An Integral Related To The Cauchy Transform On The Sierpinski Gasket

Xin-Han Dong and Ka-Sing Lau

CONTENTS

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We estimate an integral on the Sierpinski gasket and justify a theorem in the paper [Lund et al. 98]. The integral relates to the Laplace transform of the Hausdorff measure. It is fundamental and useful in some other contexts [Dong and lau xx].

1. INTRODUCTION

Let K be the Sierpinski gasket in the complex plane $\mathbb{C}$ with three vertices at $\varepsilon_k = e^{2k\pi i/3}, k = 0, 1, 2$. It is well known that K is the attractor of the iterated function system $\{S_k\}_{k=0}^2$ with $S_k z = \varepsilon_k + (z - \varepsilon_k)/2$ and the Hausdorff dimension of K is $\alpha = \log 3/\log 2$. Let μ be the Hausdorff measure $\mathcal{H}^\alpha$ normalized on K. We define the Cauchy transform of μ by

$$F(z) = \int_K \frac{d\mu(w)}{z - w} .$$

In [Lund et al. 98], Strichartz et al. initiated the study of the analytic and geometric behavior of the function F. One of the most interesting observations concerns the image of K under F. Let Δ_0 denote the unbounded connected region outside the Sierpinski gasket. The following result was claimed in [Lund et al. 98].

Theorem 1.1. *$F(-\frac{1}{2})$ lies in the interior of $F(\Delta_0)$.*

Note that the point $-1/2$ is on the boundary curve $\partial\Delta_0$ of Δ_0. The theorem implies that the image curve $F(\partial\Delta_0)$ forms a loop near $F(-\frac{1}{2})$. By self-similarity, the loops appear everywhere on the image point of each dyadic rational point on $\partial\Delta_0$ (see Figure 1). This leads to the conjecture in [Lund et al. 98] that *the boundary of $F(\Delta_0)$ is a simple closed curve and is the image of a Cantor set in $\partial\Delta_0$*. The reader can also refer to [Dong and Lau 04] for more detail.

As F is continuous and bounded on $\mathbb{C}$, $F(x) < 0$ for $x \in (-\infty, -1/2)$ and $F(-\infty) = 0$, it follows that

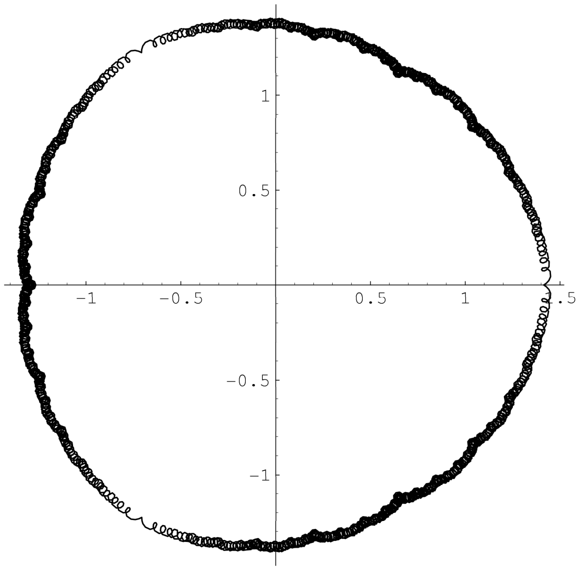

FIGURE 1. The image of the triangular boundary of the Sierpinski gasket under the mapping F.

$F([-\infty, -1/2]) = [a, 0]$ for some $a < 0$. Their proof of the theorem is to conclude $a < F(-1/2) < 0$ by showing that $F(x)$ is increasing for $x < -1/2$ and near $-1/2$. It is equivalent to show that

$$\begin{aligned} g(x) &:= F'(-(x+1/2)) = -\int_K \frac{d\mu(w)}{(1/2+x+w)^2} \\ &= \int_K \frac{v^2-(1/2+x+u)^2}{(v^2+(1/2+x+u)^2)^2} d\mu(w) > 0 \end{aligned}$$

($w = u + iv$) for small $x > 0$. The difficulty is that it is awkward to handle the integral over the fractal set K. In addition, the integrand takes both positive and negative values on K. They tried to get around this by using a clever method to show that $g(0) = \infty$, and claimed that a similar argument would imply $g(x) > 0$ for small $x > 0$. However the claim is not so direct, as it is not clear that $\lim_{x\to 0^+} g(x) = g(0)$ (in fact it is not even clear that $g(x) \neq 0$). The main purpose of this note is to justify this step. The integrals in the following are useful and appear in other contexts [Dong and Lau 04].

