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Argument Estimates of certain Multivalent Functions involving Dziok–Srivastava Operator[1]

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Abstract

The purpose of this paper is to derive some argument properties using multivalent functions in the open unit disc involving Dziok-Srivastava operator. We also investigate their integral preserving property.

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1 Introduction

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

which are *analytic* in the open unit disc $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}_p$ is said to be $p-$*valently starlike* of order α in $\mathcal{U}$, if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < p; \quad z \in \mathcal{U}). \tag{1.2}$$

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The class of all p-valently starlike functions of order α is denoted by $S_p^*(\alpha)$. A function $f \in \mathcal{A}_p$ is said to be $p-$*valently convex* of order α in $\mathcal{U}$, if it satisfies

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < p; \quad z \in \mathcal{U}). \tag{1.3}$$

The class of all p-valently convex functions of order α is denoted by $\mathcal{K}_p(\alpha)$. It follows from (1.2) and (1.3) that

$$f \in \mathcal{K}_p(\alpha) \ \textit{is equivalent with} \ zf' \in S_p^*(\alpha). \tag{1.4}$$

Further, a function $f \in \mathcal{A}_p$ is said to be a *p-valently close to convex* function of order β and type α, if there exists a function $g \in S_p^*(\alpha)$ such that

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > \beta \quad (0 \leq \alpha, \beta < p; z \in \mathcal{U}). \tag{1.5}$$

It is well known (see [13]) that every p- valently close-to-convex function is p -valent in $\mathcal{U}$. For arbitrary fixed real numbers A and B $(-1 \leq B < A \leq 1)$, let $\mathcal{P}(A,B)$ denote the class of functions of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + ... \tag{1.6}$$

which are analytic in $\mathcal{U}$ and satisfies the condition

$$\phi(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \tag{1.7}$$

where the symbol $\prec$ stands for subordination. The class $\mathcal{P}(A,B)$ was introduced and studied by Janowski [11].

We note that a function $\phi \in \mathcal{P}(A,B)$, if and only if

$$\left|\phi(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (B \neq -1, z \in \mathcal{U}). \tag{1.8}$$

For any $\phi \in \mathcal{P}(A,B)$,

$$Re\left\{\phi(z)\right\} > \frac{1-A}{2} \quad (B \neq -1, z \in \mathcal{U}). \tag{1.9}$$

For a function $f \in \mathcal{A}_p$, given by (1.1), the generalized Bernardi-Libera-Livingston integral operator F [1] is defined by

$$F(z) = \frac{\gamma + p}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)dt \quad (\gamma > -p; \quad z \in \mathcal{U}). \tag{1.10}$$

A simple computation shows that

$$F(z) = z^p + \sum_{n=1}^{\infty} \frac{\gamma + p}{\gamma + p + n} a_{n+p} z^{n+p} \quad (\gamma > -p; \quad z \in \mathcal{U}). \tag{1.11}$$

It readily follows from (1.11) that

$$f \in \mathcal{A}_p \text{ implies } F \in \mathcal{A}_p. \tag{1.12}$$

For any two analytic functions $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$, the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

For complex parameters $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ with $(\beta_j \neq 0, -1, -2, ...; j = 1, 2, ..., s)$, we define the *generalized hypergeometric function* ${}_qF_s(z)$ by

$$ {}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n ... (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n ... (\beta_s)_n (1)_n} \quad z^n \tag{1.13}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} ; z \in \mathcal{U})$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{for } n = 0 \\ \lambda \ (\lambda + 1) ... (\lambda + n - 1) & \text{for } n = 1, 2, 3... \end{cases} . \tag{1.14}$$

Corresponding to a function $h_p(\alpha_1, \alpha_2, ...\alpha_q; \beta_1, \beta_2, ...\beta_s; z)$ defined by

$$h_p(\alpha_1, \alpha_2, ...\alpha_q; \beta_1, \beta_2, ...\beta_s; z) = z^p {}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z),$$

we consider the Dziok–Srivastava operator [7]

$$H_p(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s)f(z) : \mathcal{A}_p \longrightarrow \mathcal{A}_p,$$

defined by the convolution

$$H_p(\alpha_1, \alpha_2, ...\alpha_q; \beta_1, \beta_2, ...\beta_s)f(z) = h_p(\alpha_1, \alpha_2, ...\alpha_q; \beta_1, \beta_2, ...\beta_s; z) * f(z).$$

