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A class of logarithmically completely monotonic functions related to $(1+1/x)^x$ and an application [1]

Da-Wei Niu, Jian Cao and Feng Qi

In memoriam of Associate Professor Ph. D. Luciana Lupa's

Abstract

In this paper, sufficient and necessary conditions such that the function $\left[e/(1+1/x)^{x+\beta}\right]^{x+\alpha}$ is completely monotonic or logarithmically completely monotonic in $(0,\infty)$ are established. As by-products, several inequalities related to $(1+1/x)^x$ are obtained and one of them is used to strengthen van der Corput's inequality.

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1 Introduction

It is well known [10] that a function f is said to be completely monotonic on an interval I if f has derivative of all orders on I such that $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and nonnegative integer n. In [14], it was coined that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^n[\ln f(x)]^{(n)} \geq 0$ for $x \in I$ and $n \in \mathbb{N}$. It has been proved in [12, 14] that a logarithmically completely monotonic function on an interval I must be completely monotonic on I, but not conversely. (Logarithmically) completely monotonic functions have important applications in many areas such as potential theory, physics, numerical and asymptotic analysis, number theory, and combinatorics. For more information on the logarithmically completely monotonic functions, please refer to [2, 6, 10, 11, 12, 14, 15, 16] and the references therein.

The real number e is one of the most important constants in mathematics, it plays crucial roles in many branches of mathematics and other subjects. In recent years, many papers, such as [1, 18, 20, 21], about e and function $\left(1+\frac{1}{x}\right)^x$ were emerged. In [18], the monotonicity and convexity of two functions $(x+1)\left[e-\left(1+\frac{1}{x}\right)^x\right]$ and $\left(x+\frac{1}{2}\right)\left[e-\left(1+\frac{1}{x}\right)^x\right]$ in $(0,\infty)$ were discussed. Later, in [1], the sufficient and necessary conditions such that $p_n(x)\left[e-\left(1+\frac{1}{x}\right)^x\right]$ is completely monotonic in $(0,\infty)$ were presented, where $p_n(x) = x^n + \sum_{v=0}^{n-1} a_v x^v$ is a polynomial of degree $n \geq 1$ with real coefficients a_v.

In [9], a new notion or term was coined as follows: A positive function f is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \geq 0$ for $t \in I$ and $k \in \mathbb{N}$. It was proved in [9] that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, but not conversely. Moreover, the sufficient and necessary conditions such that the function $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ or its reciprocal is logarithmically completely (absolutely) monotonic are showed also in [9], which generalizes and extends the corresponding results in [1, 17, 23].

The first aim of this paper, motivated by [1, 18], is to consider the logarithmically complete monotonicity of the function $\left[\frac{e}{(1+1/x)^{x+\beta}}\right]^{x+\alpha}$. The first main result is the following necessary and sufficient conditions.

Theorem 1.1. *Let α and β be real numbers and*

$$f(x)=\left[\frac{e}{(1+1/x)^{x+\beta}}\right]^{x+\alpha}. \tag{1}$$

Then the following three statements are equivalent each other:

(i) Either $\alpha+\beta\geq\frac{2}{3}$ and $\alpha\beta\leq 0$ or $\alpha+\beta<\frac{2}{3}$ and $\alpha\beta\leq\frac{3(\alpha+\beta)-2}{6}$ is valid.

(ii) The function $f(x)$ is logarithmically completely monotonic in $(0,\infty)$.

(iii) The function $f(x)$ is completely monotonic in $(0,\infty)$.

As direct consequences of Theorem 1.1, the following inequalities related to $\left(1+\frac{1}{x}\right)^x$ are obtained.

