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Subclasses of starlike functions associated with some hyperbola[1]

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Abstract

In this paper we define some subclasses of starlike functions associated with some hyperbola by using a generalized Sălăgean operator and we give some properties regarding these classes.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

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Let D^n be the Sălăgean differential operator (see [12]) defined as:

$$D^n : A \to A\ , \quad n \in \mathbb{N} \ \text{and} \ D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)\ , \quad D^n f(z) = D(D^{n-1} f(z)).$$

Remark 1.1. *If* $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$ *then* $D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$.

We recall here the definition of the well - known class of starlike functions

$$S^* = \left\{ f \in A : Re \frac{zf'(z)}{f(z)} > 0\ ,\ z \in U \right\}.$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt\ , \quad a \in \mathbb{C}\ , \quad Re\ a \geq 0.$$

Generalizations of the Libera-Pascu integral operator was studied by many mathematicians such are P.T. Mocanu in [7], E. Drăghici in [6] and D. Breaz in [5].

Definition 1.1.[4] *Let* $n \in \mathbb{N}$ *and* $\lambda \geq 0$. *We denote with* D_λ^n *the operator defined by*

$$D_\lambda^n : A \to A\,,$$

$$D_\lambda^0 f(z) = f(z)\ ,\ D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)\,,$$

$$D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z) \right)\ .$$

Remark 1.2.[4] *We observe that* D_λ^n *is a linear operator and for* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *we have*

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^n a_j z^j\ .$$

Also, it is easy to observe that if we consider $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator.

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [8], [9], [10]).

Theorem 1.1. *Let h convex in U and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \textit{then} \;\; p(z) \prec h(z).$$

In [1] is introduced the following operator:

Definition 1.2. *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta \, a_j z^j \,.$$

Remark 1.3. *It is easy to observe that for $\beta = n \in \mathbb{N}$ we obtain the Al-Oboudi operator D_λ^n and for $\beta = n \in \mathbb{N}\,, \lambda = 1$ we obtain the Sălăgean operator D^n.*

The purpose of this note is to define some subclasses of starlike functions associated with some hyperbola by using the operator D_λ^β defined above and to obtain some properties regarding these classes.

2 Preliminary results

Definition 2.1. *[13] A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2}-1\right) \right| < Re\left\{ \sqrt{2}\,\frac{zf'(z)}{f(z)} \right\} + 2\alpha\left(\sqrt{2}-1\right),$$

for some α $(\alpha > 0)$ and for all $z \in U$.

Remark 2.1. *Geometric interpretation:*

Let $\Omega(\alpha) = \left\{ \frac{zf'(z)}{f(z)} \,:\, z \in U\,,\, f \in SH(\alpha) \right\}$.

Then $\Omega(\alpha) = \{w = u + i \cdot v \,:\, v^2 < 4\alpha u + u^2\,,\, u > 0\}$. Note that $\Omega(\alpha)$ is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin.

Definition 2.2. *[3] Let $f \in S$ and $\alpha > 0$. We say that the function f is in the class $SH_n(\alpha)$, $n \in \mathbb{N}$, if*

$$\left| \frac{D^{n+1}f(z)}{D^n f(z)} - 2\alpha\left(\sqrt{2}-1\right) \right| < Re\left\{ \sqrt{2}\,\frac{D^{n+1}f(z)}{D^n f(z)} \right\} + 2\alpha\left(\sqrt{2}-1\right),\ z \in U.$$

Remark 2.2. *Geometric interpretation: If we denote with p_α the analytic and univalent functions with the properties $p_\alpha(0) = 1$, $p'_\alpha(0) > 0$ and $p_\alpha(U) = \Omega(\alpha)$ (see Remark 2.1), then $f \in SH_n(\alpha)$ if and only if $\frac{D^{n+1}f(z)}{D^n f(z)} \prec p_\alpha(z)$, where the symbol $\prec$ denotes the subordination in U. We have $p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-z}} - 2\alpha$, $b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $Im\,\sqrt{w} \geq 0$.*

Theorem 2.1. *[3] If $F(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$, and $f(z) = L_a F(z)$, where L_a is the integral operator defined by (1), then $f(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$.*

Theorem 2.2. *[3] Let $n \in \mathbb{N}$ and $\alpha > 0$. If $f \in SH_{n+1}(\alpha)$ then $f \in SH_n(\alpha)$.*

