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Some Diophantine Equations of the Form $ax^2 + pxy + by^2 = z^{k^n}$

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In this paper two special cases of the above diophantine equation are studied, i.e. the equations $ax^2+by^2 = z^{3^n}$ and $x^2+pxy+y^2 = z^{2^n}$. For these equations families of integral solutions such that x and y are relatively prime are determined by using the equations $ax^2+by^2 = z^3$ (see [5]) and $x^2 + pxy + y^2 = z^2$ (see [2]).

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1 Introduction

We begin by considering few particular diophantine equations. Let us consider the equation

(1) $$x^2 + y^2 = z^3.$$

Using the algebraic identity

$$(u(u^2 - 3v^2))^2 + (v(3u^2 - v^2))^2 = (u^2 + v^2)^3, \tag{2}$$

it follows that equation (1) has infinitely many solutions in positive integers such that $gcd(x, y) = 1$. Indeed, it is sufficient to choose the positive integers u or v of different parities and satisfying $gcd(u, v) = 1$.

In a similar way, the solutions in positive integers of equation

$$x^2 - y^2 = z^3, \tag{3}$$

can be obtained from the identity

$$(u(u^2 + 3v^2))^2 - (v(3u^2 + v^2))^2 = (u^2 - v^2)^3. \tag{4}$$

A. Schinzel has proved that all solutions (z, y, z) in positive integer for equation

$$x^2 + 2y^2 = z^3, \tag{5}$$

where x and y are relatively primes, can be obtained from the algebraic identity

$$(r(r^2 - 6s^2))^2 + 2(s(3r^2 - 2s^2))^2 = (r^2 + 2s^2)^3, \tag{6}$$

where $gcd(r, 2s) = 1$.

In the book [2] is studied the remarkable equation

$$x^2 + pxy + y^2 = z^2, \tag{7}$$

where p is a fixed integer. The well-known pythagorean equation is obtained for $p = 0$. Science the solutions of the equation the following result has been proved: all integral solutions to (7) are given by

$$\text{(8)} \qquad \begin{cases} x = k(pn^2 - 2mn) \\ y = k(m^2 - n^2) \\ z = k(pmn - m^2 - n^2) \end{cases},$$

where k, m, n are integral parameters. The factor k appears because the equation is homogeneous. Replace m by $-m$, we get other form of solutions:

$$\text{(9)} \qquad \begin{cases} x = k(pn^2 + 2mn) \\ y = k(m^2 - n^2) \\ z = k(m^2 + pmn + n^2) \end{cases}.$$

Also, in the book [2] some special cases of equation (7) are analyzed as well as some equations in many variables connected to (7) are discussed.

The main purpose of this paper is the study of two special cases of the general diophantine equation

$$\text{(10)} \qquad ax^2 + pxy + by^2 = z^{k^n}.$$

2 The equation $ax^2 + by^2 = z^{3^n}$

Consider the equation

$$\text{(11)} \qquad ax^2 + by^2 = z^{3^n},$$

where a, b are integers with $gcd(a, b) = 1$ and n is a positive integer. The equation (1) is obtained for $a = b = 1$ and $n = 1$, in the equation (3) we have $a = 1$, $b = -1$ and $n = 1$, and in (5) we have $a = 1, b = 2$ and $n = 1$. Therefore, these are special cases of equation (11).

Let us consider first the equation

$$ax_1^2 + by_1^2 = z_1^3, \tag{12}$$

and let us use the identity

$$a(u(au^2 - 3bv^2))^2 + b(v(3au^2 - bv^2))^2 = (au^2 + bv)^3 \tag{13}$$

in order to get the solutions

$$\begin{cases} x_1 = u(au^2 - 3bv^2) \\ y_1 = v(3au^2 - bv^2) \\ z_1 = au^2 + bv^2 \end{cases}, \tag{14}$$

where u and v are arbitrary integers.

We are interested to construct solutions of (12) with the property $gcd(x_1, y_1) = 1$. In this respect we impose the following conditions:

$$(C_1) \qquad gcd(au, bv) = 1 \text{ and } abuv \equiv 0 \pmod 2.$$

Now, let us show that x_1 and y_1 also satisfy the conditions (C_1). Indeed, we have $gcd(au, v) = gcd(au, 3au^2 - bv^2) = 1$ and $gcd(bv, u) = $ $= gcd(bv, au^2 - 3bv^2) = 1$. From the relation

$$3au^2 - bv^2 = 3(au^2 - 3bv^2) + 8bv^2,$$

using the fact that $8bv^2$ and $au^2 - 3bv^2$ are relatively primes, it follows that $gcd(3au^2 - bv^2, au^2 - 3bv^2) = 1$. Also, form this property and from (14) it is clear that $abx_1y_1 \equiv 0 \pmod 2$. Hence, the integers x_1 and y_1 also satisfying (C_1).