We observe that, for a function f of the form (1.1), we have

$$H_p(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s)f(z) = z^p + \sum_{n=k}^{\infty} \Gamma_n a_n z^n \tag{1.15}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-p}(\alpha_2)_{n-p}, ..., (\alpha_q)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p}, ..., (\beta_s)_{n-p}(1)_{n-p}}. \tag{1.16}$$

For convenience, we write

$$H_p(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) := H_{p,q,s}(\alpha_1) \tag{1.17}$$

Thus, through a simple calculations, we obtain

$$z\left(H_{p,q,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{p,q,s}(\alpha_1 + 1)f(z) - (\alpha_1 - p)H_{p,q,s}(\alpha_1)f(z). \tag{1.18}$$

The Dziok–Srivastava operator $H_p(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s)$ includes various other linear operators which were considered in earlier works in the literature. In particular, for $p = 2, q = 1$, the Dziok–Srivastava operator reduces to the operator $\mathcal{L}_p(\alpha_1, \alpha_2; \beta_1)f(z)$, studied by Saitoh and Nunokawa [19]. For $p = s = 1$ and $q = 2$, we obtain the linear operator:

$$\mathcal{F}(\alpha_1, \alpha_2; \beta_1)f(z) = H_1(\alpha_1, \alpha_2; \beta_1)f(z),$$

which was introduced by Hohlov [9]. Moreover, putting $\alpha_2 = 1$, we obtain the Carlson-Shaffer operator [3]:

$$\mathcal{L}(\alpha_1, \beta_1)f(z) = H_1(\alpha_1, 1; \beta_1)f(z).$$

Ruscheweyh [18] introduced an operator

$$\mathcal{D}^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda}} * f(z) \quad (\lambda \geq -1; f \in \mathcal{A}_p). \tag{1.19}$$

From the equation (1.18), we have

$$\mathcal{D}^{\lambda} f(z) = H_1(\lambda+1, 1; 1) f(z). \tag{1.20}$$

A detailed investigation on argument estimates using $L_p(a, c)$ was discussed by Cho et al. [6]. In this paper, motivated by the work of Cho et al. [6], we give some argument properties of function in certain subclasses of $\mathcal{A}_p$ involving the Dziok–Srivastava operator $H_{p,q,s}(\alpha_1)$. An application of certain integral operator is also considered . The results obtained here, besides extending the works of Bulboacă [2], Nunokawa [16], Chichra [4], Libera [12] and Sakaguchi [20], also yields a number of new results.

2 Main Results

To establish the main results we need the following lemmas.

Lemma 2.1. *[14]. Let $h(z)$ be convex univalent in $\mathcal{U}$ and let $\psi(z)$ be analytic in $\mathcal{U}$ with $Re\{\psi(z)\} \geq 0$. If $\phi(z)$ is analytic in $\mathcal{U}$ and $\psi(0) = \phi(0)$, then*

$$\phi(z) + \psi(z) z \phi'(z) \prec h(z) \quad (z \in \mathcal{U}) \tag{2.1}$$

implies

$$\phi(z) \prec h(z) \quad (z \in \mathcal{U}). \tag{2.2}$$

Lemma 2.2. *[17]. Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$. If there exists two points $z_1, z_2 \in \mathcal{U}$ such that*

$$-\frac{\eta_1 \pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\eta_2 \pi}{2} \tag{2.3}$$

for $\eta_1 > 0$, $\eta_2 > 0$, *and for* $|z| < |z_1| = |z_2|$, *then we have*

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\eta_1 + \eta_2}{2} m \tag{2.4}$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\eta_1 + \eta_2}{2} m \tag{2.5}$$

where

$$m \geq \frac{1-|a|}{1+|a|} \quad \text{and} \quad a := i\tan\left(\frac{\pi}{4}\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right). \tag{2.6}$$