3 Main results

Definition 3.1. *Let $\beta \geq 0$, $\lambda \geq 0$, $\alpha > 0$ and $p_\alpha(z) = (1+2\alpha)\sqrt{\dfrac{1+bz}{1-z}} - 2\alpha$, where $b = b(\alpha) = \dfrac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $Im\sqrt{w} \geq 0$. We say that a function $f(z) \in S$ is in the class $SH_{\beta,\lambda}(\alpha)$ if*

$$\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \prec p_\alpha(z)\,,\ z \in U\,.$$

Remark 3.1. *Geometric interpretation: $f(z) \in SH_{\beta,\lambda}(\alpha)$ if and only if $\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)}$ take all values in the domain $\Omega(\alpha)$ which is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin (see Remark 2.1 and Remark 2.2).*

Remark 3.2. *It is easy to observe that for $\beta = n \in \mathbb{N}$ and $\lambda = 1$ we obtain in the above definition we obtain the class $SH_n(\alpha)$ studied in [3] and for $\lambda = 1$, $\beta = 0$ we obtain the class $SH(\alpha)$ studied in [13].*

Theorem 3.1. *Let $\beta \geq 0$, $\alpha > 0$ and $\lambda > 0$. We have*

$$SH_{\beta+1,\lambda}(\alpha) \subset SH_{\beta,\lambda}(\alpha)\,.$$

Proof. Let $f(z) \in SH_{\beta+1,\lambda}(\alpha)$.

With notation

$$p(z) = \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)}\,,\ p(0) = 1\,,$$

we obtain

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} \cdot \frac{D_\lambda^{\beta} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} \tag{2}$$

Also, we have

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} = \frac{z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+2} a_j z^j}{z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta} a_j z^j}$$

and

$$zp'(z) = \frac{z\left(D_\lambda^{\beta+1} f(z)\right)'}{D_\lambda^{\beta} f(z)} - \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} f(z)} \cdot \frac{z\left(D_\lambda^{\beta} f(z)\right)'}{D_\lambda^{\beta} f(z)} =$$

$$= \frac{z\left(1 + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1} j a_j z^{j-1}\right)}{D_\lambda^{\beta} f(z)} -$$

$$-p(z) \cdot \frac{z\left(1 + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta} j a_j z^{j-1}\right)}{D_\lambda^{\beta} f(z)}$$

or

$$zp'(z) = \frac{z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta+1} a_j z^j}{D_\lambda^{\beta} f(z)} - \tag{3}$$

$$-p(z) \cdot \frac{z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta} a_j z^j}{D_\lambda^{\beta} f(z)}.$$

We have

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta+1} a_j z^j =$$

$$= z + \sum_{j=2}^{\infty} ((j-1)+1)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j =$$

$$= z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1}\, a_j z^j + \sum_{j=2}^{\infty}(j-1)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j =$$

$$= z + D_{\lambda}^{\beta+1} f(z) - z + \sum_{j=2}^{\infty}(j-1)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j =$$

$$= D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty} ((j-1)\lambda)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j =$$

$$= D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda-1)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j =$$

$$= D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1}\, a_j z^j + \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+2}\, a_j z^j =$$

$$= D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda}\left(D_{\lambda}^{\beta+1} f(z) - z\right) + \frac{1}{\lambda}\left(D_{\lambda}^{\beta+2} f(z) - z\right) =$$

$$= D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} D_{\lambda}^{\beta+1} f(z) + \frac{z}{\lambda} + \frac{1}{\lambda} D_{\lambda}^{\beta+2} f(z) - \frac{z}{\lambda} =$$

$$= \frac{\lambda-1}{\lambda} D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda} D_{\lambda}^{\beta+2} f(z) =$$

$$= \frac{1}{\lambda}\left((\lambda-1) D_{\lambda}^{\beta+1} f(z) + D_{\lambda}^{\beta+2} f(z)\right).$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta}\, a_j z^j = \frac{1}{\lambda}\left((\lambda-1) D_{\lambda}^{\beta} f(z) + D_{\lambda}^{\beta+1} f(z)\right).$$

From (3) we obtain

$$zp'(z) =$$

$$= \frac{1}{\lambda}\left(\frac{(\lambda-1) D_{\lambda}^{\beta+1} f(z) + D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta} f(z)} - p(z)\frac{(\lambda-1) D_{\lambda}^{\beta} f(z) + D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)}\right) =$$

$$= \frac{1}{\lambda}\left((\lambda-1)p(z) + \frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta}f(z)} - p(z)\left((\lambda-1)+p(z)\right)\right) =$$

$$= \frac{1}{\lambda}\left(\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta}f(z)} - p(z)^2\right)$$

Thus

$$\lambda z p'(z) = \frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta}f(z)} - p(z)^2$$

or

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta}f(z)} = p(z)^2 + \lambda z p'(z)\,.$$

From (2) we obtain

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} = \frac{1}{p(z)}\left(p(z)^2 + \lambda z p'(z)\right) = p(z) + \lambda\frac{zp'(z)}{p(z)}\,,$$

where $\lambda > 0$.