Corollary 1 *If either $\alpha+\beta\geq\frac{2}{3}$ and $\alpha\beta\leq 0$ or $\alpha+\beta<\frac{2}{3}$ and $\alpha\beta\leq\frac{3(\alpha+\beta)-2}{6}$, then inequality*

$$\frac{e}{(1+1/x)^x}>\left(1+\frac{1}{x}\right)^{\beta}\exp\frac{1-2\beta}{2(x+\alpha)} \tag{2}$$

holds in $(0,\infty)$. In particular,

$$\frac{e}{(1+1/x)^x}>\exp\frac{1}{2x+4/3} \tag{3}$$

and

$$\frac{e}{(1+1/x)^x}>\left(1+\frac{1}{x}\right)^{\frac{2}{3}}\exp\frac{1}{6x}. \tag{4}$$

hold for $x\in(0,\infty)$.

As a generalization of Theorem 1.1, we obtain the following Theorem 1.2.

Theorem 1.2. *Let* $p_n(x) = x^n + \sum_{v=0}^{n-1} c_v x^v$ *be a polynomial of degree* $n \geq 1$ *with real coefficients* c_v. *Then the function*

$$g(x) = \left[\frac{e}{(1+1/x)^{x+\beta}}\right]^{p_n(x)} \tag{5}$$

is logarithmically completely monotonic in $(0, \infty)$ *if and only if* $n = 1$, β *and* c_0 *satisfies either* $c_0 + \beta \geq \frac{2}{3}$ *and* $c_0\beta \leq 0$ *or* $c_0 + \beta < \frac{2}{3}$ *and* $c_0\beta \leq \frac{3(c_0+\beta)-2}{6}$.

The van der Corput's inequality [19] reads that inequality

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/H_n} < e^{1+\gamma}\sum_{n=1}^{\infty}(n+1)a_n \tag{6}$$

holds true for $a_n \geq 0$ such that $0 < \sum_{n=1}^{\infty}(n+1)a_n < \infty$, where $H_n = \sum_{m=1}^{n}\frac{1}{m}$ is the harmonic number and $\gamma = 0.57721566\cdots$ stands for Euler-Mascheroni's constant.

As an application of Corollary 1, we are about to present a strengthened van der Corput's inequality below.

Theorem 1.3. *For* $n \in \mathbb{N}$, *let* $H_n = \sum_{m=1}^{n}\frac{1}{m}$. *If* $a_n \geq 0$ *and*

$$0 < \sum_{n=1}^{\infty} n\left(1 - \frac{\ln n}{2n + \ln n + 4/3}\right)a_n < \infty,$$

then

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/H_n} < e^{1+\gamma}\sum_{n=1}^{\infty} e^{-\frac{3\gamma}{6n+4}} n\left(1 - \frac{\ln n}{2n + \ln n + 4/3}\right)a_n, \tag{7}$$

where $\gamma = 0.57721566\cdots$ *is the Euler-Mascheroni's constant.*

Remark 1.1. *Recently, some strengthened van der Corput's inequalities are given in [3, 4, 7, 8, 22], but they are not better than* (7).

2 Lemmas

In order to prove our theorems, the following two lemmas are necessary.

Lema 2.1. *The function* $\frac{2}{3} + 3x + 2x^2 - 2x(1+x)^2 \ln\left(1 + \frac{1}{x}\right)$ *is completely monotonic and*

$$0 < 2x(1+x)^2 \ln\left(1+\frac{1}{x}\right) - 2x^2 - 3x < \frac{2}{3} \tag{8}$$

in $(0, \infty)$.

Proof. Let $\varphi(x) = 2x(1+x)^2 \ln(1+\frac{1}{x}) - 2x^2 - 3x$ is defined in $(0,\infty)$. Then

$$\begin{aligned}
\varphi'(x) &= 2(3x^2+4x+1)\ln\left(1+\frac{1}{x}\right) - 6x - 5,\\
\varphi''(x) &= 2\left[2(3x+2)\ln\left(1+\frac{1}{x}\right) - \frac{6x+1}{x}\right],\\
\varphi'''(x) &= 2\left[6\ln\left(1+\frac{1}{x}\right) + \frac{1-3x-6x^2}{x^2(x+1)}\right],\\
\varphi^{(4)}(x) &= -\frac{4}{x^3(x+1)^2}.
\end{aligned}$$