Theorem 2.3. *Let* $\alpha_1 > 0, -1 \leq B < A \leq 1, f \in \mathcal{A}_p$, *and suppose that* $g \in \mathcal{A}_p$ *satisfies*

$$\frac{H_{p,q,s}(\alpha_1 + 1)g(z)}{H_{p,q,s}(\alpha_1)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathfrak{U}). \tag{2.7}$$

If

$$-\frac{\pi}{2}\delta_1 < \arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1+1)g(z)} - \beta\right\} < \frac{\pi}{2}\delta_2 \tag{2.8}$$

$$(\lambda \geq 0; \quad 0 \leq \beta < 1; \quad 0 < \delta_1, \delta_2 \leq 1; \quad z \in \mathfrak{U}),$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\left\{\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} - \beta\right\} < \frac{\pi}{2}\eta_2 \quad (z \in \mathfrak{U}) \tag{2.9}$$

where $\eta_1\,(0 < \eta_1 \leq 1)$ *and* $\eta_2\,(0 < \eta_2 \leq 1)$ *are the solutions of the equations*

$$\delta_1 = \begin{cases} \eta_1 + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2\alpha_1(1+A)/(1+B)(1+|a|)+\lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right], \textit{for } B \neq -1 \\ \eta_1\,, \textit{for } B = -1 \end{cases} \tag{2.10}$$

$$\delta_2 = \begin{cases} \eta_2 + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2\alpha_1(1+A)/(1+B)(1+|a|)+\lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right], \textit{for } B \neq -1 \\ \eta_2\,, \textit{for } B = -1 \end{cases} \tag{2.11}$$

and t_1 is given by

$$t_1 = \frac{2}{\pi}\sin^{-1}\left\{\frac{A-B}{1-AB}\right\}. \tag{2.12}$$

Proof. Let

$$\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} = \beta + (1-\beta)\phi(z). \tag{2.13}$$

Then the function is analytic $\phi(z)$ analytic in $\mathcal{U}$ with $\phi(0) = 1$. On differentiating with respect to z both sides of (2.13) and using the identity (1.18), we get

$$\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1+1)g(z)} - \beta\right\} = \tag{2.14}$$

$= (1-\beta)\left\{\phi(z) + \frac{\lambda z\phi'(z)}{\alpha_1 r(z)}\right\}$
where,

$$r(z) = \frac{H_{p,q,s}(\alpha_1+1)g(z)}{H_{p,q,s}(\alpha_1)g(z)} \tag{2.15}$$

Let

$$r(z) = \rho e^{(\pi\theta/2)i} \tag{2.16}$$

then from (2.7) followed by (1.8) and (1.9), it follows that

$$\frac{1-A}{1-B} < \rho < \frac{1+A}{1+B},$$
$$-t_1 < \theta < t_1 \quad \text{for} \quad B \neq -1,$$

where t_1 is given by (2.12), and

$$\frac{1-A}{2} < \rho < \infty,$$
$$-1 < \theta < 1 \quad \text{for} \quad B \neq -1. \tag{2.17}$$

Let $h(z)$ be the function which maps the open unit disc $\mathcal{U}$ onto the angular domain $\left\{\omega : -\frac{\pi}{2}\delta_1 < \arg(\omega) < -\frac{\pi}{2}\delta_2\right\}$ with $h(0) = 1$. Applying Lemma 2.1 for this $h(z)$ with $\psi(z) = \frac{\lambda}{\alpha r(z)}$, we see that $Re\,(\phi(z)) > 0$ in $\mathcal{U}$ and hence $\phi(z) \neq 0$ in $\mathcal{U}$. Suppose there exists points z_1 and z_2 such that (2.3) is satisfied. Then by Lemma 2.2, we obtain (2.4) and (2.5) with the restrictions (2.6). For the case $B \neq -1$, we have

