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# On Salagean-type harmonic multivalent functions

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**Abstract**

We define and investigate a new class of Salagean-type harmonic multivalent functions. we obtain coefficient inequalities, extreme points and distortion bounds for the functions in our classes.

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## 1 Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both $u$ and $v$ are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $\mathcal{D}$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathcal{D}$. See also Clunie and Sheil-Small [1].

Denote by H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ so that $f = h + \bar{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$

Recently, Ahuja and Jahangiri [9] defined the class $H_p(n)$ $(p, n \in \mathbb{N} = \{1, 2, ..\})$ consisting of all p-valent harmonic functions $f = h + \bar{g}$ that are sense preserving in $\mathbb{U}$ and $h$ and $g$ are of the form

$$(1) \quad h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1.$$

The differential operator $D^m$ was introduced by Salagean [5]. For $f = h + \bar{g}$ given by (1), Jahangiri et al. [4] defined the modified Salagean operator of $f$ as

$$(2) \qquad D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}; \qquad p > m$$

where

$$D^m h(z) = z^p + \sum_{k=2}^{\infty} (\frac{k+p-1}{p})^m a_{k+p-1} z^{k+p-1}$$

$$D^m g(z) = \sum_{k=1}^{\infty} (\frac{k+p-1}{p})^m b_{k+p-1} z^{k+p-1}.$$

For $0 \le \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in \mathbb{U}$, let $H_p(m, n, \alpha)$ denote the family of harmonic functions $f$ of the form (1) such that

$$(3) \qquad Re \left\{ \frac{D^m f(z)}{D^n f(z)} \right\} > \alpha.$$

where $D^m$ is defined by (2). Let we denote the subclass $\overline{H_p}(m, n, \alpha)$ consist of harmonic functions $f_m = h + \overline{g_m}$ in $\overline{H_p}(m, n, \alpha)$ so that $h$ and $g_m$ are of the form

$$(4)\quad h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \qquad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}$$

where $a_{k+p-1}, b_{k+p-1} \geq 0,\ |b_p| < 1$.

The families $H_p(m,n,\alpha)$ and $\overline{H_p}(m,n,\alpha)$ include a variety of well-known classes of harmonic functions as well as many new ones. For example $\overline{H_1}(1,0,\alpha) \equiv HS(\alpha)$ is the class of sense-preserving, harmonic univalent functions $f$ which are starlike of order $\alpha$ in $\mathbb{U}$, $\overline{H_1}(2,1,\alpha) \equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions $f$ which are convex of order $\alpha$ in $\mathbb{U}$ and $\overline{H_1}(n+1,n,\alpha) \equiv \overline{H}(n,\alpha)$ is the class of Salagean-type harmonic univalent functions.

For the harmonic functions $f$ of the form (1) with $b_1 = 0$, Avcı and Zlotkiewicz [2] showed that if $\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 1$ then $f \in HS(0)$ and if $\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1$ then $f \in HK(0)$. Silverman [6] proved that the above two coefficient conditions are also necessary if $f = h + \overline{g}$ has negative coefficients. Later, Silverman and Silvia [7] improved the results of [2] and [6] to the case $b_1$ not necessarily zero.

For the harmonic functions $f$ of the form (4) with $m = 1$, Jahangiri [3] showed that $f \in HS(\alpha)$ if and only if

$$\sum_{k=2}^{\infty} (k-\alpha)|a_k| + \sum_{k=1}^{\infty} (k+\alpha)|b_k| \leq 1 - \alpha$$

and $f \in \overline{H_1}(2,1,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| + \sum_{k=1}^{\infty} k(k+\alpha)|b_k| \leq 1 - \alpha.$$

In this paper, the coefficient conditions for the classes $HS(\alpha)$ and $HK(\alpha)$ are extended to the class $H_p(m,n,\alpha)$,of the forms (3) above. Furthermore, we determine extreme points and distortion theorem for the functions in $\overline{H_p}(m,n,\alpha)$.

# 2 Main Results

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H_p(m,n,\alpha)$.