Let $T = 1 - K$ be the relocation of the Sierpinski gasket with the new vertices at $0, \sqrt{3}e^{\pi i/6}, \sqrt{3}e^{-\pi i/6}$, and let $T_j = 1 - K_j$ where $K_j = S_j K,\ j = 0, 1, 2$. Let

$$A_0 = \bigcup_{n=-\infty}^{\infty} 2^n (T_1 \cup T_2)$$

be the "Sierpinski cone" generated by T (see Figure 2). It is easy to see that $T = A_0 \cap T = A_0 \cap \{z = x + yi : x \leq 3/2\}$ and $A_0 = \lim_{r\to+\infty} A_0 \cap \{z = x + yi : x \leq r\}$. We

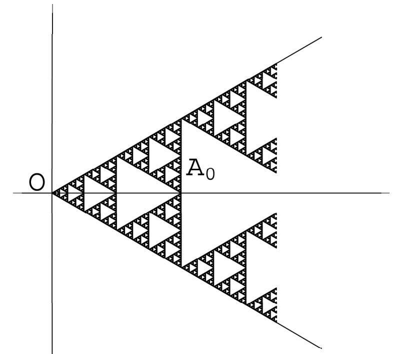

FIGURE 2. The Sierpinski cone A_0.

still use μ to denote the normalized Hausdorff measure (i.e., $\mu(T) = 1$) on $\mathbb{C}$. We define

$$H(x) := \int_{e^{\pi i/3}A_0 \cup e^{-\pi i/3}A_0} \frac{d\mu(w)}{(x+w)^2}\ .$$

(see Figure 3 for the domain of integration of H, the union of the rotations of A_0 by $e^{\pi i/3}$ and $e^{-\pi i/3}$). We can reduce the consideration of F' to H as follows:

Proposition 1.2. $F'(-(x+1/2)) = -H(x) + \psi(x),\ x > 0$ *for some real function* $\psi(x)$*, bounded and continuous for* $x \geq 0$.

Our main result is the following:

Proposition 1.3. $H(x)$ *is continuous and is* < 0 *for* $x > 0$.

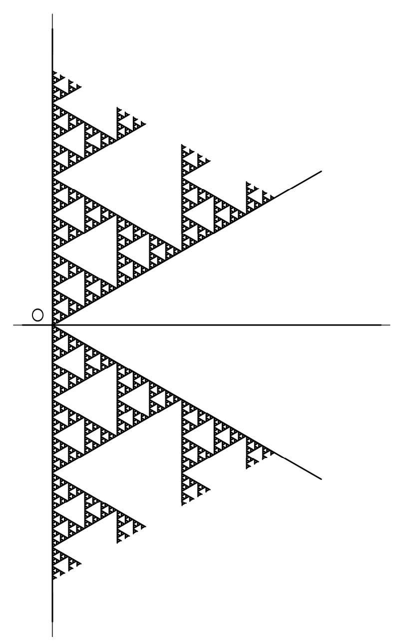

FIGURE 3. The region $e^{\pi i/3}A_0 \cup e^{-\pi i/3}A_0$.

By using $\mu(2E) = 3\mu(E)$, it is easy to show that $H(2x) = (3/4)H(x)$. Combining this with Proposition 1.3, we have the following:

Corollary 1.4. $\lim_{x\to 0^+} H(x) = -\infty$.

It follows immediately from Proposition 1.2 and Corollary 1.4 that $F'(-(x+1/2)) > 0$ for small $x > 0$, hence Theorem 1.1 holds.

The major part of the proof is to show that $H(x) < 0$ in Proposition 1.3. We overcome the difficulty in [Lund et al. 98] by considering the Laplace transform $\Phi(t)$ of μ on A_0, which is given by an infinite product of simple functions [Dong and Lau 03]. We use Mathematica and MATLAB to help prove the following interesting fact: $0.4715 < t^\alpha\Phi(t) < 0.4795$ for all $t > 0$. (It is known that $t^\alpha\Phi(t)$ is not a constant [Dong and Lau 03, Theorem 5.6]). This small variation in the values of $t^\alpha\Phi(t)$ allows us to prove Proposition 1.3.