$$\begin{aligned}
&\arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z_1)}{H_{p,q,s}(\alpha_1)g(z_1)} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z_1)}{H_{p,q,s}(\alpha_1+1)g(z_1)} - \beta\right\}\\
&= \arg\phi(z_1) + \arg\left\{1 + \frac{\lambda}{\alpha_1 r(z_1)}\frac{z_1\phi'(z_1}{\phi(z_1}\right\}\\
&= \frac{-\pi}{2}\eta_1 + \arg\left[1 - i\frac{\lambda e^{i(\frac{-\pi\theta}{2})}}{\rho\alpha_1}\frac{\eta_1+\eta_2}{2}m\right]\\
&\leq \frac{-\pi}{2}\eta_1 - \tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)m\sin\frac{\pi}{2}(1-\theta))}{2\alpha_1\rho + \lambda(\eta_1+\eta_2)m\cos\frac{\pi}{2}(1-\theta)}\right] \leq \frac{-\pi}{2}\eta_1 -\\
&- \tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2\alpha_1(1+A)/(1+B)(1+|a|) + \lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right]\\
&= -\frac{\pi}{2}\delta_1
\end{aligned}$$

and

$$\begin{aligned}
&\arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z_2)}{H_{p,q,s}(\alpha_1)g(z_2)} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z_2)}{H_{p,q,s}(\alpha_1+1)g(z_2)} - \beta\right\}\\
&= \arg\phi(z_2) + \arg\left\{1 + \frac{\lambda}{\alpha_1 r(z_2)}\frac{z_2\phi'(z_2}{\phi(z_2}\right\} \geq \frac{\pi}{2}\eta_2 +\\
&+ \tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2\alpha_1(1+A)/(1+B)(1+|a|) + \lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right]\\
&= \frac{\pi}{2}\delta_2.
\end{aligned}$$

Similarly for the case B = –1, we have

$$\arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1+1)g(z)} - \beta\right\} \leq -\frac{\pi}{2}\delta_1$$

and

$$\arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z_2)}{H_{p,q,s}(\alpha_1)g(z_2)}+\lambda\frac{H_{p,q,s}(\alpha_1+1)f(z_2)}{H_{p,q,s}(\alpha_1+1)g(z_2)}-\beta\right\}\geq\frac{\pi}{2}\delta_2$$

where we have used the inequality (2.6) with δ_1, δ_2 and t_1 are as given in (2.10), (2.11) and (2.12) respectively. These obviously contradict the assumption of Theorem 2.3. The proof of Theorem 2.3 is thus completed. If we let $\delta_1 = \delta_2 = \delta$ in Theorem 2.3, we readily obtain the following.

Corollary 2.4. *Let* $\alpha_1 > 0$, $-1 \leq B < A \leq 1$, $f \in \mathcal{A}_p$, *and suppose that* $g \in \mathcal{A}_p$ *satisfies*

$$\frac{H_{p,q,s}(\alpha_1+1)g(z)}{H_{p,q,s}(\alpha_1)g(z)}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathfrak{U}). \tag{2.18}$$

If

$$\left|\arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)}+\lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1+1)g(z)}-\beta\right\}\right|<\frac{\pi}{2}\delta \tag{2.19}$$

$$(\lambda\geq 0;\quad 0\leq\beta<1;\quad 0<\delta\leq 1;\quad z\in\mathfrak{U}),$$

then

$$\left|\arg\left\{\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)}-\beta\right\}\right|<\frac{\pi}{2}\eta\quad(z\in\mathfrak{U}) \tag{2.20}$$

where $\eta\,(0<\eta\leq 1)$ *is the solution of the equation*

$$\delta=\begin{cases}\eta+\frac{2}{\pi}\tan^{-1}\left[\frac{\lambda\eta\sin\frac{\pi}{2}(1-t_1)}{\alpha_1(1+A)/(1+B)+\lambda\eta\cos\frac{\pi}{2}(1-t_1)}\right] & \text{for } B\neq -1\\ \eta & \text{for } B=-1\end{cases} \tag{2.21}$$

and t_1 *is given by (2.12).*

For $s = q = 1$ in Theorem 2.3 , we have the following corollary.