Now, we choose the integers

$$(15) \qquad \begin{cases} x_2 = x_1(ax_1^2 - 3by_1^2) \\ y_2 = y_1(3ax_1^2 - by_1^2) \\ z_2 = ax_1^2 + by_1^2 \end{cases} .$$

Because the integers x_1, y_1 satisfy (C_1), it follows that the integers x_2, y_2 have the same property, and from (15) we get

$$(16) \qquad ax_2^2 + by_2^2 = z_2^3.$$

Taking into account that $z_2 = ax_1^2 + by_1^2 = z_1^3$, we obtain

$$(17) \qquad ax_2^2 + by_2^2 = z_1^9 \text{ and } gcd(x_2, y_2) = 1.$$

By continuing this procedure we can construct recursively a family of solutions to equation (11) with the property $gcd(x, y) = 1$. Let x_{j-1}, y_{j-1} be integers such that

$$(18) \qquad ax_{j-1}^2 + by_{j-1}^2 = z_1^{3^{j-1}}, \ j \geq 2$$

and which satisfy conditions (C_1), i.e. $gcd(ax_{j-1}, by_{j-1}) = 1$ and $abx_{j-1}y_{j-1} \equiv 0 \pmod 2$.

Define

$$(19) \qquad \begin{cases} x_j = x_{j-1}(ax_{j-1}^2 - 3by_{j-1}^2) \\ y_j = y_{j-1}(3ax_{j-1}^2 - by_{j-1}^2) \\ z_j = ax_{j-1}^2 + by_{j-1}^2 \end{cases}$$

We have $gcd(ax_j, by_j) = 1$, $abx_jy_j \equiv 0 \pmod 2$ and $ax_j^2 + by_j^2 = z_j^3 =$ $= (ax_{j-1}^2 + by_{j-1}^2)^3$. Using (18) it follows

$$(20) \qquad ax_j^2 + by_j^2 = (z_1^{3^{j-1}})^3 = z_1^3.$$

From $gcd(ax_j, by_j) = 1$ we get $gcd(x_j, y_j) = 1$, hence the following result is proved:

Theorem 1. *The equation (11) has infinitely many solutions in integers (or in positive integers) such that $gcd(x, y) = 1$. An infinite family of solutions is given by $(x_n(u, v), y_n(u, v), z_1(u, v))$ where $x_n(u, v)$ and $y_n(u, v)$ are constructed by the previous algorithm.*

We will indicate the effective construction of an infinite family of integral solutions to the equation

$$x^2 + y^2 = z^9, \tag{21}$$

such that $gcd(x, y) = 1$.

In this case we have $a = b = 1$ and $n = 2$. Take u, v arbitrary integers such that $gcd(u, v) = 1$ and $uv \equiv 0 \pmod 2$. Then

$$\begin{cases} x_1 = u(u^2 - 3v^2) \\ y_1 = v(3u^2 - v^2) \\ z_1 = u^2 + v^2 \end{cases} \tag{22}$$

and

$$\begin{cases} x_2 = x_1(x_1^2 - 3y_1^2) = u(u^2 - 3v^2)[u^2(u^2 - 3v^2)^2 - 3v^2(3u^2 - v^2)^2] \\ y_2 = y_1(3x_1^2 - y_1^2) = v(3u^2 - v^2)[3u^2(u^2 - 3v^2)^2 - v^2(3u^2 - v^2)^2] \\ z_2 = x_1^2 + y_1^2 = u^2(u^2 - 3v^2)^2 + v^2(3u^2 - v^2)^2. \end{cases} \tag{23}$$

For instance, if $u = 2$ and $v = 1$, we obtain solution $x = 1199$, $y = 718$, $z = 5$ and $gcd(x, y) = 1$.

3 The equation $x^2 + pxy + y^2 = z^{2^n}$

Consider the equation

$$x^2 + pxy + y^2 = z^{2^n} \tag{24}$$

where p is an integer and n is a positive integer. We will construct an infinite family of solutions (x, y, z) such that $gcd(x, y) = 1$.