2. THE PROOFS

By using the scaling property $\mu(2E) = 3\mu(E)$ and the rotational invariance of μ, we have

$$\begin{aligned} H(x) &= 2\mathrm{Re}\int_{A_0}\frac{d\mu(w)}{(x+we^{\pi i/3})^2} \\ &= 2\mathrm{Re}\sum_{n=-\infty}^{\infty}\frac{1}{3^n}\int_{T_1\cup T_2}\frac{d\mu(w)}{(x+2^{-n}we^{\pi i/3})^2} \\ &= 2\mathrm{Re}\Big(\sum_{n=0}^{\infty}\frac{1}{3^n}\int_{T_1\cup T_2}\frac{d\mu(w)}{(x+2^{-n}we^{\pi i/3})^2} \\ &\quad+\sum_{n=1}^{\infty}(\frac{3}{4})^n\int_{T_1\cup T_2}\frac{d\mu(w)}{(2^{-n}x+we^{\pi i/3})^2}\Big). \end{aligned}$$

It follows that the above series converges absolutely and uniformly on each compact subset of $\mathbb{R}^+$, therefore $H(x)$ is well defined for $x > 0$ and is continuous.

Proof of Proposition 1.2: For $T = 1 - K$, we let $T_j = 1 - K_j$ where $K_j = S_jK$, $j = 0, 1, 2$. It is easy to see that $T_0 = \bigcup_{n\le -1} 2^n(T_1\cup T_2)\cup\{0\}$, as the "cap" of the Sierpinski cone A_0. Note that

$$F'(-(x+1/2)) = -\int_{K+1/2}\frac{d\mu(w)}{(x+w)^2}$$

and $K + 1/2 = (e^{\pi i/3}T_0)\cup(e^{-\pi i/3}T_0)\cup(K_0 + 1/2)$. We have

$$F(-(x+1/2)) = -H(x) + \Big(\int_{\tilde{A}} - \int_{K_0+1/2}\Big)\frac{d\mu(w)}{(x+w)^2},$$

where $\tilde{A} = (e^{\pi i/3}(A_0\backslash T_0))\cup(e^{-\pi i/3}(A_0\backslash T_0))$. Let $\psi(x)$ be the above integral. Since $\tilde{A}$ is bounded away from 0 and the integrand is integrable on $\tilde{A}$ for each $x \ge 0$ (use the same argument as in the above series expression of $H(x)$), it is easy to see that $\psi(x)$ is bounded and continuous for $x \ge 0$. □

The remaining task is to prove $H(x) < 0$ in Proposition 1.3. We need to establish a few lemmas. Let $\Phi(t) = \int_{A_0} e^{-tw}d\mu(w)$, $t > 0$ be the Laplace transform of μ on A_0 [Dong and Lau 03, page 78]. Similar to $H(x)$, it is easy to see that

$$\Phi(t) = \sum_{n=-\infty}^{\infty}\frac{1}{3^n}\int_{T_1\cup T_2} e^{-t2^{-n}w}d\mu(w)$$

and $\Phi(t)$ is continuous. In [Dong and Lau 03, Example 2], we proved that

$$\Phi(t) = \prod_{k=1}^{\infty} q(2^k t)\prod_{k=0}^{\infty}\frac{q(2^{-k}t)}{3}, \qquad t > 0, \tag{2–1}$$

where

$$q(t) = 1 + 2e^{-3t/4}\cos(\frac{\sqrt{3}t}{4}). \tag{2–2}$$

Let $\Phi_0(t) = t^\alpha\Phi(t)$. Since $\Phi_0(2t) = \Phi_0(t)$ and Φ_0 is continuous, Φ_0 is bounded on $\mathbb{R}^+$. Let

$$M = \max_{1/2\le t\le 1}\Phi_0(t) = \sup_{t\in\mathbb{R}^+}\Phi_0(t),$$
$$m = \min_{1/2\le t\le 1}\Phi_0(t) = \inf_{t\in\mathbb{R}^+}\Phi_0(t).$$

Lemma 2.1. $0.4715 < m \le M < 0.4795$.

Proof: We approximate $\Phi_0(t)$ by the finite product

$$f(t) = t^\alpha\prod_{k=1}^{4} q(2^k t)\prod_{k=0}^{5}\frac{q(2^{-k}t)}{3}. \tag{2–3}$$

For this elementary function f, we can use "fminbnd" of MATLAB to obtain the maximum and minimum estimation on $[1/2, 1]$:

$$0.4790 < f(t) < 0.4832. \tag{2–4}$$

Our main estimation is on the two truncated parts of $\Phi_0(t)$. From (2–2), we have

$$1 - 2e^{-3\cdot 2^{k-3}} \le q(2^k t) \le 1 + 2e^{-3\cdot 2^{k-3}}, \qquad 1/2 \le t \le 1.$$

Consider $1 - 2e^{-3x}$; we look for a d_1 such that

$$-d_1x^{-7} \le \log(1 - 2e^{-3x}), \quad x \ge 4.$$

By a direct differentiation of $g(x) = \log(1 - 2e^{-3x}) + d_1 x^{-7}$, we have

$$g'(x) = \frac{6}{e^{3x} - 2}\Big(1 - \frac{7d_1(e^{3x} - 2)}{6x^8}\Big), \quad x \geq 4.$$