Corollary 2.5. *Let $\alpha_1 > 0, -1 \leq B < A \leq 1, f \in \mathcal{A}_p$, and suppose that $g \in \mathcal{A}_p$ satisfies*

$$\frac{L_p(\alpha_1+1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}). \tag{2.22}$$

If

(2.23)
$$-\frac{\pi}{2}\delta_1 < \arg\left\{(1-\lambda)\frac{L_p(\alpha_1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} + \lambda\frac{L_p(\alpha_1+1,\beta_1)f(z)}{L_p(\alpha_1+1,\beta_1)g(z)} - \beta\right\} < \frac{\pi}{2}\delta_2$$

$$(\lambda \geq 0; \quad 0 \leq \beta < 1; \quad 0 < \delta_1, \delta_2 \leq 1; \quad z \in \mathcal{U}),$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\left\{\frac{L_p(\alpha_1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} - \beta\right\} < \frac{\pi}{2}\eta_2 \quad (z \in \mathcal{U}) \tag{2.24}$$

where $\eta_1\,(0 \leq \eta_1 \leq 1)$ and $\eta_2\,(0 \leq \eta_2 \leq 1)$ are the solutions of the equations

(2.25)
$$\delta_1 = \begin{cases} \eta_1 + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2\alpha_1(1+A)/(1+B)(1+|a|)+\lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right], for B \neq -1 \\ \eta_1\,,\ for\ B = -1 \end{cases}$$

(2.26)
$$\delta_2 = \begin{cases} \eta_2 + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2\alpha_1(1+A)/(1+B)(1+|a|)+\lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right], for\ B \neq -1 \\ \eta_2\,,\ for\ B = -1 \end{cases}$$

and t_1 is given by (2.12).

For $\delta_1 = \delta_2 = \delta, \quad s = q = 1$ in Theorem 2.3, we get the following result.

Corollary 2.6. *Let $\alpha_1 > 0, -1 \leq B < A \leq 1, f \in \mathcal{A}_p$, and suppose that $g \in \mathcal{A}_p$ satisfies*

$$\frac{L_p(\alpha_1+1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}). \tag{2.27}$$

If

$$\left|\arg\left\{(1-\lambda)\frac{L_p(\alpha_1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} + \lambda\frac{L_p(\alpha_1+1,\beta_1)f(z)}{L_p(\alpha_1+1,\beta_1)g(z)} - \beta\right\}\right| < \frac{\pi}{2}\delta \tag{2.28}$$

$$(\lambda \geq 0; \quad 0 \leq \beta < 1; \quad 0 < \delta \leq 1; \quad z \in \mathfrak{U}),$$

then

$$\left|\arg\left\{\frac{L_p(\alpha_1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} - \beta\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathfrak{U}) \tag{2.29}$$

where $\eta\,(0 < \eta \leq 1)$ *is the solution of the equation*

$$\delta = \begin{cases} \eta + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda\eta\sin\frac{\pi}{2}(1-t_1)}{\alpha_1(1+A)/(1+B)(1+|a|)+\lambda\eta\cos\frac{\pi}{2}(1-t_1)}\right] & \text{for } B \neq -1 \\ \eta & \text{for } B = -1 \end{cases} \tag{2.30}$$

and t_1 *is given by (2.12).*

Remark 2.7. *For* $\delta_1 = \delta_2, \quad s = q = 1, \quad \alpha_1 = \beta_1 = p, \quad A = 1, \quad B = -1$ *and* $\lambda = 1$ *in Theorem 2.3, we get the result obtained by Nunokawa [16].*

Taking $s = q = 1, \quad \alpha_1 = \mu + p, \quad \beta_1 = 1, \quad A = 1$, and $B = 0$ in Theorem 2.3, we have the following corollary.

