

# Optimal combined quadrature formulas in Schmeisser's sense

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## Abstract

In this paper we study the optimal combined quadrature formulas in Schmeisser's sense ([8]).

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## 1 Combined quadrature formulas

In [1] - [3] we introduced the combined quadrature formulas.

We consider the family of elementary quadrature formulas ([3])

$$(1) \quad \int_a^b f(x)dx = \sum_{h=0}^{n-1} \sum_{i=0}^{m_j} A_{h,i}^{[j]} f^{(h)}(x_{i,j}) + R_j(f)$$

with

$$a = x_{0,j} \leq x_{1,j} < x_{2,j} < \dots < x_{m_j,j} \leq x_{m_j+1,j} = b$$

and

$$R_j(x^k) = 0, k = 0, 1, 2, \dots, n - 1$$

for  $j = 1, 2, \dots, r$ . It results that the elementary quadrature formulas (1) have the algebraical degree of exactness  $n - 1$ .

Now, we divide the interval  $[a, b]$  by the points

$$(2) \quad a = u_0 < u_1 < \dots < u_{r-1} < u_r = b$$

into the subintervals  $[d_{j-1}, u_j]$ ,  $j = \overline{1, r}$ , having the length  $d_j = u_j - u_{j-1}$ ,  $j = \overline{1, r}$ . Having in view the identity

$$\int_a^b f(x)dx = \sum_{j=1}^r \int_{a_{j-1}}^{a_j} f(x)dx$$

and computing the integral  $\int_{d_{j-1}}^{u_j} f(x)dx$  with the quadrature formula  $j$  by the family (1) of the quadrature formulas,  $j = \overline{1, r}$ , we obtain the quadrature formula

$$(3) \quad \int_a^b f(x)dx = \sum_{j=1}^r \sum_{n=0}^{n-1} \sum_{i=0}^{m_j} \left( \frac{d_j}{b-a} \right)^{h+1} A_{h,i}^{[j]} f^{(h)} \left( u_{j-1} + d_j \frac{x_{i,j} - a}{b-a} \right) + \varphi(t)$$

with

$$(4) \quad \rho(f) = \sum_{j=1}^r \frac{d_j}{b-a} R_j \left( f \left( u_{j-1} + d_j \frac{x-a}{b-a} \right) \right)$$

The rule (3) with the remainder given (4) we call it the **combined quadrature formula** connected to the family of the elementary formula (1).

**Remark 1.** Every permutation of the elementary quadrature rules by the family (1) determines a combined quadrature formula.

**Remark 2.** Evidently, when the all  $r$  the rule form the family (1) coincide with the same elementary quadrature formula, then the combined quadrature formulas reduces to the **generalized composed quadrature formulas** which was studied in [5].

**Remark 3.** The combined quadrature formula (3) has the algebraical degree of exactness  $n - 1$ .

Now, we suppose the function  $f$  to be from  $C^n[a, b]$  - the set of all functions  $f$  having on the interval  $[a, b]$  continuous derivatives up to the order  $n$ .

**Theorem 1.** If every influence function  $\phi_j(x)$ , (see [5]), corresponding to the quadrature formula  $j$ ,  $j = \overline{1, r}$ , by the family (1) is semidefinite and sign  $\phi_1(x) = \text{sign } \phi_2(x) = \dots = \text{sign } \phi_r(x)$ , for any  $x$  from  $[a, b]$ , then for  $f \in C^n[a, b]$  the remainder of combined quadrature formula (3) has the form:

$$(5) \quad \rho(f) = \left[ \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \int_a^b \phi_j(x) dx \right] f^{(n)}(\xi), \quad \xi \in [a, b]$$

**Proof.** From (4) and the asumption of the theorem we have:

$$\begin{aligned} \rho(f) &= \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \int_a^b \phi_j(x) f^{(n)} \left( u_{j-1} + d_j \frac{x-a}{b-a} \right) dx = \\ &= \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} f^{(n)}(\xi_j) \int_a^b \phi_j(x) dx = \end{aligned}$$

$$\begin{aligned}
&= \left[ \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \int_a^b \phi_j(x) dx \right] \cdot \frac{\sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \left| \int_a^b \phi_j(x) dx \right| f^{(n)}(\xi_j)}{\sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \left| \int_a^b \phi_j(x) dx \right|} = \\
&= \left[ \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \int_a^b \phi_j(x) dx \right] f^{(n)}(\xi), \quad \xi \in (a, b).
\end{aligned}$$

## 2 Optimal combined quadrature formulas in Schmeisser's sense

From Peano's result we have that if the influence function (Peano's kernel) is semidefinite (it has constant sign), then the remainder of quadrature formula with the algebraical degree of exactness  $n - 1$  has the form

$$(6) \quad R_n(f) = C f^{(n)}(\xi), \quad \xi \in [a, b]$$

(see [4], [5], [6]).

In [8] G. Schmeisser formulated the problem of finding the quadrature formula for which  $C$  has a minimum value.

We observe that in the conditions of the Theorem 1 the remainder of combined quadrature formulas (3) is the form (6).

