General Mathematics Vol. 13, No. 2 (2005), 73–82

Applications of Jack′s lemma for certain subclasses of analytic functions

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Two subclasses $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of certain analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ are considered . The object of the present paper is to discuss some properties for $f(z)$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ by applying Jack′s lemma .

2000 Mathematical Subject Classification: Primary 30C45.

Key words and phrases: Jack′s lemma, analytic function, starlike function, convex function.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}. \tag{1.2}$$

Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}) \tag{1.3}$$

for some α $(0 < \alpha < 1)$. We also define $f(z) \in \mathcal{N}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}(\alpha)$.

It is easy to see that if $f(z) \in \mathcal{M}(\alpha)$, then

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{1.4}$$

Therefore, we know that if $f(z) \in \mathcal{N}(\alpha)$, then

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{1.5}$$

To discuss some properties for functions $f(z)$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, we recall here the following lemma due to Jack [1].

Lemma. *Let $w(z)$ be regular in the open unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, then we have*

$$z_0 w'(z_0) = k w(z_0) \tag{1.6}$$

where $k \geq 1$ is a real number.

2 Properties of the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$

Our first result for the class $\mathcal{M}(\alpha)$ is contained in

Theorem 1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < 1 - 2\alpha \quad (z \in \mathbb{U}) \tag{2.1}$$

for some α $\left(\frac{1}{4} \leq \alpha < \frac{1}{2}\right)$, then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < \frac{1}{2\alpha} - 1 \quad (z \in \mathbb{U}), \tag{2.2}$$

therefore, $f(z) \in \mathcal{M}(\alpha)$.

Proof. Let us define the function $w(z)$ in $\mathbb{U}$ by

$$w(z) = \frac{\frac{2\alpha f(z)}{zf'(z)} - 1}{2\alpha - 1} - 1.$$

Then, clearly, $w(0) = 0$ and $w(z)$ is analytic in $\mathbb{U}$. We want to prove that $w(z)$ satisfies $|w(z)| < 1$ in $\mathbb{U}$. By the definition for $w(z)$, we have that

$$\frac{2\alpha f(z)}{zf'(z)} - 2\alpha = (2\alpha - 1)w(z), \tag{2.3}$$

that is, that

$$\frac{f(z)}{zf'(z)} = \frac{2\alpha - 1}{2\alpha}w(z) + 1 = \frac{(2\alpha - 1)w(z) + 2\alpha}{2\alpha}. \tag{2.4}$$

Differentiating both sides of (2.4) logarithmically, we obtain

$$\frac{zf'(z)}{f(z)} - 1 - \frac{zf''(z)}{f'(z)} = \frac{(2\alpha - 1)zw'(z)}{(2\alpha - 1)w(z) + 2\alpha}, \tag{2.5}$$

and hence

$$\left|\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| = |2\alpha - 1|\left|\frac{zw'(z)}{(2\alpha - 1)w(z) + 2\alpha}\right| < \tag{2.6}$$

$$< 1-2\alpha$$

by the condition of the theorem. Suppose that there exists a point z_0 in $\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1. \tag{2.7}$$

Applying Jack's lemma to $w(z)$ at the point z_0, we can write that $w(z_0) = e^{i\theta}$ and

$\dfrac{z_0 w'(z_0)}{w(z_0)} = k \quad (k \geq 1)$. This gives us that

$$\left|\frac{z_0 f'(z_0)}{f(z_0)} - 1 - \frac{z_0 f''(z_0)}{f'(z_0)}\right| = |2\alpha-1|\left|\frac{k}{(2\alpha-1)+2\alpha e^{-i\theta}}\right| \geq \tag{2.8}$$

$$\geq |2\alpha-1|\left|\frac{1}{(2\alpha-1)+2\alpha e^{-i\theta}}\right|.$$

This implies that

$$\left|\frac{z_0 f'(z_0)}{f(z_0)} - 1 - \frac{z_0 f''(z_0)}{f'(z_0)}\right|^2 \geq \frac{(1-2\alpha)^2}{(2\alpha-1)^2+4\alpha^2+4\alpha(2\alpha-1)\cos\theta}. \tag{2.9}$$