**Theorem 1.** *Let $f = h + \bar{g}$ be given by (1). Furthermore, let*

$$\sum_{k=1}^{\infty} \{\Psi(m,n,p,\alpha)\,|a_{k+p-1}| + \Theta(m,n,p,\alpha)\,|b_{k+p-1}|\} \leq 2 \tag{5}$$

*where*

$$\Psi(m,n,p,\alpha) = \frac{(\frac{k+p-1}{p})^m - \alpha(\frac{k+p-1}{p})^n}{1-\alpha}$$

$$\Theta(m,n,p,\alpha) = \frac{(\frac{k+p-1}{p})^m - (-1)^{m-n}\alpha(\frac{k+p-1}{p})^n}{1-\alpha},$$

$a_p = 1$, *$\alpha(0 \leq \alpha < 1)$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $m > n$. Then $f$ is sense-preserving in $\mathbb{U}$ and $f \in H_p(m,n,\alpha)$.*

**Proof.** According to (2) and (3) we only need to show that

$$Re\left(\frac{D^m f(z) - \alpha D^n f(z)}{D^n f(z)}\right) \geq 0$$

The case $r = 0$ is obvious. For $0 \leq r < 1$, it follows that

$$Re\left(\frac{D^m f(z) - \alpha D^n f(z)}{D^n f(z)}\right) =$$

$$Re\{\frac{z^p(1-\alpha)+\sum_{k=2}^{\infty}[(\frac{k+p-1}{p})^m-\alpha(\frac{k+p-1}{p})^n]a_{k+p-1}z^{k+p-1}}{z^p+\sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}z^{k+p-1}+(-1)^n\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n\overline{b}_{k+p-1}\overline{z}^{k+p-1}}$$

$$+\frac{(-1)^m\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}\alpha(\frac{k+p-1}{p})^n]\overline{b}_{k+p-1}\overline{z}^{k+p-1}}{z^p+\sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}z^{k+p-1}+(-1)^n\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n\overline{b}_{k+p-1}\overline{z}^{k+p-1}}\}$$

$$=Re\{\frac{(1-\alpha)+\sum_{k=2}^{\infty}[(\frac{k+p-1}{p})^m-\alpha(\frac{k+p-1}{p})^n]a_{k+p-1}z^{k-1}}{1+\sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}z^{k-1}+(-1)^n\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}$$

$$+\frac{(-1)^m\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}\alpha(\frac{k+p-1}{p})^m]\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}{1+\sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}z^{k-1}+(-1)^n\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\}$$

$$=Re\left[\frac{(1-\alpha)+A(z)}{1+B(z)}\right]$$

For $z=re^{i\theta}$ we have

$$A(re^{i\theta})=\sum_{k=2}^{\infty}[(\frac{k+p-1}{p})^m-\alpha(\frac{k+p-1}{p})^n]a_{k+p-1}r^{k-1}e^{(k-1)\theta i}$$

$$+(-1)^m\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}\alpha(\frac{k+p-1}{p})^n]\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}$$

$$\begin{aligned}B(re^{i\theta}) &= \sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}r^{k-1}e^{(k-1)\theta i}\\ &+ (-1)^n\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}\end{aligned}$$

Setting

$$\frac{(1-\alpha)+A(z)}{1+B(z)}=(1-\alpha)\frac{1+w(z)}{1-w(z)}$$

the proof will be complete if we can show that $|w(z)|\leq r<1$. This is the case since,by the condition (5), we can write

$$|w(z)|=\left|\frac{A(z)-(1-\alpha)B(z)}{A(z)+(1-\alpha)B(z)+2(1-\alpha)}\right|$$

$$=\left|\frac{\sum_{k=2}^{\infty}[(\frac{k+p-1}{p})^m-(\frac{k+p-1}{p})^n]a_{k+p-1}r^{k-1}e^{(k-1)\theta i}}{2(1-\alpha)+\sum_{k=2}^{\infty}C(m,n,p,\alpha)a_{k+p-1}r^{k-}e^{(k-1)\theta i}+(-1)^m\sum_{k=1}^{\infty}D(m,n,p,\alpha)\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}}\right.$$

$$\left.+\frac{(-1)^m\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}(\frac{k+p-1}{p})^n]\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}}{2(1-\alpha)+\sum_{k=2}^{\infty}C(m,n,p,\alpha)a_{k+p-1}r^{k-1}e^{(k-1)\theta i}+(-1)^m\sum_{k=1}^{\infty}D(m,n,p,\alpha)\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}}\right|$$