Since $\varphi^{(4)}(x) < 0$ in $(0,\infty)$, then $\varphi'''(x)$ is decreasing in $(0,\infty)$. It is clear that $\lim_{x\to\infty}\varphi'''(x) = 0$. From this, it follows that $\varphi'''(x) > 0$ and $\varphi''(x)$ increases. It is also clear that $\lim_{x\to\infty}\varphi''(x) = 0$, thus $\varphi''(x) < 0$ in $(0,\infty)$, therefore $\varphi'(x)$ is decreasing. By L'Hôspital's rule, it is easy to obtain that $\lim_{x\to\infty}\varphi'(x) = 0$. Hence $\varphi'(x) > 0$ and $\varphi(x)$ is increasing in $(0,\infty)$. Using L'Hôspital's rule once again yields $\lim_{x\to 0}\varphi(x) = 0$ and $\lim_{x\to\infty}\varphi(x) = \frac{2}{3}$, this means $0 < \varphi(x) < \frac{2}{3}$ in $(0,\infty)$.

Let $\omega(x) = \frac{2}{3} - \varphi(x)$ in $(0,\infty)$. By the argument above, it easy to see that $(-1)^i\omega^{(i)}(x) > 0$ for $i = 0,1,2,3,4$ and $\omega^{(4)}(x) = \frac{4}{x^3(x+1)^2}$. Since the functions $\frac{1}{x^3}$ and $\frac{1}{(x+1)^2}$ are completely monotonic in $(0,\infty)$ and the product of finite completely monotonic functions is also completely monotonic, then $\omega^{(4)}(x)$ is completely monotonic in $(0,\infty)$. The proof of Lemma 2.1 is complete.

Lema 2.2.*[[5, 13]] For $n \in \mathbb{N}$,*

$$\frac{1}{2n+1/(1-\gamma)-2} < H_n - \ln n - \gamma < \frac{1}{2n+1/3}. \tag{9}$$

The constants $\frac{1}{1-\gamma}-2$ and $\frac{1}{3}$ in (9) are the best possible.

3 Proofs of theorems and Corollary 1

Proof. [Proof of Theorem 1.1] It is sufficient to prove the circle (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii)

Direct computation yields

$$\ln f(x) = (x+\alpha)\left[1-(x+\beta)\ln\left(1+\frac{1}{x}\right)\right], \tag{10}$$

$$[\ln f(x)]' = 1-(2x+\alpha+\beta)\ln\left(1+\frac{1}{x}\right)+\frac{(x+\alpha)(x+\beta)}{x(1+x)},$$

(11)

$$[\ln f(x)]'' = -2\ln\left(1+\frac{1}{x}\right)+\frac{2-\alpha-\beta}{1+x}+\frac{\alpha+\beta}{x}+\frac{1+\alpha\beta-\alpha-\beta}{(1+x)^2}-\frac{\alpha\beta}{x^2}.$$

Using the well-known formulas

$$\ln\left(1+\frac{1}{x}\right) = \int_0^\infty \frac{e^{-xt}-e^{-(x+1)t}}{t}\,\mathrm{d}t \tag{12}$$

and

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t \tag{13}$$

for $n \in \mathbb{N}$ and $x > 0$ leads to

$$[\ln f(x)]'' = \int_0^\infty \frac{2(e^{-(x+1)t}-e^{-xt})}{t}\,\mathrm{d}t + (2-\alpha-\beta)\int_0^\infty e^{-(1+x)t}\,\mathrm{d}t +$$

$$+(\alpha+\beta)\int_0^\infty e^{-xt}\,\mathrm{d}t-\alpha\beta\int_0^\infty te^{-xt}\,\mathrm{d}t+$$

$$+(1+\alpha\beta-\alpha-\beta)\int_0^\infty te^{-(1+x)t}\,\mathrm{d}t=$$

$$=\int_0^\infty \frac{1}{t}\big\{\big[(\alpha+\beta)t-2-\alpha\beta t^2\big]e^{-xt}+$$

$$+\big[2+(2-\alpha-\beta)t+(1+\alpha\beta-\alpha-\beta)t^2\big]e^{-(1+x)t}\big\}\mathrm{d}t=$$