Since the right hand side of (2.9) takes its minimum value for $\cos\theta = -1$, we have that

$$\left|\frac{z_0 f'(z_0)}{f(z_0)} - 1 - \frac{z_0 f''(z_0)}{f'(z_0)}\right|^2 \geq \frac{(1-2\alpha)^2}{(2\alpha-1-2\alpha)^2} =$$

$$= (1-2\alpha)^2$$

This contradicts our condition of the theorem. Thus, there is no point z_0 in $\mathbb{U}$ which satisfies (2.7). This shows that

$$|w(z)| = \left|\frac{\dfrac{2\alpha f(z)}{zf'(z)}-1}{2\alpha-1} - 1\right| < 1 \tag{2.10}$$

for all $z \in \mathbb{U}$. This implies that

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < \frac{1}{2\alpha} - 1 \quad (z \in \mathbb{U}) \tag{2.11}$$

for $\frac{1}{4} \leq \alpha < \frac{1}{2}$. Since $\frac{1}{2\alpha} > 1$ for $\frac{1}{4} \leq \alpha < \frac{1}{2}$, (2.11) satisfies

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}) \tag{2.12}$$

for $\frac{1}{4} \leq \alpha < \frac{1}{2}$, so $f(z) \in \mathcal{M}(\alpha)$.

Noting that $f(z) \in \mathcal{N}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}(\alpha)$, we have

Corollary 1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)}\right| < 1 - 2\alpha \quad (z \in \mathbb{U}) \tag{2.13}$$

for some α $\left(\frac{1}{4} \leq \alpha < \frac{1}{2}\right)$, then

$$\left|\frac{f'(z)}{f'(z) + zf''(z)} - 1\right| < \frac{1}{2\alpha} - 1 \quad (z \in \mathbb{U}), \tag{2.14}$$

therefore, $f(z) \in \mathcal{N}(\alpha)$.

Example. Let us consider the function $f(z)$ given by

$$f(z) = z + a_2 z^2 \quad (z \in \mathbb{U})$$

with

$$a_2 = \frac{2 - 3\alpha - \sqrt{2 - 4\alpha + \alpha^2}}{2(1 - 2\alpha)}$$

for $\frac{1}{4} \leq \alpha < \frac{1}{2}$. Then we see that $0 < a_2 < \frac{1}{2}$ and

$$\left|\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| = \left|\frac{-a_2 z}{(1 + a_2 z)(1 + 2a_2 z)}\right| <$$

$$< \frac{a_2}{(1-a_2)(1-2a_2)} = 1-2\alpha.$$

Therefore, the function $f(z)$ satisfies the condition (2.1) of Theorem 1. Also, if we take the function $g(z) = z + b_2z^2 \ (z \in \mathbb{U})$

with

$$b_2 = \frac{2-3\alpha-\sqrt{2-4\alpha+\alpha^2}}{4(1-2\alpha)}$$

for $\frac{1}{4} \leq \alpha < \frac{1}{2}$, then $g(z)$ satisfies the condition (2.13) of Corollary 1.

Remark. In view of the proof of Theorem 1, we know that there exists some β such that $f(z) \in \mathcal{M}(\beta)$ for the function $f(z)$ satisfying the condition (2.1) of Theorem 1. But we don't find such β in this paper.

Next, we consider

Theorem 2. *If* $f(z) \in \mathcal{M}(\alpha) \quad (\frac{1}{2} \leq \alpha < 1)$, *then*

$$\left| \left(\frac{z}{f(z)}\right)^{\beta} - 1 \right| < 1-\gamma \quad (z \in \mathbb{U}), \tag{2.15}$$

where, $0 \leq \gamma < 1$ *and* $0 < \beta \leq 1-\gamma$.

Proof. Let us define the function $w(z)$ in $\mathbb{U}$ by

$$w(z) = \frac{\left(\frac{z}{f(z)}\right)^{\beta} - 1}{1-\gamma}. \tag{2.16}$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. We need to prove that $|w(z)| < 1$ in $\mathbb{U}$. Since

$$\left(\frac{z}{f(z)}\right)^{\beta} = (1-\gamma)w(z) + 1, \tag{2.17}$$

we obtain that

$$\beta\left(1 - \frac{zf'(z)}{f(z)}\right) = \frac{(1-\gamma)zw'(z)}{(1-\gamma)w(z)+1}. \tag{2.18}$$

This gives us that

$$\frac{zf'(z)}{f(z)} = 1 - \frac{(1-\gamma)zw'(z)}{\beta(1-\gamma)w(z)+\beta} = \tag{2.19}$$

$$= \frac{\beta(1-\gamma)w(z) - (1-\gamma)zw'(z) + \beta}{\beta(1-\gamma)w(z)+\beta},$$

or

$$\frac{2\alpha f(z)}{zf'(z)} - 1 = \frac{\beta(1-\gamma)(2\alpha-1)w(z) + (1-\gamma)zw'(z) + \beta(2\alpha-1)}{\beta(1-\gamma)w(z) - (1-\gamma)zw'(z) + \beta}. \tag{2.20}$$

