General Mathematics Vol. 12, No. 2 (2004), 47–52

Inequalities for logarithmic and exponential functions

Eugen Constantinescu

Abstract

In [10] it was announced the precise inequality : for $x \geq \frac{1}{4}$

$$\ln\left(\frac{12x^2+6x+1-\frac{1}{60x^2+2}}{12x^2-6x+1-\frac{1}{60x^2+2}}\right) < \frac{1}{x} < \ln\left(\frac{12x^2+6x+1-\frac{1}{60x^2+1}}{12x^2-6x+1-\frac{1}{60x^2+1}}\right).$$

The aim of the paper is to improve as well as to give a generalization of above inequalities, see [4].

2000 Mathematical Subject Classification: 26D05,26D07,33B15

Let $A(x) := 12x^2 + 6x + 1$, $\quad b(x) = \dfrac{1}{60x^2+2}$, $\quad B(x) = \dfrac{1}{60x^2+1}$.
A question raised by Slavko Simić in [10] is to prove following nice inequalities

$$\boxed{\ln\left(\frac{A(x)-b(x)}{A(-x)-b(x)}\right) < \frac{1}{x} < \ln\left(\frac{A(x)-B(x)}{A(-x)-B(x)}\right)}, \quad x \geq \frac{1}{4}. \tag{1}$$

Further we need following polynomials

$$Q_3(x) := 120x^3+60x^2+12x+1 \ , \quad Q_4(x) := 1680x^4+840x^3+180x^2+20x+1.$$

Observe that for $x \geq [\frac{1}{4}, \infty)$ we have

$$\frac{A(x)-b(x)}{A(-x)-b(x)} < \frac{Q_4(x)}{Q_4(-x)} \quad \text{and} \quad \frac{A(x)-B(x)}{A(-x)-B(x)} = \frac{Q_3(x)}{-Q_3(-x)} \; .$$

In fact $\dfrac{Q_4(x)}{Q_4(-x)} - \dfrac{A(x)-b(x)}{A(-x)-b(x)} = \dfrac{4096x}{C(x)D(x)}$ where

$$\begin{aligned} C(x) &= (4x-1)^2(1680x^2+75)+70(4x-1)+11 \\ D(x) &= (4x-1)^2(720x^2+39)+30(4x-1)+7 \; . \end{aligned}$$

We shall prove the following ,,better" estimate

$$\ln\left(\frac{Q_4(x)}{Q_4(-x)}\right) < \frac{1}{x} < \ln\left(\frac{Q_3(x)}{-Q_3(-x)}\right) \; , \; x \geq \frac{1}{4}. \tag{2}$$

However, inequalities (2) are equivalent with case $n=2, m=1$ of a more general result:

Theorem 1. *Suppose that* $c_k(n) := \binom{n}{k}\frac{(2n-k)!}{n!} = \frac{(2n)!}{n!}\frac{(-n)_k}{(-2n)_k}$ *and let*

$$\boxed{Q_n(x) := \sum_{k=0}^{n} c_k(n) x^{n-k}} \; . \tag{3}$$

If n, m are positive integers, then for $x \geq \dfrac{1}{2m+2}$ *the following inequalities are valid*

$$\boxed{\frac{Q_{2n}(x)}{Q_{2n}(-x)} < e^{1/x} < \frac{Q_{2m+1}(x)}{-Q_{2m+1}(-x)}} \; . \tag{4}$$

Proof. Note that $Q_n(x) = \dfrac{(2n)!}{n!}x^n + \ldots + 1$. The first polynomials Q_n are:

$$Q_0(x) = 1 \; , \; Q_1(x) = 2x+1 \; , \; Q_2(x) = 12x^2+6x+1$$
$$Q_3(x) = 120x^3+60x^2+12x+1.$$

These polynomials are connected to some special polynomials as : Legendre polynomial,Laguerre polynomial, Bessel polynomial (see [1],[2],[8]). For instance, following integral representations are valid

$$Q_n(x) = \int_0^\infty e^{-t} P_n(1+2tx)\,dt = \frac{(-1)^n n!}{(2n)!x^n} \int_0^\infty e^{-t}(1+tx)^{2n} L_n(t)\,dt$$

where P_n is the Legendre polynomial and L_n denotes Laguerre polynomial. Likewise

$$(5)\qquad Q_n(x) = \frac{1}{n!}\int_0^\infty e^{-t} t^n (1+tx)^n\,dt = \frac{(2n)!}{n!} x^n {}_1F_1(-n\,; -2n\,; 1/x)\ .$$

Some properties of polynomials Q_n are listed in following proposition:

Lemma 1. *Let $(Q_n)_{n\geq 1}$ be the polynomial sequence defined as in (3). Then for $n = 1, 2, ...$*

- (Recurrence relation): $Q_{n+1}(x) = (4n+2)xQ_n(x) + Q_{n-1}(x)\ .$
- (The roots): *Q_{2n} has not real roots. Q_{2n+1} has only one real root, namely in $(-\infty, 0)$. If $Q_n(x_j) = 0$, then*

$$\frac{1}{n(n+1)} \leq |x_k| \leq \frac{1}{n+1}\ , \qquad \mathrm{Re}(x_k) < 0\ , \quad k \in \{1, 2, ..., n\}.$$