$$=\int_0^\infty \frac{1}{te^{(1+x)t}}\big\{\big[(\alpha+\beta)t-2-\alpha\beta t^2\big]e^{t}+2+$$

$$+(2-\alpha-\beta)t+(1+\alpha\beta-\alpha-\beta)t^2\big\}\mathrm{d}t\triangleq\int_0^\infty \frac{g(t)}{t}e^{-(1+x)t}\,\mathrm{d}t,$$

$$(-1)^n[\ln f(x)]^{(n)}=\int_0^\infty \frac{x^{n-2}g(t)}{t}e^{-(1+x)t}\,\mathrm{d}t.$$

For $n\geq 2$, and

$$\begin{aligned}
g'(t)&=[-2+\alpha+\beta+(\alpha+\beta-2\alpha\beta)t-\alpha\beta t^2]e^t\\
&\quad+2-\alpha-\beta+2(1+\alpha\beta-\alpha-\beta)t,\\
g''(t)&=[-2+2(\alpha+\beta)-2\alpha\beta+(\alpha+\beta-4\alpha\beta)t-\alpha\beta t^2]e^t\\
&\quad+2(1+\alpha\beta-\alpha-\beta),\\
g'''(t)&=[-2+3(\alpha+\beta)-6\alpha\beta+(\alpha+\beta-6\alpha\beta)t-\alpha\beta t^2]e^t\\
&\triangleq h(t)e^t.
\end{aligned}$$

If $h(t)\geq 0$, then $g'''(t)\geq 0$ and $g''(t)$ is increasing in $(0,\infty)$. From $g''(0)=0$, it is concluded that $g''(t)>0$ and $g'(t)$ is increasing in $(0,\infty)$. From $g'(0)=0$, it is deduced that $g'(t)>0$ and $g(t)$ is increasing in $(0,\infty)$. From $g(0)=0$, it is followed that $g(t)>0$ in $(0,\infty)$. This means that $(-1)^n[\ln f(x)]^{(n)}>0$ for $n\geq 2$.

Now we are in a position to find the conditions such that $h(t)\geq 0$ in $(0,\infty)$. Let

$$C_1\triangleq\begin{cases}\alpha\beta<0,\\ \Delta\leq 0,\end{cases}\qquad C_2\triangleq\begin{cases}\alpha\beta<0,\\ \Delta\geq 0,\\ y_1\leq 0,\end{cases}\qquad C_3\triangleq\begin{cases}\alpha\beta=0,\\ \alpha+\beta\geq \frac{2}{3},\end{cases}\tag{14}$$

where $\Delta = \alpha^2 + \beta^2 + 12\alpha^2\beta^2 - 6\alpha\beta$ is the discriminant of the root of the equation $h(t) = 0$ and

$$y_1 = \frac{6\alpha\beta - \alpha - \beta + \sqrt{\Delta}}{-2\alpha\beta}$$

is the larger root of the equation $h(t) = 0$. it is clear to say that the function $h(t)$ is positive is equivalent to say that either C_1 or C_2 or C_3 holds. By simplifying them we find that C_1 has no solution and that C_2 is equivalent to either

(15)

$$\begin{cases} \alpha + \beta \geq 1, \\ \alpha\beta < 0, \end{cases} \quad or \quad \begin{cases} \frac{2}{3} \leq \alpha + \beta < 1, \\ \alpha\beta < 0, \end{cases} \quad or \quad \begin{cases} \alpha + \beta < \frac{2}{3}, \\ \alpha\beta \leq \frac{3(\alpha+\beta)-2}{6}. \end{cases}$$

From (14) and (15), we conclude that the function $h(t)$ being positive is equivalent to

$$\begin{cases} \alpha + \beta \geq \frac{2}{3} \\ \alpha\beta \leq 0, \end{cases} \quad or \quad \begin{cases} \alpha + \beta < \frac{2}{3} \\ \alpha\beta \leq \frac{3(\alpha+\beta)-2}{6}. \end{cases} \tag{16}$$

