}$$

Thus, $x_0x_mx_{2m}\ldots x_{(k-1)m} = \epsilon$ and likewise $(x_mx_{2m}\ldots x_{(k-1)m}x_0)^{-1} = \epsilon$. So $x_1 = \epsilon$ and the group is trivial. □

It is easy to see that this can be proved using Andrews-Curtis moves.

For $k = 1$, these presentations are the uninteresting m-generator trivial presentations. For all other k, these are irreducible cyclic presentations (mainly new). They all have as cyclic associated polynomial $f_w(t) = \pm t^i$. This theorem leads to other constructions of irreducible cyclic presentations. We merely present two corollaries.

Corollary 2.2. *Let $v = \Pi_{i=0}^{k-1}x_{im}$ and let $u = (v\theta^m)^{-1}$. Then each of the balanced presentations derived from the word $w = x_s^{-1}u^{\pm 1}v^{\mp 1}$ on km generators (where $k > 1$, $m \geq 1$ and $\gcd(s, m) = 1$) is irreducible and defines the trivial group. In a similar way we can replace v by Πv_j for $v_j = v\theta^{i_j}$.*

What this means is you can take any standard v from the basic theorem and apply a power of the automorphism to it. Furthermore, you can multiply any v's obtained this way (constructing suitable u's). The simplest example comes from raising a given v to any nonzero power q, thus: $w = x_s^{-1} u^{\pm q} v^{\mp q}$. A more complicated example on four generators is:

$$w = x_1^{-1}(x_0x_1x_2x_3)^{-1}(x_3x_0x_1x_2)^{-2}(x_0x_1x_2x_3)^{-1} \times (x_1x_2x_3x_0)(x_0x_1x_2x_3)^2(x_1x_2x_3x_0).$$

Before giving our next corollary, we define the composition of two cyclic presentations. This is based on a construction in [Neumann 79] (see also [Havas and Ramsay 00]) for one particular cyclic presentation of the trivial group.

Consider the groups $G_n(w_1)$ and $G_n(w_2)$. Define their composition $G_n(w_1 * w_2)$ to be the cyclically presented group where $w_1 * w_2$ is obtained by replacing the generator i in w_1 with the relator i of $G_n(w_2)$. Notice that if $G_n(w_1)$ and $G_n(w_2)$ are trivial, then $G_n(w_1 * w_2)$ is trivial with $f_{w_1 * w_2}(t) = f_{w_1}(t) f_{w_2}(t) \mod (t^n - 1)$.

For example, to take the composition of

$$\begin{aligned} G_3(w) = \langle x_0, x_1, x_2 \mid\ & x_1^{-1}(x_0x_1x_2)^{-1}(x_1x_2x_0), \\ & x_2^{-1}(x_1x_2x_0)^{-1}(x_2x_0x_1), \\ & x_0^{-1}(x_2x_0x_1)^{-1}(x_0x_1x_2)\rangle \end{aligned}$$

with itself, form $G_3(w * w)$ where $w * w$ is obtained by substituting

$$\begin{aligned} x_1 &= a_1^{-1}(a_0a_1a_2)^{-1}(a_1a_2a_0), \\ x_2 &= a_2^{-1}(a_1a_2a_0)^{-1}(a_2a_0a_1), \\ x_0 &= a_0^{-1}(a_2a_0a_1)^{-1}(a_0a_1a_2) \end{aligned}$$

into w. (That $G_3(w)$ is the trivial group is a consequence of Theorem 2.1.)

We can now give a second corollary.

Corollary 2.3. *The composition of cyclic presentations of trivial groups gives new cyclic presentations of trivial groups. If at least one of the components is irreducible, the composition is irreducible.*

We now comment on the proofs of Theorems 1.3–1.5 and some consequences.

Theorem 1.3 follows from Corollary 2.2 above. Further infinite collections of irreducible cyclic presentations of the trivial group on n generators can be constructed by applying the composition construction. The examples produced by Corollary 2.2, and also presentations from application of the composition construction to those examples, all have polynomial $\pm t^i$, so Theorem 1.4 follows.

One family of positive answers to Question 1.2 (of Edjvet, Hammond, and Thomas) comes from choosing $n = 5$ in our Theorem 1.4. Alternatively, we note that it can also be answered by applying the composition construction to the first two groups $G_5(x_0^{-1}x_1^{-1}x_3x_2x_1)$ and $G_5(x_0^{-1}x_2x_3^{-1}x_0x_4)$ in their list, since $(-1+t^2+t^3)(t^2 - t^3 + t^4) \equiv t^3 \mod (t^5 - 1)$.

Theorem 1.5 follows by noting that we can take $G_5(x_0^{-1}x_1^{-1}x_3x_2x_1)$ with polynomial $f_w(t) = -1 + t^2 + t^3$ and construct from it $G_{5k}(x_0^{-1}x_k^{-1}x_{3k}x_{2k}x_k)$. This group has polynomial $f_w(t) = -1 + t^{2k} + t^{3k}$ and is reducible for $k > 1$. Taking the composition of this group with a $5k$-generator group given by the construction of Theorem 2.1 gives an example of a $5k$-generator irreducible cyclically presented group with polynomial $\neq \pm t^i$. Repeated composition yields an infinite number of such groups.

As far as the Andrews-Curtis conjecture is concerned, our constructions will not produce counterexamples. The presentations constructed in Theorem 2.1 satisfy the conjecture. Furthermore, applying composition to two presentations which satisfy the conjecture leads to a presentation satisfying the conjecture; Miasnikov [Miasnikov 99, Section 6] proves this.

3. DERIVATION OF OUR CONSTRUCTION

Edjvet, Hammond, and Thomas chose to do a computer search for suitable presentations of the split extension $H_n(w)$ of $G_n(w)$ by the cyclic group of order n, which has a presentation

$$H_n(w) = \langle x, t \mid t^n, w(x,t)\rangle.$$

The size of the search space is exponential in the length of $w(x,t)$, and they restricted themselves to lengths up to 15.

We observed that by looking at the groups $G_n(w)$ themselves we could investigate alternative search spaces, this time with sizes exponential in the length of w. Note that the length of w is significantly less than the length of $w(x,t)$, which makes our search space more tractable.

Thus, we wrote a program in GAP [GAP4 03] which enumerated inequivalent words w. Our definition of equivalence was designed to eliminate words that generated groups easily determined to have presentations that could be transformed into presentations otherwise considered. For each such word w, we first tested whether the corresponding group $G_n(w)$ was perfect—a very fast test that eliminated most words from further study. For perfect groups $G_n(w)$, we tried to determine the group order using coset enumeration. Since at least some of the enumerations figured to be difficult, we used the ACE package [Havas et al. 03] in GAP.

Since nothing was known for six generators, we started by considering words w of odd length for six generators. Our very first run on presentations of length five produced a collection of presentations for the trivial group, some of which were reducible, but also some irreducible ones, including equivalents of $w = x_1^{-1}(x_0x_3)^{-1}x_3x_0$. From this one example, which is an instance of Theorem 2.1, we were able to deduce the pattern which led to that theorem.

The reason that Edjvet, Hammond, and Thomas did not find this presentation is that it was outside their search space. The length of $w(x,t)$ is 17, the first possible length outside of their range; however, the length of w is only five, comfortably within our range.

Subsequently, we found other instances of Theorem 2.1 for various numbers of generators and lengths. We have not found any other new presentations of interest. We did produce equivalents of the presentations found by Edjvet, Hammond, and Thomas and we also found various reducible presentations. Note that the four-generator presentations produced by them are instances of our Theorem 2.1 or Corollary 2.2; however, their five-generator presentations are different.

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