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Proof of the best bounds in Wallis' inequality

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Let $n \geq 1$ be an integer, then

$$\frac{1}{\sqrt{\pi(n+4\pi^{-1}-1)}} \leq \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} < \frac{1}{\sqrt{\pi(n+1/4)}}.$$

The constants $4\pi^{-1}-1$ and $1/4$ are the best possible.

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The sine has the infinite product representation

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right). \tag{1}$$

Taking in (1) $x = \pi/2$ gives well known the Wallis formula

(2) $$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n-1)(2n+1)}\right].$$

Motivated by (2), Kazarinoff [2] proved that

(3) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}$$

for $n \in \mathbb{N}$, the set of positive integers. We here show that, for $n \in \mathbb{N}$,

(4) $$\frac{1}{\sqrt{\pi(n+4\pi^{-1}-1)}} \le \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} < \frac{1}{\sqrt{\pi(n+1/4)}},$$

improving the lower bound and confirming the upper in (3), by a very simple argument. We also prove that the bounds in (4) are the best possible.

Proof. It is clear that

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad 2^n n! = (2n)!!.$$

To prove the right hand inequality of (4), it suffices to show that

(5) $$R_n = \frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{n+\frac{1}{4}}}{\Gamma(n+1)} < 1.$$

Using the recurrence relation for the gamma function $\Gamma(x+1) = x\Gamma(x)$ we conclude that

$$\frac{R_n}{R_{n+1}} = \sqrt{\frac{n+\frac{1}{4}}{n+\frac{5}{4}}}\,\frac{n+1}{n+\frac{1}{2}} < 1 \quad \text{for} \quad n \ge 1.$$

Hence, the sequence $\{R_n\}_{n=1}^{\infty}$ is strictly increasing with $n \in \mathbb{N}$.

From the asymptotic expansion [1, p. 257]

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\big(x^{-2}\big), \tag{6}$$

we conclude that $\lim_{n\to\infty} R_n = 1$, thus inequality (5) holds for all $n \in \mathbb{N}$.

The left hand side of inequality (4) is equivalent to

$$L_n = \frac{\Gamma\big(n+\frac{1}{2}\big)\sqrt{n+\frac{4}{\pi}-1}}{\Gamma(n+1)} \geq 1. \tag{7}$$

It is easy to see that

$$\frac{L_n}{L_{n+1}} = \sqrt{\frac{n+\frac{4}{\pi}-1}{n+\frac{4}{\pi}}}\,\frac{n+1}{n+\frac{1}{2}} > 1 \quad \text{for} \quad n \geq 2.$$

Hence, the sequence $\{L_n\}_{n=1}^{\infty}$ is strictly decreasing for $n \geq 2$. By (6), we conclude that $\lim_{n\to\infty} L_n = 1$, thus inequality (7) holds strictly for all $n \geq 2$. Clearly, the sign of equality in (7) holds for $n = 1$. The proof is complete.

References

[1] M. Abramowitz, I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.

[2] D. K. Kazarinoff, *On Wallis' formula*, Edinburgh. Math. Soc. Notes, No. 40 (1956), 19–21.

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