For to find the optimal combined quadrature formula in Schmeisser's sense among the quadrature formulas (3) we must determine the parameters  $d_1, d_2, \dots, d_r$  with

$$\sum_{i=1}^r \frac{d_i}{b-a} = 1,$$

such that the expression:

$$|C| = \sum_{i=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \int_a^b |\phi_j(x)| dx$$

has a minimum value. We have a problem of conditional extremum. Using the method of lagrange multipliers we find:

**Theorem 2.** *If are verified the conditions of Theorem 1, then for*

$$(7) \quad d_i = \frac{b-a}{\sqrt[n]{\int_a^b |\phi_i(x)| dx} \cdot \sum_{i=1}^r \frac{1}{\sqrt[n]{\int_a^b |\phi_j(x)| dx}}}, \quad i = \overline{1, r}$$

we obtain the optimal combined quadrature rule in Schmeisser sense with

$$(8) \quad \min_{d_1, d_2, \dots, d_r} \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \int_a^b |\phi_j(x)| dx =$$

$$= \frac{1}{\left( \sum_{j=1}^r \frac{1}{\sqrt[n]{\int_a^b |\phi_j(x)| dx}} \right)^n}$$

### 3 Particular cases

#### 3.1 Generalized quadrature formulae

From Theorem 2 for the generalized quadrature formulae we find

$$d_1 = d_2 = \dots = d_r = \frac{b-a}{r}$$

and

$$\begin{aligned} \min_{d_1, d_2, \dots, d_r} \left[ \sum_{j=1}^r \left( \frac{d_j}{b-a} \right)^{n+1} \right] \int_a^b |\phi(x)| dx = \\ = \frac{1}{r^n} \int_a^b |\phi(x)| dx, \end{aligned}$$

where  $\phi(x)$  are the influence functions corresponding to the quadrature formula which generates the generalized quadrature formula.

### 3.2 Combined quadrature formula by type Simpson - Newton

In [1] (see and [3]) we introduced a combined quadrature formula by type Simpson - Newton.

Such, by applying the Simpson's formula

$$\begin{aligned} \int_a^b f(x) dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) \right] - \\ - \frac{(b-a)^5}{2880} f^{(iv)}(\xi_1), \quad \xi_1 \in [a, b] \end{aligned}$$

to the interval  $[d_{j-1}, u_j]$ ,  $j = \overline{1, k}$ ,  $0 \leq k \leq r$ , and the Newton's formula

$$\begin{aligned} \int_a^b f(x) dx = \frac{b-a}{8} \left[ f(a) + 3f\left(a + \frac{b-a}{3}\right) + 3f\left(a + \frac{2(b-a)}{3}\right) + f(b) \right] - \\ - \frac{(b-a)^5}{6480} f^{(iv)}(\xi_2), \quad \xi_2 \in (a, b) \end{aligned}$$

to  $[u_{j-1}, u_j]$ ,  $j = \overline{k+1, n}$ , where  $f \in C^4[a, b]$ , we obtain the combined quadrature formula

$$(9) \quad \int_a^b f(x) dx = \frac{d_1}{a} f(a) + \sum_{j=1}^{k-1} \frac{d_j + d_{j+1}}{8} f(a + d_1 + \dots + d_j) +$$

$$\begin{aligned}
& + \sum_{j=1}^k \frac{2d_j}{3} f\left(a + d_j + \dots + d_{j-1} + \frac{d_j}{2}\right) + \\
& + \left(\frac{d_k}{6} + \frac{d_{k+1}}{8}\right) f(a + d_1 + \dots + d_k) + \\
& + \sum_{j=k+1}^{r-1} \frac{d_j + d_{j+1}}{8} f(a + d_1 + \dots + d_j) + \\
& + \sum_{j=k+1}^r \frac{3d_j}{8} f\left(a + d_1 + \dots + d_{j-1} + \frac{d_j}{3}\right) + \\
& + \sum_{j=k+1}^r \frac{3d_j}{8} f\left(a + d_1 + \dots + d_{j-1} + \frac{2d_j}{3}\right) + \frac{d_r}{8} f(b) + \rho_{S_k, N_{r-k}}(f),
\end{aligned}$$

with

$$\rho_{S_k, N_{r-k}}(f) = -\frac{1}{6!} \left( \frac{1}{4} \sum_{j=1}^k d_j^5 + \frac{1}{9} \sum_{j=k+1}^r d_j^5 \right) f^{(iv)}(\xi), \quad \xi \in (a, b).$$

Using the Theorem 2 we find: among the all the combined quadrature formulas (9) that which is optimal in Schmeisser's sense is given by:

$$d_i = \frac{\sqrt{2}(b-a)}{k\sqrt{2} + (r-k)\sqrt{3}}, \quad i = \overline{1, k}$$

$$d_i = \frac{\sqrt{2}(b-a)}{k\sqrt{2} + (r-k)\sqrt{3}}, \quad i = \overline{k+1, r}$$

with

$$\min_{d_1, \dots, d_r} \frac{1}{6!} \left[ \frac{1}{4} \sum_{j=1}^k d_j^5 + \frac{1}{9} \sum_{j=1}^{k+1} d_j^5 \right] = \frac{(b-a)^5}{6! [k\sqrt{2} + (r-k)\sqrt{3}]^4}, \quad h = \overline{0, r}.$$

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