If we take $d_1 = (6 \cdot 4^8)/(7(e^{12} - 2))$, then $g'(x) < 0$; from $g(\infty) = 0$, we conclude that $g(x) > 0$ for $x \geq 4$ as needed. Similarly we can take $d_2 = (6 \cdot 4^8)/(7(e^{12} + 2))$ so that

$$\log(1 + 2e^{-3x}) \leq d_2 x^{-7}, \quad x \geq 4.$$

Combining these estimates, we have

$$\begin{aligned} e^{-d_1'} &= e^{-d_1 \sum_{k=5}^{\infty} 2^{-7(k-3)}} \\ &\leq \prod_{k=5}^{\infty} q(2^k t) \\ &\leq e^{d_2 \sum_{k=5}^{\infty} 2^{-7(k-3)}} = e^{d_2'} \end{aligned} \tag{2–5}$$

for $1/2 \leq t \leq 1$, where $d_i' = d_i/(2^7 \cdot (2^7 - 1)),\ i = 1, 2$.

Next we estimate $\prod_{k=6}^{\infty}(q(2^{-k}t)/3)$. It is easy to check that for $0 \leq x \leq 1/64$,

$$\begin{aligned} \big(3e^{-cx} - q(x)\big)' &= \frac{3}{2}e^{-3x/4} \\ &\quad \times \Big(\frac{2\sqrt{3}}{3}\cos(\frac{\pi}{6} - \frac{\sqrt{3}x}{4}) - 2ce^{(3/4-c)x}\Big) \\ &\geq \frac{3}{2}e^{-3x/4}\big(1 - 2ce^{(3/4-c)x}\big). \end{aligned}$$

If we take $c = 2^{-1}e^{-3/256} = 0.494175\cdots$, the above expression is positive, hence

$$q(x) = 1 + \cos(\sqrt{3}x/4)e^{-3x/4} \leq 3e^{-cx}, \qquad 0 < x \leq 1/64.$$

Combining this and (5.10) in [Dong and Lau 03], we have

$$3e^{-1/2^{(k+1)}} \leq q(2^{-k}t) \leq 3e^{-c/2^{(k+1)}}$$

for $k \geq 6$ and $1/2 \leq t \leq 1$; hence

$$e^{-1/2^6} \leq \prod_{k=6}^{\infty} q(2^{-k}t)/3 \leq e^{-c/2^6}, \qquad 1/2 \leq t \leq 1. \tag{2–6}$$

By (2–1) and (2–3)–(2–6)

$$\begin{aligned} 0.4715 < 0.4790\ e^{-d_1' - 1/2^6} &< \Phi_0(t) \\ &< 0.4832\ e^{d_2' - c/2^6} < 0.4795 \end{aligned}$$

and Lemma 2.1 follows. □

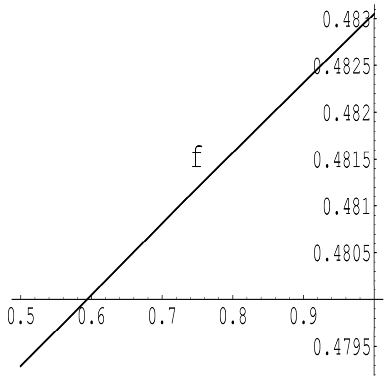

FIGURE 4. $f(t)$ for $\frac{1}{2} \leq t \leq 1$.

We remark that the choice of the number of factors in $f(t)$ and the x^{-7} are by trial and error so as to get two bounds accurate enough to fit in Lemma 2.3 in the sequel to get a positive value. We also remark that for the $f(t)$ in (2–3), we can actually show that $f'(t) > 0$ for $1/2 \leq t \leq 1$, hence $f(1/2) \leq f(t) \leq f(1)$ for $t \in [1/2, 1]$ (see Figure 4). However, the proof is lengthy and does not have much significance, so the above MATLAB approximation is enough for our purpose.