Note that $f(z) \in \mathcal{M}(\alpha)$ satisfies

$$\left|\frac{2\alpha f(z)}{zf'(z)} - 1\right| < 1 \quad (z \in \mathbb{U}). \tag{2.21}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

Jack's lemma gives $w(z_0) = e^{i\theta}$ and $\dfrac{z_0 w'(z_0)}{w(z_0)} = k \quad (k \geq 1)$.
Therefore, we have that

$$\left|\frac{2\alpha f(z_0)}{z_0 f'(z_0)} - 1\right| = \left|\frac{\beta(1-\gamma)(2\alpha-1) + (1-\gamma)k + \beta(2\alpha-1)e^{-i\theta}}{\beta(1-\gamma) - (1-\gamma)k + \beta e^{-i\theta}}\right|, \tag{2.22}$$

that is, that

$$\left|\frac{2\alpha f(z_0)}{z_0 f'(z_0)} - 1\right|^2 = \tag{2.23}$$

$$= \frac{(1-\gamma)^2(2\alpha\beta+k-\beta)^2 + \beta^2(2\alpha-1)^2 + 2\beta(1-\gamma)(2\alpha-1)(2\alpha\beta+k-\beta)\cos\theta}{(1-\gamma)^2(\beta-k)^2 + \beta^2 + 2\beta(1-\gamma)(\beta-k)\cos\theta}.$$

Let us define the function $g(t)$ by

$$(2.24) \qquad g(t) =$$

$$= \frac{(1-\gamma)^2(2\alpha\beta+k-\beta)^2+\beta^2(2\alpha-1)^2+2\beta(1-\gamma)(2\alpha-1)(2\alpha\beta+k-\beta)t}{(1-\gamma)^2(\beta-k)^2+\beta^2+2\beta(1-\gamma)(\beta-k)t}$$

with $t = \cos\theta$.

Taking the differentiation of (2.24) for t, the numerator of $g'(t)$ becomes that

$$(2.25) \qquad 4\alpha\beta k(1-\gamma)\left\{\beta^2(2\alpha-1)-(1-\gamma)^2(\beta-k)(2\alpha\beta+k-\beta)\right\} =$$

$$= 4\alpha\beta k(1-\gamma)\left\{\beta^2(2\alpha-1)-(1-\gamma)^2(\beta-k)\beta(2\alpha-1)-(1-\gamma)^2(\beta-k)k\right\} > 0,$$

because $\beta - k < 0$ from $0 \leq \gamma < 1$, $0 < \beta \leq 1-\gamma$, $k \geq 1$.

Thus, $g(t)$ is monotone increasing for t where $\frac{1}{2} \leq \alpha < 1$.

Therefore,

$$(2.26) \qquad g(t) \geq g(-1) = \frac{(1-\gamma)(2\alpha+k-\beta)-\beta(2\alpha-1)}{\beta-(1-\gamma)(\beta-k)} =$$

$$= 1 + \frac{2\alpha(1-\beta-\gamma)}{(1-\gamma)(k-\beta)+\beta} \geq 1.$$

This contradicts the condition $f(z) \in \mathcal{M}(\alpha)$.

Therefore, there is no point $z_0 \in \mathbb{U}$ such that $\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$.

Hence,

$$(2.27) \qquad |w(z)| = \left|\frac{\left(\frac{z}{f(z)}\right)^\beta - 1}{1-\gamma}\right| < 1 \quad (z \in \mathbb{U}),$$

or

$$(2.28) \qquad \left|\left(\frac{z}{f(z)}\right)^\beta - 1\right| < 1-\gamma \quad (z \in \mathbb{U}).$$

Taking $\gamma = 0$ in Theorem 2, we have

Corollary 2. *If* $f(z) \in \mathcal{M}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, *then*

$$\left| \left(\frac{z}{f(z)} \right)^{\beta} - 1 \right| < 1 \quad (z \in \mathbb{U}) \tag{2.29}$$

where $0 < \beta \leq 1$.

For $f(z) \in \mathcal{N}(\alpha)$, we also have

Theorem 3. *If* $f(z) \in \mathcal{N}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, *then*

$$\left| \left(\frac{1}{f'(z)} \right)^{\beta} - 1 \right| < 1 - \gamma \quad (z \in \mathbb{U}), \tag{2.30}$$

where $0 \leq \gamma < 1$ *and* $0 < \beta \leq 1 - \gamma$.

Making $\gamma = 0$ in Theorem 3, we see

Corollary 3. *If* $f(z) \in \mathcal{N}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, *then*

$$\left| \left(\frac{1}{f'(z)} \right)^{\beta} - 1 \right| < 1 \quad (z \in \mathbb{U}), \tag{2.31}$$

where $0 < \beta \leq 1$.

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