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Properties regarding the trace of a matrix [1]

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

There exist in many collection of mathematics problems applications concerning the trace of a matrix (ex. ...). We understand by the trace of a matrix the sum of all elements that are on the matrix first diagonal. The aim of this article is to present some properties regarding the trace of a matrix.

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Through out the paper if A is an $n \times n$ matrix, we write $tr\ A$ to denote the trace of A and $det\ A$ for the determinant of A. If A is positive definite we write $A > 0$.

Application 1. Let $A \in \mathcal{M}_2(\mathbb{C})$ and $n \in \mathbb{N}^*$ with $A^n = I_2$. Show that if $A + det\ A$ is a real matrix, then $tr\ A$ and $det\ A$ are real numbers.

Proof. Let $f(x) = X^n - tr\ A \cdot X + det\ A$ be the characteristic polyoma

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of the A matrix and λ_1, λ_2 its roots. Because $\lambda_1^n = \lambda_2^n = 1$, results that $|\lambda_1| = |\lambda_2| = 1$. By the hypothesis $trA + detA = \lambda_1 + \lambda_2 + \lambda_1\lambda_2$ is a real number, so $\overline{\lambda_1} + \overline{\lambda_2} + \overline{\lambda_1\lambda_2}$ is a real number. We obtain $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_1\lambda 2}$ from $\mathbb{R}$, so $\frac{tr\ A+1}{det\ A}$ is real number. Then $1 + \frac{tr\ A+1}{det\ A} = \frac{tr\ A+det\ A+1}{det\ A}$ is real number and $tr\ A + det\ A + 1$ also, so $det\ A$ and then $tr\ A$ is real number.

Application 2. If $A > 0$ and $B > 0$, then

$$0 < tr\ (AB)^m \leq (tr\ (AB))^m \qquad \text{for all} \quad m \in \mathbb{N}^*$$

Proof. The equality takes place for $n = 1$. If $n > 1$, for $B = I$ the inequality is true because $0 < tr\ (A^n) \leq (tr\ A)^n$, become

$$\sum_{i=1}^{n} \lambda_i^m \leq \left(\sum_{i=1}^{n} \lambda_i\right)^m ,$$

where $\lambda_1\, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

If $A \mapsto AB$, the result has been proved.

Application 3. If $A_i > 0$ and $B_i > 0$ $(i = 1, 2, \ldots, k)$ then

$$\left(tr\ \sum_{i=1}^{k} A_i B_i\right)^n \leq \left(tr\ \sum_{i=1}^{k} A_i^n\right)\left(tr\ \sum_{i=1}^{k} B_i^n\right)$$

If $A_i B_i > 0$ $(i = 1, 2, \ldots, k)$, then

$$\left(tr\ \sum_{i=1}^{k} A_i B_i\right)^n \leq \left(tr\ \sum_{i=1}^{k} A_i^n\right)\left(tr\ \sum_{i=1}^{k} B_i^n\right)$$

Proof. Because

$$0 \leq tr\ \left(\sum_{i=1}^{k} \alpha A_i + B_i\right)^n = \alpha^n tr\ \left(\sum_{i=1}^{k} A_i^n\right) +$$

$$+2\alpha tr\ \left(tr\ \sum_{i=1}^{k} A_i B_i\right) + tr\ \left(tr\ \sum_{i=1}^{k} B_i^n\right)$$

the result has been proved.

In order to demonstrate the second inequality it is sufficient to demonstrate that

$$(1) \qquad tr \left(\sum_{i=1}^{k} A_i B_i\right)^n \leq \left(tr \sum_{i=1}^{k} A_i B_i\right)$$

Because $A_i B_i > 0$ for all $i = 1, 2, \ldots, k$, we have $U = \sum_{i=1}^{k} A_i B_i > 0$. The inequality (1) will result by the fact that $tr\ (U)^n \leq (tr\ U)^n$, for all $U > 0$.

Application 4. If $A > 0$ and $B > 0$ then

$$n(det\ A\ det\ B)^{\frac{m}{n}} \leq tr\ (A^n B^n)$$

for any positive integer m.

Proof. Since A is diagonaligable, there exists an orthodiagonal matrix P and a diagonal matrix D such that $D = P^\top A$ (see [2]). So if the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then$D = diag\ (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Let $b_{11}(m), b_{22}(m), \ldots, b_{nn}(m)$ denote the elements of $(PBP^\top)^n$. Then

$$\frac{1}{n} tr\ (A^n B^n) = \frac{1}{n} tr\ (PD^n P^\top B^n) = \frac{1}{n} tr\ (D^n P^\top B^n P) =$$

$$= \frac{1}{n} tr\ [D^n (P^\top BP)^n] = \frac{1}{n} [\lambda_1^m b_{11}(m) + \lambda_2^m b_{22}(m) + \ldots + \lambda_n^m b_{nn}(m)].$$

Using the arithmetic - mean geometric - mean inequality, we get

$$(2) \qquad \frac{1}{n} tr\ (A^n B^n) \leq [\lambda_1^m \lambda_2^m \ldots \lambda_n^m]^{\frac{1}{n}} [b_{11}(m)\ b_{22}(m) \ldots b_{nn}(m)]^{\frac{1}{n}}\ .$$

Since $det\ A \leq a_{11} a_{22} \ldots a_{nn}$ for any positive definite matrix A, we conclude that

$$det\ (P^\top BP)^n \leq b_{11}(m) b_{22}(m) \ldots b_{nn}(m)$$

and

$$det\ D^n \leq \lambda_1^m \lambda_2^m \ldots \lambda_n^m\ .$$

Therefore from (2) if fellows that

$$\begin{aligned}\frac{1}{n}tr\ (A^nB^n) &\le [det\ (D^n)]^{\frac{1}{n}}[det\ (P^\top BP)^m]^{\frac{1}{n}} = \\ &= [det\ (P^\top AP)]^{\frac{m}{n}}[det\ (P^\top BP)]^{\frac{m}{n}} = (det\ A\ det\ B)^{\frac{m}{n}}.\end{aligned}$$

Here we used the fact that $A > 0$ and $B > 0$. The proof is complete.

Corollary 1.*Let A and X be positive definite $n \times n$ - matrices such that $det\ X = 1$. Then*

$$n(det\ A)\frac{1}{n} \le tr\ (AX)\ .$$

Proof. *Take $B = X$ and $m = 1$ in Applications 4.*

References

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