where

$$C(m,n,p,\alpha)=(\frac{k+p-1}{p})^m+(1-2\alpha)(\frac{k+p-1}{p})^n$$

and

$$D(m,n,p,\alpha)=(\frac{k+p-1}{p})^m+(-1)^{m-n}(1-2\alpha)(\frac{k+p-1}{p})^n$$

$$\leq\frac{\sum_{k=2}^{\infty}[(\frac{k+p-1}{p})^m-(\frac{k+p1}{p})^n]|a_{k+p-1}|r^{k-1}}{2(1-\alpha)-\sum_{k=2}^{\infty}C(m,n,p,\alpha)|a_{k+p-1}|r^{k-1}-\sum_{k=1}^{\infty}D(m,n,p,\alpha)|b_{k+p-1}|r^{k-1}}$$

$$+\frac{\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}(\frac{k+p-1}{p})^n]|b_{k+p-1}|r^{k-1}}{2(1-\alpha)-\sum_{k=2}^{\infty}C(m,n,p,\alpha)|a_{k+p-1}|r^{k-1}-\sum_{k=1}^{\infty}D(m,n,p,\alpha)|b_{k+p-1}|r^{k-1}}$$

$$=\frac{\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(\frac{k+p-1}{p})^n]|a_{k+p-1}|r^{k-1}}{4(1-\alpha)-\sum_{k=1}^{\infty}\{C(m,n,p,\alpha)|a_{k+p-1}|+D(m,n,p,\alpha)|b_{k+p-1}|\}r^{k-1}}$$

$$+\frac{\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}(\frac{k+p-1}{p})^n]|b_{k+p-1}|r^{k-1}}{4(1-\alpha)-\sum_{k=1}^{\infty}\{C(m,n,p,\alpha)|a_{k+p-1}|+D(m,n,p,\alpha)|b_{k+p-1}|\}r^{k-1}}$$

$$<\frac{\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(\frac{k+p-1}{p})^n]|a_{k+p-1}|}{4(1-\alpha)-\sum_{k=1}^{\infty}\{C(m,n,p,\alpha)|a_{k+p-1}|+D(m,n,p,\alpha)|b_{k+p-1}|\}}$$

$$+\frac{\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m-(-1)^{m-n}(\frac{k+p-1}{p})^n]|b_{k+p-1}|}{4(1-\alpha)-\sum_{k=1}^{\infty}\{C(m,n,p,\alpha)|a_{k+p-1}|+D(m,n,p,\alpha)|b_{k+p-1}|\}}$$

$$\leq 1$$

The harmonic univalent functions

$$f(z)=z^p+\sum_{k=2}^{\infty}\frac{1}{\Psi(m,n,p,\alpha)}x_kz^{k+p-1}+\sum_{k=1}^{\infty}\frac{1}{\Theta(m,n,p,\alpha)}\overline{y_kz^{k+p-1}} \tag{6}$$

where $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m \geq n$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (5) is sharp. The functions of the form (6) are in $H_p(m, n, \alpha)$ because

$$\sum_{k=1}^{\infty} \{\Psi(m,n,p,\alpha)\, |a_{k+p-1}| + \Theta(m,n,p,\alpha)\, |b_{k+p-1}|\}$$

$$= 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.$$

In the following theorem it is shown that the condition (5) is also necessary for functions $f_m = h + \overline{g_m}$ where $h$ and $g_m$ are of the form (4).

**Theorem 2.** *Let $f_m = h + \overline{g_m}$ be given by (4). Then $f_m \in \overline{H}_p(m, n, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \{\Psi(m,n,p,\alpha)a_{k+p-1} + \Theta(m,n,p,\alpha)b_{k+p-1}\} \leq 2 \tag{7}$$

*where $a_p = 1$, $0 \leq \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $m > n$.*

**Proof.** Since $\overline{H}_p(m, n, \alpha) \subset H_p(m, n, \alpha)$, we only need to prove the "only if" part of the theorem. For functions $f_m$ of the form (4), we note that the condition

$$Re\left\{\frac{D^m f_m(z)}{D^n f_m(z)}\right\} > \alpha.$$

is equivalent to

(8)

$$Re\{\frac{(1-\alpha)z^p - \sum_{k=2}^{\infty}\left[(\frac{k+p-1}{p})^m - \alpha(\frac{k+p-1}{p})^n\right] a_{k+p-1}z^{k+p-1}}{z^p - \sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}z^{k+p-1} + (-1)^{m+n-1}\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n b_{k+p-1}\overline{z}^{k+p-1}}$$

$$+\frac{(-1)^{2m-1}\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m - (-1)^{m-n}\alpha(\frac{k+p-1}{p})^n]b_{k+p-1}\overline{z}^{k+p-1}}{z^p - \sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}z^{k+p-1} + (-1)^{m+n-1}\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n b_{k+p-1}\overline{z}^{k+p-1}}\} \geq 0$$