From $f(z) \in SH_{\beta+1,\lambda}(\alpha)$ we have

$$p(z) + \lambda\frac{zp'(z)}{p(z)} \prec p_\alpha(z)\,,$$

with $p(0) = p_\alpha(0) = 1$, $\alpha > 0$, $\beta \geq 0$, $\lambda > 0$, and $Re\, p_\alpha(z) > 0$ from here construction. In this conditions from Theorem 1.1, we obtain

$$p(z) \prec p_\alpha(z)$$

or

$$\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} \prec p_\alpha(z)\,.$$

This means $f(z) \in SH_{\beta,\lambda}(\alpha)$.

Theorem 3.2. *Let $\beta \geq 0$, $\alpha > 0$ and $\lambda \geq 1$. If $F(z) \in SH_{\beta,\lambda}(\alpha)$ then $f(z) = L_a F(z) \in SH_{\beta,\lambda}(\alpha)$, where L_a is the Libera-Pascu integral operator defined by (1).*

Proof. From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator $D_\lambda^{\beta+1}$, we obtain

$$(1+a)D_\lambda^{\beta+1}F(z) = aD_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+1}\left(z + \sum_{j=2}^{\infty} ja_j z^j\right) =$$

$$= aD_\lambda^{\beta+1}f(z) + z + \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}\, ja_j z^j$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta+1} a_j z^j = \frac{1}{\lambda}\Big((\lambda-1)D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\Big)$$

Thus

$$(1+a)D_\lambda^{\beta+1}F(z) = aD_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}\Big((\lambda-1)D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\Big) =$$

$$= \left(a + \frac{\lambda-1}{\lambda}\right)D_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}D_\lambda^{\beta+2}f(z)$$

or

$$\lambda(1+a)D_\lambda^{\beta+1}F(z) = ((a+1)\lambda-1)\,D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\,.$$

Similarly, we obtain

$$\lambda(1+a)D_\lambda^{\beta}F(z) = ((a+1)\lambda-1)\,D_\lambda^{\beta}f(z) + D_\lambda^{\beta+1}f(z)\,.$$

Then

$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^{\beta}F(z)} = \frac{\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}\cdot\dfrac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} + ((a+1)\lambda-1)\cdot\dfrac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}}{\dfrac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} + ((a+1)\lambda-1)}\,.$$

With notation
$$\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^\beta f(z)}=p(z)\,,\; p(0)=1\,,$$
we obtain
$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^\beta F(z)}=\frac{\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}\cdot p(z)+((a+1)\lambda-1)\cdot p(z)}{p(z)+((a+1)\lambda-1)}. \tag{4}$$
We have (see the proof of the above theorem)
$$\lambda zp'(z)=\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}\cdot\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^\beta f(z)}-p(z)^2=$$
$$=\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}\cdot p(z)-p(z)^2\,.$$
Thus
$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}=\frac{1}{p(z)}\cdot\left(p(z)^2+\lambda zp'(z)\right).$$
Then, from (4), we obtain
$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^\beta F(z)}=\frac{p(z)^2+\lambda zp'(z)+((a+1)\lambda-1)\,p(z)}{p(z)+((a+1)\lambda-1)}=$$
$$=p(z)+\lambda\frac{zp'(z)}{p(z)+((a+1)\lambda-1)}\,,$$
where $a\in\mathbb{C}$, $Re\,a\geq 0$, $\beta\geq 0$, and $\lambda\geq 1$.

From $F(z)\in SH_{\beta,\lambda}(\alpha)$ we have
$$p(z)+\frac{zp'(z)}{\frac{1}{\lambda}\left(p(z)+((a+1)\lambda-1)\right)}\prec p_\alpha(z)\,,$$
where $a\in\mathbb{C}$, $Re\,a\geq 0$, $\alpha>0$, $\beta\geq 0$, $\lambda\geq 1$, and from her construction, we have $Re\,p_\alpha(z)>0$. In this conditions we have from Theorem 1.1 we obtain
$$p(z)\prec p_\alpha(z)$$

or

$$\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \prec p_\alpha(z) \, .$$

This means $f(z) = L_a F(z) \in SH_{\beta,\lambda}(\alpha)$.

Remark 3.3. *If we consider $\beta = n \in \mathbb{N}$ in the previously results we obtain the Theorem 3.1 and Theorem 3.2 from [2].*

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