Lemma 2.2. *There exists a constant $C > 0$ such that for $x > 0$,*

$$\begin{aligned} x^{2-\alpha}H(x) &= \\ &- C\int_0^{\infty} \Phi_0\big(\frac{2\pi t}{\sqrt{3}x}\big)\ t^{1-\alpha}e^{-\pi t/\sqrt{3}}\sin(\frac{\pi}{6} - \pi t)\ dt \\ &:= -C\phi(x). \end{aligned}$$

Proof: Let $x > 0$ be fixed. Using integration by parts, we have

$$\begin{aligned} &\int_{A_0}\int_0^{\infty} |te^{-t(w+xe^{-\pi i/3})}|dt\ d\mu(w) \\ &\qquad = \int_{A_0} \frac{1}{(\mathrm{Re}w + x/2)^2}d\mu(w) < +\infty. \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} \int_0^{\infty} \Phi(t)te^{-txe^{-\pi i/3}}dt &= \int_{A_0}\Big(\int_0^{\infty} te^{-t(w+xe^{-\pi i/3})}dt\Big)d\mu(w) \\ &= \int_{A_0} \frac{d\mu(w)}{(w + xe^{-\pi i/3})^2}. \end{aligned}$$

It follows from the definition of $H(x)$ that

$$\begin{aligned}H(x) &= 2\mathrm{Re}\int_{A_0} \frac{d\mu(w)}{(x+we^{\pi i/3})^2} \\ &= 2\mathrm{Re}\Big(e^{-2\pi i/3}\int_{A_0} \frac{d\mu(w)}{(w+xe^{-\pi i/3})^2}\Big) \\ &= -2\mathrm{Re}\int_0^\infty \Phi(t)\ te^{\pi i/3 - txe^{-\pi i/3}}\ dt \\ &= -2\int_0^\infty \Phi(t)\ te^{-tx/2}\cos\big(\frac{\pi}{3}+\frac{\sqrt{3}tx}{2}\big)\ dt \\ &= -Cx^{\alpha-2}\int_0^\infty \Phi_0\big(\frac{2\pi t}{\sqrt{3}x}\big)\ t^{1-\alpha}e^{-\pi t/\sqrt{3}} \\ &\qquad \times \sin\big(\frac{\pi}{6}-\pi t\big)dt.\end{aligned}$$

The last equality follows by a change of variable and by replacing Φ with Φ_0. □

Lemma 2.3. *Let $\phi(x)$ be the integral given in Lemma 2.2 and let*

$$\begin{aligned}a &= \int_0^{1/6} t^{1-\alpha}e^{-\pi t/\sqrt{3}}\sin(\frac{\pi}{6}-\pi t)\ dt, \\ b &= \int_{1/6}^{7/6} t^{1-\alpha}e^{-\pi t/\sqrt{3}}\sin(\pi t-\frac{\pi}{6})\ dt.\end{aligned}$$

Then $\phi(x) > ma - Mb$ for all $x > 0$.

Proof: Let

$$t_n = t + n + 1/6,$$

and let

$$c_n = \int_0^1 t_n^{1-\alpha}e^{-\pi t/\sqrt{3}}\sin(\pi t)dt.$$

Obviously $c_n > c_{n+1} > 0$. By using the 2π periodicity of the sine function, we have

$$\begin{aligned}\phi(x) &= \\ &\Big(\int_0^{1/6}+\int_{1/6}^{7/6}\Big)\ \Phi_0\big(\frac{2\pi t}{\sqrt{3}x}\big)\ t^{1-\alpha}e^{-\pi t/\sqrt{3}}\sin(\frac{\pi}{6}-\pi t)\ dt \\ &\quad + e^{-\pi/(6\sqrt{3})}\sum_{n=1}^\infty(-1)^{n-1}e^{-n\pi/\sqrt{3}} \\ &\qquad \times\int_0^1 \Phi_0\big(\frac{2\pi t_n}{\sqrt{3}x}\big)\ t_n^{1-\alpha}e^{-\pi t/\sqrt{3}}\sin(\pi t)\ dt \\ &> ma - Mb \\ &\quad + (m - Me^{-\pi/\sqrt{3}})e^{-\pi/(6\sqrt{3})}\sum_{k=1}^\infty e^{-(2k-1)\pi/\sqrt{3}}c_{2k-1}.\end{aligned}$$

Lemma 2.1 implies that the last term is positive. Therefore $\phi(x) > ma - Mb$. □

Proof of Proposition 1.3: The continuity follows from the remark in the beginning of this section. We use Mathematica to estimate the two constants a and b in Lemma 2.3: $a > 0.3890$, $b < 0.3270$. This together with Lemma 2.1 implies that $\phi(x) > ma - Mb > 0.025$. By Lemma 2.2, $H(x) < 0$ for $x > 0$. □

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X. H. Dong, Department of Mathematics, Hunan Normal University, Changsha, 410081, China and Department of Mathematics, The Chinese University of Hong Kong, Hong Kong (xhdong@hunnu.edu.cn)

K. S. Lau, Department of Mathematics, The Chinese University of Hong Kong, Hong Kong (kslau@math.cuhk.edu.hk)

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