For $n = 1$, we have proved $[\ln f(x)]'' \geq 0$ under the condition (16), this implies $[\ln f(x)]' \geq 0$ is increasing in $(0, \infty)$. Applying the Talyor expansion

$$\ln\left(1 + \frac{1}{x}\right) = \frac{2}{2x+1} + \frac{2}{3(2x+1)^3} + \frac{2}{5(2x+1)^5} + \cdots \tag{17}$$

for $x > 0$ yields $\lim_{x\to\infty}[\ln f(x)]' = 0$, this means that if (16) holds then $[\ln f(x)]' < 0$. Hence, for $n \geq 1$ we show $(-1)^n[\ln f(x)]^{(n)} > 0$ in $(0, \infty)$. So, the sufficient condition of the function $f(x)$ being logarithmically completely monotonic is (16).

(ii) ⇒ (iii)

This follows from a fact obtained in [2, 9, 12, 14] that a logarithmically completely monotonic function must be completely monotonic.

(iii) ⇒ (i)

If $f(x)$ is completely monotonic in $(0,\infty)$, we have

$$f''(x) = \left[e^{\ln f(x)}\right]'' = f(x)[\ln f(x)]'' + f(x)\{[\ln f(x)]'\}^2 > 0,$$

that is, $\{[\ln f(x)]'\}^2 > -[\ln f(x)]''$, this implies $[\ln f(x)]'' > 0$. From (11), it follows that

(18)
$$[\ln f(x)]'' = -2\ln\left(1+\frac{1}{x}\right)+\frac{2-\alpha-\beta}{1+x}+\frac{\alpha+\beta}{x}+\frac{1+\alpha\beta-\alpha-\beta}{(1+x)^2}-\frac{\alpha\beta}{x^2} > 0,$$

which can be rearranged as

$$\alpha+\beta-\frac{\alpha\beta}{x/(1+2x)} > 2x(1+x)^2\ln\left(1+\frac{1}{x}\right)-2x(1+x)-x = \varphi(x). \tag{19}$$

Making use of Lemma 2.1 leads to

$$\alpha+\beta-\frac{\alpha\beta}{x/(1+2x)} \geq \sup_{x\in(0,\infty)}\varphi(x) = \frac{2}{3}. \tag{20}$$

On the other hand, since the function $\phi(x) = -\frac{x}{1+2x}$ is decreasing in $(0,\infty)$, it follows that $-\frac{1}{2} < \phi(x) < 0$. In virtue of (20) we obtain

$$\phi(x)\left(\frac{2}{3}-\alpha-\beta\right) \geq \alpha\beta. \tag{21}$$

If $\alpha+\beta = \frac{2}{3}$, then $\alpha\beta \leq 0$. If $\alpha+\beta < \frac{2}{3}$, then (21) is equivalent to $\phi(x) \geq \frac{\alpha\beta}{2/3-\alpha-\beta}$. Since $-\frac{1}{2} < \phi(x) < 0$, we deduce

$$\frac{\alpha\beta}{2/3-\alpha-\beta} \leq \inf_{x\in(0,\infty)}\phi(x) = -\frac{1}{2},$$

that is, $6\alpha\beta \leq 3(\alpha+\beta)-2$. If $\alpha+\beta > \frac{2}{3}$, then (21) is equivalent to $\phi(x) \leq \frac{\alpha\beta}{2/3-\alpha-\beta}$. Since $-\frac{1}{2} < \phi(x) < 0$, we have

$$\frac{\alpha\beta}{2/3-\alpha-\beta} \geq \sup_{x\in(0,\infty)}\phi(x) = 0,$$

that is, $\alpha\beta \le 0$. These cases are equivalent to (16). The proof of Theorem 1.1 is complete.