The above required condition (8) must hold for all values of $z$ in $\mathbb{U}$. Upon choosing the values of $z$ on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha) - \sum_{k=2}^{\infty}\left[(\frac{k+p-1}{p})^m - \alpha(\frac{k+p-1}{p})^n\right] a_{k+p-1}r^{k-1}}{1 - \sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}r^{k-1} - (-1)^{m-n}\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n b_{k+p-1}r^{k-1}} \tag{9}$$

$$+\frac{-\sum_{k=1}^{\infty}[(\frac{k+p-1}{p})^m - (-1)^{m-n}\alpha(\frac{k+p-1}{p})^n]b_{k+p-1}r^{k-1}}{1 - \sum_{k=2}^{\infty}(\frac{k+p-1}{p})^n a_{k+p-1}r^{k-1} - (-1)^{m-n}\sum_{k=1}^{\infty}(\frac{k+p-1}{p})^n b_{k+p-1}r^{k-1}} \geq 0$$

If the condition (7) does not hold, then the expression in (9) is negative for $r$ sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient in (9) is negative. This contradicts the required condition for $f_m \in \overline{H}_p(m,n,\alpha)$. And so the proof is complete.

Next we determine the extreme points of the closed convex hull of $\overline{H}_p(m,n,\alpha)$, denoted by $clco\overline{H}_p(m,n,\alpha)$.

**Theorem 3.** *Let $f_m$ be given by (4). Then $f_m \in \overline{H}_p(m,n,\alpha)$ if and only if*

$$f_m(z) = \sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{m_{k+p-1}}(z)\right]$$

*where*

$$h_p(z) = z^p, h_{k+p-1}(z) = z^p - \frac{1}{\Psi(m,n,p,\alpha)}z^{k+p-1}; \qquad (k = 2,3,\dots)$$

*and*

$$g_{m_{k+p-1}}(z) = z^p + (-1)^{m-1}\frac{1}{\Theta(m,n,p,\alpha)}\overline{z}^{k+p-1}; \qquad (k = 1,2,3,\dots)$$

$x_{k+p-1} \geq 0$ , $y_{k+p-1} \geq 0$ , $x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}$. *In particular, the extreme points of $\overline{H}_p(m,n,\alpha)$ are $\{h_{k+p-1}\}$ and $\{g_{k+p-1}\}$.*

**Proof.** For functions $f_m$ of the form (5)

$$\begin{aligned} f_m(z) &= \sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + (y_{k+p-1}g_{m_{k+p-1}}(z)\right] \\ &= \sum_{k=1}^{\infty}(x_{k+p-1}+y_{k+p-1})z^p - \sum_{k=2}^{\infty}\frac{1}{\Psi(m,n,p,\alpha)}x_{k+p-1}z^{k+p-1} \\ &+ (-1)^{m-1}\sum_{k=1}^{\infty}\frac{1}{\Theta(m,n,p,\alpha)}y_{k+p-1}\overline{z}^{k+p-1} \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\Psi(m,n,p,\alpha)\left(\frac{1}{\Psi(m,n,p,\alpha)}x_{k+p-1}\right)+\sum_{k=1}^{\infty}\Theta(m,n,p,\alpha)\left(\frac{1}{\Theta(m,n,p,\alpha)}y_{k+p-1}\right)$$

$$=\sum_{k=2}^{\infty}x_{k+p-1}+\sum_{k=1}^{\infty}y_{k+p-1}=1-x_p\le 1$$

and so $f_m(z)\in clco\overline{H}_p(m,n,\alpha)$.

Conversely, suppose $f_m(z)\in clco\overline{H}_p(m,n,\alpha,\beta)$. Letting $x_p = 1-\sum_{k=2}^{\infty}x_{k+p-1}-\sum_{k=1}^{\infty}y_{k+p-1}$ Set

$$x_{k+p-1}=\Psi(m,n,p,\alpha)a_{k+p-1}, \quad (k=2,3,\dots)$$

and

$$y_{k+p-1}=\Theta(m,n,p,\alpha)b_{k+p-1}, \quad (k=1,2,3,\dots)$$

we obtain the required representation ,since

$$f_m(z)=z^p-\sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1}+(-1)^{m-1}\sum_{k=1}^{\infty}b_{k+p-1}\overline{z}^{k+p-1}$$