Corollary 2.8. *Let* $-1 \leq B < A \leq 1, f \in \mathcal{A}_p$, *and suppose that* $g \in \mathcal{A}_p$ *satisfies*

$$\frac{D^{\mu+p}g(z)}{D^{\mu+p-1}g(z)} \prec 1 + z \quad (z \in \mathfrak{U}). \tag{2.31}$$

If

$$-\frac{\pi}{2}\delta_1 < \arg\left\{(1-\lambda)\frac{D^{\mu+p-1}f(z)}{D^{\mu+p-1}g(z)} + \lambda\frac{D^{\mu+p}f(z)}{D^{\mu+p}g(z)} - \beta\right\} < \frac{\pi}{2}\delta_2 \tag{2.32}$$

$$(\lambda \geq 0; \quad 0 \leq \beta < 1; \quad 0 < \delta_1, \delta_2 \leq 1; \quad z \in \mathfrak{U}),$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\left\{\frac{D^{\mu+p-1}f(z)}{D^{\mu+p-1}g(z)} - \beta\right\} < \frac{\pi}{2}\eta_2 \quad (z \in \mathfrak{U}) \tag{2.33}$$

where $\eta_1\,(0<\eta_1\le 1)$ *and* $\eta_2\,(0<\eta_2\le 1)$ *are the solutions of the equations*

(2.34)

$$\delta_1=\begin{cases}\eta_1+\frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2(\mu+p)(1+|a|)+\lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right] & \text{for } B\neq -1\\ \eta_1 & \text{for } B=-1\end{cases}$$

(2.35)

$$\delta_2=\begin{cases}\eta_2+\frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)\sin\frac{\pi}{2}(1-t_1)}{2(\mu+p)(1+|a|)+\lambda(\eta_1+\eta_2)(1-|a|)\cos\frac{\pi}{2}(1-t_1)}\right] & \text{for } B\neq -1\\ \eta_2 & \text{for } B=-1\end{cases}$$

and t_1 *is given by*

$$t_1=\frac{2}{\pi}\sin^{-1}\left\{\frac{A-B}{1-AB}\right\}. \tag{2.36}$$

Letting $B\to A$ and $g(z)=z^p$ in Theorem 2.3, we get the following corollary

Corollary 2.9. *Let* $f\in\mathcal{A}_p$. *If*

(2.37)

$$-\frac{\pi}{2}\delta_1<\arg\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{z^p}-\beta\right\}<\frac{\pi}{2}\delta_2$$

$$(\alpha_1>0,\quad \lambda\ge 0;\quad 0\le\beta<1;\quad 0<\delta_1,\delta_2\le 1;\quad z\in\mathfrak{U}), \tag{2.38}$$

then,

$$-\frac{\pi}{2}\eta_1<\arg\left\{\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p}-\beta\right\}<\frac{\pi}{2}\eta_2(z\in\mathfrak{U}) \tag{2.39}$$

where $\eta_1\,(0<\eta_1\le 1)$ *and* $\eta_2\,(0<\eta_2\le 1)$ *are the solutions of the equations*

$$\delta_1=\eta_1+\frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)}{2\alpha_1(1+|a|)}\right] \tag{2.40}$$

and

$$\delta_2=\eta_2+\frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)}{2\alpha_1(1+|a|)}\right]. \tag{2.41}$$

Corollary 2.10. *Under the hypothesis of Corollary 2.9, we have*

$$-\frac{\pi}{2}\,\eta_1 < \arg\{G'(z)-\beta\} < \frac{\pi}{2}\,\eta_2 \qquad (z \in \mathcal{U}) \tag{2.42}$$

where the function $G(z)$ is defined in $\mathcal{U}$ by

$$G(z) = \int_0^z \frac{H_{p,q,s}(\alpha_1)f(t)}{t^p}dt \tag{2.43}$$

and $\eta_1\,(0<\eta_1\le 1)$ and $\eta_2\,(0<\eta_2\le 1)$ are the solutions of the equations (2.40) and (2.41).

For $q = s = 1$, $B \to A$ and $g(z) = z^p$, in Corollary 2.9, we have the following corollary.