Let us consider first the equation (see [2])

$$x_1^2 + px_1y_1 + y_1^2 = z_1^2. \tag{25}$$

This equation is symmetric in x_1 and y_1, and all its integral solutions are

$$\begin{cases} x_1 = pu^2 + 2uv \\ y_1 = v^2 - u^2 \\ z_1 = u^2 + puv + v^2 \end{cases}, \tag{26}$$

where u, v are arbitrary integral parameters.

Assume that the following conditions are satisfies

$$(C_2) \qquad gcd(u, v) = 1 \text{ and } p^2 - 4|u.$$

From these two conditions we obtain $gcd(v, p \pm 2) = 1$.

Let us show that x_1, y_1 in (26) also satisfying the conditions (C_2). Indeed, we have $x_1 = u(pu + 2v)$, $y_1 = (v + u)(v - u)$ and $gcd(u, v \pm u) = 1$. Also, from relations

$$pu + 2v = p(u + v) + (2 - p)v = p(u - v) + (2 + p)^v, \quad gcd(u + v, (2 - p)v) = 1$$

and

$$(u - v, (2 + p)v) = 1,$$

we obtain $gcd(pu+2v, u \pm v) = 1$, i.e. $gcd(pu+2v, u^2-v^2) = 1$. It follows $gcd(x_1, y_1) = 1$ and from $x_1 = u(pu+2v)$, $p^2-4|u$, we get $p^2-4|x_1$. Therefore x_1, y_1 satisfy conditions (C_2).

We consider now the equation

$$x_2^2 + px_2y_2 + y_2^2 = z_2^2, \tag{27}$$

with solutions

$$\begin{cases} x_2 = px_1^2 + 2x_1y_1 \\ y_2 = y_1^2 - x_1^2 \\ z_2 = x_1^2 + px_1y_1 + y_1^2 \end{cases}, \tag{28}$$

where x_1, y_1 satisfy equation (25) and conditions (C_2).

It follows that x_2, y_2 also verify conditions (C_2), we have

$$x_2^2 + px_2y_2 + y_2^2 = z_2^2 = (x_1^2 + px_1y_1 + y_1^2)^2 = z_1^4 \quad \text{and} \quad gcd(x_2, y_2) = 1. \tag{29}$$

Recursively, consider the equation:

$$x_j^2 + px_jy_j + y_j^2 = z_1^2, \tag{30}$$

with solution (x_j, y_j, z_1), where x_j, y_j satisfy the conditions (C_2). Define

$$\begin{cases} x_{j+1} = px_j^2 + 2x_jy_j \\ y_{j+1} = y_j^2 - x_j^2 \\ z_{j+1} = x_j^2 + px_jy_j + y_j^2 \end{cases}. \tag{31}$$

It is clear that x_{j+1}, y_{j+1} satisfy conditions (C_2) and we have

$$x_{j+1}^2 + px_{j+1}y_{j+1} + y_{j+1}^2 = (x_j^2 + px_jy_j + y_j^2)^2 = z_1^{2^{j+1}},$$

where $gcd(x_{j+1}, y_{j+1}) = 1$.

We obtained the following result:

Theorem 2. *The equation (24) has infinitely many solutions in integers such that* $gcd(x, y) = 1$. *As infinite family of integral solutions is given by* $(x_n(u, v), y_n(u, v), z_1(u, v))$, *where* $x_n(u, v)$ *and* $y_n(u, v)$ *are constructed by the algorithm described above.*

We will indicate the construction of an infinite family of integral solutions to the equation

$$x^2 + xy + y^2 = z^4 \tag{32}$$

In this case we have $p = 1$ and $n = 2$. Consider u, v integers such that $gcd(u, v) = 1$ and $u \equiv 0 \pmod 2$. Then

$$\begin{cases} x_1 = u^2 + 2uv \\ y_1 = v^2 - u^2 \\ z_1 = u^2 + uv + v^2 \end{cases}, \tag{33}$$

and

$$\begin{cases} x_2 = x_1^2 + 2x_1y_1 = (u^2 + 2uv)^2 + 2(u^2 + 2uv)(v^2 - u^2) \\ y_2 = y_1^2 - x_1^2 = (v^2 - u^2)^2 - (u^2 + 2uv)^2 \\ z_2 = x_1^2 + x_1y_1 + y_1^2 = (u^2 + 2uv)^2 + (u^2 + 2uv)(v^2 - u^2) + (v^2 - u^2) \end{cases}. \tag{34}$$

For instance, if $u = 3, v = -1$, we obtain solution $x = -39$, $y = 55$, $z = 7$ and $gcd(x, y) = 1$.

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