Proof. [Proof of Corollary 1] Using Taylor expansion (17) and (10), we have

$$\ln f(x) = (x+\alpha)\left[1 - \frac{2(x+\beta)}{2x+1}\right] + O\left(\frac{1}{x}\right)$$

as $x \to \infty$. Therefore,

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{(1-2\beta)(x+\alpha)}{2x+1} = \frac{1}{2} - \beta.$$

This implies

$$\lim_{x\to\infty} \left[\frac{e}{(1+1/x)^{x+\beta}}\right]^{x+\alpha} = e^{\frac{1}{2}-\beta}.$$

Since (16) holds, inequality (2) follows from Theorem 1.1.

Substituting either $\alpha = \frac{2}{3}$ and $\beta = 0$ or $\alpha = 0$ and $\beta = \frac{2}{3}$ into (2) respectively yield either (3) or (4). The proof is complete.

Proof. [Proof of Theorem 1.2] Let

$$I_1 = \{x \mid p_n(x) > 0 \text{ and } x \ge 0\} \quad \text{and} \quad I_2 = \{x \mid p_n(x) < 0 \text{ and } x \ge 0\}.$$

If $x \in I_1$ and $\beta \le 0$, the complete monotonicity of $g(x)$ in $(0,\infty)$ gives

$$g'(x) = g(x)\left\{p'_{n(x)}\left[1 - (x+\beta)\ln\left(1+\frac{1}{x}\right)\right] + \right.$$

$$\left. + p_n(x)\left[\frac{x+\beta}{x(1+x)} - \ln\left(1+\frac{1}{x}\right)\right]\right\} \le 0$$

which is equivalent to

$$\frac{xp'_n(x)}{p_n(x)}\left[1 - (x+\beta)\ln\left(1+\frac{1}{x}\right)\right] + \left[\frac{x+\beta}{x+1} - x\ln\left(1+\frac{1}{x}\right)\right] \le 0,$$

that is,

$$\frac{xp'_n(x)}{p_n(x)} \le \frac{x\ln\left(1+\frac{1}{x}\right) - \frac{x+\beta}{x+1}}{1 - (x+\beta)\ln\left(1+\frac{1}{x}\right)}.$$

Using (17) leads to

$$x\ln\left(1+\frac{1}{x}\right)-\frac{x+\beta}{x+1}=1-(x+\beta)\ln\left(1+\frac{1}{x}\right)+O\left(\frac{1}{x}\right)$$

as $x\to\infty$, that is,

$$\lim_{x\to\infty}\frac{x\ln\left(1+\frac{1}{x}\right)-\frac{x+\beta}{x+1}}{1-(x+\beta)\ln\left(1+\frac{1}{x}\right)}=1.$$

Therefore, it follows that $n=\lim_{x\to\infty}\frac{xp_n'(x)}{p_n(x)}\le 1$, which means $n=1$.

If $x\in I_2$ and $\beta>0$, the same procedure as above can be employed to obtain the result of $n=1$ also.

In one word, if $\beta p_n(x)\le 0$, then $n=1$.

By Theorem 1.1, the sufficient condition of $g(x)$ being completely monotonic is $n=1$ and

$$\text{(22)}\qquad \begin{cases}\beta p_n(x)=(x+c_0)\beta\le 0,\\ c_0+\beta\ge\frac{2}{3},\\ c_0\beta\le 0,\end{cases}\qquad\text{or}\qquad\begin{cases}\beta p_n(x)=(x+c_0)\beta\le 0,\\ c_0+\beta<\frac{2}{3},\\ c_0\beta\le\frac{3(c_0+\beta)-2}{6}.\end{cases}$$

Now consider the inequality $(x+c_0)\beta\le 0$, that is, $c_0\beta\le -x\beta$. For $x\ge 0$, if $\beta\ge 0$, then $c_0\beta\le -x\beta\le 0$; if $\beta<0$, then $-x\beta>0$, which implies that $c_0\beta\le\inf -x\beta=0$. This tells us that (22) is equivalent to (16) ($c_0=\alpha$). Thus the sufficiency is proved. The necessary condition is clear by Theorem 1.1 also.