$$=z^p-\sum_{k=2}^{\infty}\frac{1}{\Psi(m,n,p,\alpha)}x_{k+p-1}z^{k+p-1}+(-1)^{m-1}\sum_{k=1}^{\infty}\frac{1}{\Theta(m,n,p,\alpha)}y_{k+p-1}\overline{z^{k+p-1}}$$

$$= z^p - \sum_{k=2}^{\infty}\left[z^p - h_{k+p-1}(z)\right]x_{k+p-1} - \sum_{k=1}^{\infty}\left[z^p - g_{m_{k+p-1}}(z)\right]y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty}x_{k+p-1} - \sum_{k=1}^{\infty}y_{k+p-1}\right]z^p + \sum_{k=2}^{\infty}x_{k+p-1}h_{k+p-1}(z) + \sum_{k=1}^{\infty}y_{k+p-1}g_{m_{k+p-1}}(z)$$

$$= \sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{m_{k+p-1}}(z)\right].$$

The following theorem gives the distortion bounds for functions in $\overline{H}_p(m,n,\alpha)$ which yields a covering results for this class.

**Theorem 4.** *Let $f_m \in \overline{H}_p(m,n,\alpha)$. Then for $|z| = r < 1$ we have*

$$|f_m(z)| \le (1+b_p)r^p + \left\{\Phi(m,n,p,\alpha) - \Omega(m,n,p,\alpha)b_p\right\}r^{n+p}$$

*and*

$$|f_m(z)| \ge (1-b_p)r^p - \left\{\Phi(m,n,p,\alpha) - \Omega(m,n,p,\alpha)b_p\right\}r^{n+p}$$

*where,*

$$\Phi(m,n,p,\alpha) = \frac{1-\alpha}{(\frac{p+1}{p})^m - \alpha(\frac{p+1}{p})^n}$$

$$\Omega(m,n,p,\alpha) = \frac{1-(-1)^{m-n}\alpha}{(\frac{p+1}{p})^m - \alpha(\frac{p+1}{p})^n}$$

**Proof.** We prove the right hand side inequality for $|f_m|$. The proof for the left hand inequality can be done using similar arguments. Let $f_m \in \overline{H}_p(m,n,\alpha)$. Taking the absolute value of $f_m$ then by Theorem 2, we obtain:

$$|f_m(z)| = \left|z^p - \sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1} + (-1)^{m-1}\sum_{k=1}^{\infty}b_{k+p-1}\overline{z}^{k+p-1}\right|$$

$$\le r^p + \sum_{k=2}^{\infty}a_{k+p-1}r^{k+p-1} + \sum_{k=1}^{\infty}b_{k+p-1}r^{k+p-1}$$

$$= r^p + b_p r^p + \sum_{k=2}^{\infty} (a_{k+p-1} + b_{k+p-1}) r^{k+p-1}$$

$$\leq r^p + b_p r^p + \sum_{k=2}^{\infty} (a_{k+p-1} + b_{k+p-1}) r^{p+1}$$

$$= (1 + b_p) r^p + \Phi(m,n,p,\alpha) \sum_{k=2}^{\infty} \frac{1}{\Phi(m,n,p,\alpha)} (a_{k+p-1} + b_{k+p-1}) r^{p+1}$$

$$\leq (1 + b_p) r^p + \Phi(m,n,p,\alpha) r^{n+p} \left[ \sum_{k=2}^{\infty} \Psi(m,n,p,\alpha) a_{k+p-1} + \Theta(m,n,p,\alpha) b_{k+p-1} \right]$$

$$\leq (1 + b_p) r^p + \{\Phi(m,n,p,\alpha) - \Omega(m,n,p,\alpha) b_p\} r^{n+p}.$$

The following covering result follows from the left hand inequality in Theorem 4.

**Corollary 1.** *Let $f_m \in \overline{H}_p(m,n,\alpha)$, then for $|z| = r < 1$ we have*

$$\{w : |w| < 1 - b_p - [\Phi(m,n,p,\alpha) - \Omega(m,n,p,\alpha) b_p] \subset f_m(\mathbb{U})\}.$$

**Remark 1.** *If we take $m = 1$, $n = 0$ and $p = 1$, then the above covering result given in [3]. Furthermore, taking $m = n + 1$ and $p = 1$ we obtain the results given in [4].*

**Remark 2.** *The results of this paper, for $p = 1$, coincide with the results in [8].*

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