- (Two identities): $Q_n(x)Q_n(y) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}(x+y)^k Q_k\left(\frac{xy}{x+y}\right),$

$$(6)\qquad \begin{cases} (-1)^n Q_n(-x)e^{1/x} = Q_n(x) + \varepsilon_n(x), \\ \text{where}\quad \varepsilon_n(x) := \dfrac{(-1)^n}{(2n)!} \displaystyle\int_0^{1/x} t^n(1-tx)^n e^t\,dt. \end{cases}$$

- (Approximating $e^{1/x}$): *If* $Q_n(x) \neq 0$, *then*

$$e^{1/x} = \frac{Q_n(x)}{(-1)^n Q_n(-x)} + r_n(x) \tag{7}$$

where

$$r_n(x) := \frac{\varepsilon_n(x)}{(-1)^n Q_n(-x)} = (-1)^n e^{1/x} \int_x^{\infty} \frac{e^{-1/t}}{|tQ_n(-t)|^2}\, dt \ .$$

The identity (6) is obtained using repeated integrating by parts (see [4]-[5]). Equality (7) is a consequence of (6). The roots are investigated using Eneström -Kakeya theorem (see [5] or N.Obreschkoff's monograph [7].) Other identities were established using theory of Special Functions[1],[2],[8]. From above Lemma the proof of Theorem is complete (see (6)-(7).

In case $n = 3$, $m = 2$ from (4) we find

Corollary 1. *For* $x \geq \frac{1}{6}$ *the following inequalities hold :*

$$\ln\left(\frac{665280x^6 + 332640x^5 + 75600x^4 + 10080x^3 + 840x^2 + 42x + 1}{665280x^6 - 332640x^5 + 75600x^4 - 10080x^3 + 840x^2 - 42x + 1}\right) <$$

$$< \frac{1}{x} < \ln\left(\frac{30240x^5 + 15120x^4 + 3360x^3 + 420x^2 + 30x + 1}{30240x^5 - 15120x^4 + 3360x^3 - 420x^2 + 30x - 1}\right)$$

When $x = 1$, the recurrence relation from Lemma 1 implies

Corollary 2. *If* $(Y_n(\lambda))_{n=0}^{\infty}$ *is defined by* $Y_0(\lambda) = 1$, $Y_1(\lambda) = \lambda$, *and*

$$\boxed{Y_{n+1}(\lambda) = (4n+2)Y_n(\lambda) + Y_{n-1}(\lambda)} \ , \ n = 1, 2, ...,$$

then for n large we have $\dfrac{Y_n(3)}{Y_n(1)} \approx e$. *More precisely*

$$\frac{Y_{2n}(3)}{Y_{2n}(1)} < e < \frac{Y_{2m+1}(3)}{Y_{2m+1}(1)} \ , \ \ n, m = 0, 1, 2, ...$$

and

$$\left|e - \frac{Y_n(3)}{Y_n(1)}\right| < 4\left(\frac{e}{4n}\right)^{2n+1} \ , \ (n \geq 4).$$

Observe that $\frac{Y_n(3)}{Y_n(1)}$ is a rational number. If $n = 1000$, then $Y_n(3)$ is an integer having 3169 digits and $Y_n(1)$ is an integer with 3168 digits. Using a PC, approximation $\frac{Y_{1000}(3)}{Y_{1000}(1)} \approx e$ gives 6338 correct digits of Napier's constant ,, e ".

References

[1] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G., *Higher Transcendental Functions, vol.I- vol. III= Bateman Manuscript Project.* McGraw-Hill, New York, **1953**.

[2] Grosswald E., *Bessel Polynomials.* Lecture Notes in Mathematics- 698, Springer Verlag, Berlin, **1978**.

[3] Hummel P.M., Seebeck C.L., *A generalization of Taylor's expansion.* Amer.Math.Monthly **56**(1949) 243-247.

[4] Lupaş A., *Calculul valorilor unor funcţii elementare (I).* Gazeta Matematică (seria A) , **VII**, 1, (1986) 15-26.

[5] Lupaş A., *Numerical Methods.*(Romanian), Ed.Constant, Sibiu,2001.

[6] Obreschkoff N., *Neue Quadraturformeln.* Abhandlugen der Preussischen Akademie der Wissenschaften, (1940), 6-26.

[7] Obreschkoff N., *Verteilung und Berechnung der Nullstellen reeler Polynome.* VEB , Band 55 , Berlin, **1963**.

[8] Rainville D., *Special Functions.* MacMillan, New York, **1960**.

[9] Saff E.B. , Varga R.S., *On the zeros and poles of Padé approximants to* e^z *(II) .* in E.B.Saff & R.S.Varga (eds.), ,, Padé and Rational Approximation", Academic Press, New York,pp.195-214, **1976** .

[10] Simić S. , *Problem 55*,Publikacije Elektrotehničkog Fakulteta, Serija: Matematika 14(2003) p.112.

,,Lucian Blaga" University of Sibiu
Faculty of Sciences
Department of Mathematics
Str. I.Raţiu 7,
550012-Sibiu, Romania
E-mail: *egnconst68@yahoo.com*