Proof. [Proof of Theorem 1.3] Let

$$B_n=\left[\frac{(n+1)H_{n+1}}{nH_n}\right]^{nH_n}.$$

By applying inequality (3), we have

$$\text{(23)}\qquad \begin{aligned}(B_n)^{1/(H_n+1)}&=\left(1+\frac{H_n+1}{nH_n}\right)^{nH_n/(H_n+1)}\\ &<\exp\left[1-\frac{H_n+1}{2nH_n+4(H_n+1)/3}\right]\\ &<\exp\left(1-\frac{1}{2n+4/3}\right).\end{aligned}$$

By using Lemma 2.2 in (23), we obtain

$$
\begin{aligned}
B_n &< \exp\left[(H_n+1)\left(1-\frac{1}{2n+4/3}\right)\right] \\
&< \exp\left(1+\gamma+\ln n+\frac{1}{2n+1/3}-\frac{1+\gamma+\ln n+\frac{1}{2n+1/3}}{2n+4/3}\right) \\
&= ne^{1+\gamma}\exp\left(-\frac{\gamma}{2n+4/3}\right)\exp\left(-\frac{\ln n}{2n+4/3}\right).
\end{aligned} \tag{24}
$$

In virtue of inequality $e^{-x} \le \frac{1}{1+x}$ for $x > -1$ in (24), we have

$$
B_n < ne^{1+\gamma-\frac{3\gamma}{6n+4}}\left(1-\frac{\ln n}{2n+\ln n+4/3}\right). \tag{25}
$$

To prove inequality (7), it suffices to prove

$$
\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/H_n} \le \sum_{n=1}^{\infty} B_n a_n. \tag{26}
$$

For $k \le 1 \le n$, let

$$
c_k = \frac{[(k+1)H_{k+1}]^{kH_k}}{(kH_k)^{kH_{k-1}}} \tag{27}
$$

with assumption $S_0 = 0$. Then

$$
\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-1/H_n} = \frac{1}{(n+1)H_{n+1}}. \tag{28}
$$

By using the discrete weighted arithmetic-geometric mean inequality and interchanging the order of summations,

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{1/H_n} &= \sum_{n=1}^{\infty}\left[\prod_{k=1}^{n}(c_ka_k)^{1/k}\right]^{1/H_n}\left(\prod_{k=1}^{n}c_k^{1/k}\right)^{-1/H_n}\\
&\le \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}c_k^{1/k}\right)^{-1/H_n}\frac{1}{H_n}\sum_{k=1}^{n}\frac{c_ka_k}{k}\\
&\le \sum_{n=1}^{\infty}\frac{1}{(n+1)H_{n+1}H_n}\sum_{k=1}^{n}\frac{c_ka_k}{k}\\
&= \sum_{k=1}^{\infty}\frac{c_ka_k}{k}\sum_{n=k}^{\infty}\frac{1}{(n+1)H_nH_{n+1}}\\
&= \sum_{k=1}^{\infty}\frac{c_ka_k}{k}\sum_{n=k}^{\infty}\left(\frac{1}{H_n}-\frac{1}{H_{n+1}}\right)\\
&= \sum_{k=1}^{\infty}\frac{c_ka_k}{kH_k}\\
&= \sum_{k=1}^{\infty}\left[\frac{(k+1)H_{k+1}}{kH_k}\right]^{kH_k}a_k\\
&= \sum_{n=1}^{\infty}B_na_n.
\end{aligned}$$

Therefore, inequality (26) is proved.

Substituting (25) into (26) leads to (7). The proof of Theorem 1.3 is complete.

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School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City, Henan Province, 454010, China
E-mail address: *nnddww@163.com, nnddww@tom.com,*
nnddww@hotmail.com

School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City, Henan Province, 454010,China
E-mail address: *21caojian@163.com, goodfriendforeve@163.com,*
goodfriendforeve@163.com

Research Institute of Mathematical Inequality Theory,
Henan Polytechnic University,
Jiaozuo City, Henan Province, 454010, China
E-mail address: *qifeng@hpu.edu.cn, qifeng618@hotmail.com,*
qifeng618@msn.com, 316020821@qq.com,
fengqi618@member.ams.org
http://rgmia.vu.edu.au/qi.html