Corollary 2.11. *If $f \in \mathcal{A}_p$, satisfies*

$$-\frac{\pi}{2}\,\delta_1 < \arg\left\{(1-\lambda)\frac{L_p(\alpha_1+1,\beta_1)f(z)}{z^p} + \lambda\frac{L_p(\alpha_1+1,\beta_1)f(z)}{z^p} - \beta\right\} < \frac{\pi}{2}\,\delta_2 \tag{2.44}$$

$$(\alpha_1 > 0, \quad \lambda \ge 0; \quad 0 \le \beta < 1; \quad 0 < \delta_1, \delta_2 \le 1; \quad z \in \mathcal{U}), \tag{2.45}$$

then

$$-\frac{\pi}{2}\,\eta_1 < \arg\left\{\frac{L_p(\alpha_1,\beta_1)f(z)}{z^p} - \beta\right\} < \frac{\pi}{2}\,\eta_2 (z \in \mathcal{U}) \tag{2.46}$$

where $\eta_1\,(0<\eta_1\le 1)$ and $\eta_2\,(0<\eta_2\le 1)$ are the solutions of the equations

$$\delta_1 = \eta_1 + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)}{2\alpha_1(1+|a|)}\right] \tag{2.47}$$

$$\delta_2 = \eta_2 + \frac{2}{\pi}\tan^{-1}\left[\frac{\lambda(\eta_1+\eta_2)(1-|a|)}{2\alpha_1(1+|a|)}\right]. \tag{2.48}$$

Remark 2.12. *Taking $q = s$, $p = \alpha_1 = \beta_1$, $\lambda = 1$, and $\beta = 0$ in Corollary 2.9 and $q = s = p = \alpha_1 = \alpha_2$, and $\beta = 0$ in Corollary 2.10, we get the results obtained by Cho et al. [5]*

Setting $A = 1 - \frac{2\alpha}{p}$ $(0 \le \alpha < p)$, $B = -1$ and $\delta_1 = \delta_2 = 1$ in Theorem 2.3, we get the following corollary.

Corollary 2.13. *Let $\alpha_1 > 0, f \in \mathcal{A}_p$, and suppose that $g \in \mathcal{K}_p(\alpha)$. If*

$$Re\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} + \lambda\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1+1)g(z)}\right\} > \beta \tag{2.49}$$

$$(\lambda \ge 0; \quad 0 \le \beta < 1; \quad z \in \mathcal{U}),$$

then

$$Re\left\{\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)}\right\} > \beta \qquad (z \in \mathcal{U}). \tag{2.50}$$

Corollary 2.14. *Let $\alpha_1 > 0, -1 \le B < A \le 1, f \in \mathcal{A}_p$, and suppose that $g \in \mathcal{K}_p(\alpha)$. If*

$$Re\left\{(1-\lambda)\frac{L_p(\alpha_1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} + \lambda\frac{L_p(\alpha_1+1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)} - \beta\right\} > \beta \tag{2.51}$$

$$(\lambda \ge 0; \quad 0 \le \beta < 1; \quad z \in \mathcal{U}),$$

then,

$$Re\left\{\frac{L_p(\alpha_1,\beta_1)f(z)}{L_p(\alpha_1,\beta_1)g(z)}\right\} > \beta \quad (z \in \mathcal{U}). \tag{2.52}$$

Remark 2.15. *For $q = s = 1, \quad \alpha_1 = \beta_1 = p = 1$ and $\alpha = 0$, Corollary 2.13, is the result obtained by Bulboacă [2]. If we put $q = s = 1, \quad \alpha_1 = \beta_1 = p = 1$ and $\beta = 0$,and $g(z) = z$ in Corollary 2.13, then we have the result due to Chichra [4]. Further,taking $q = s = 1, \quad \alpha_1 = \beta_1 = p, \lambda = 1$ and $\alpha = \beta = 0$ in 2.13, we get the corresponding results obtained by Libera [12] and Sakaguchi [20].*

Theorem 2.16. *If $f \in \mathcal{A}_p$, satisfies*

$$-\frac{\pi}{2}\delta_1 < \arg\left\{\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} - \beta\right\} < \frac{\pi}{2}\delta_2 \tag{2.53}$$

$$(\alpha_1 > 0, \quad \lambda \geq 0; \quad 0 \leq \beta < 1; \quad 0 < \delta_1, \delta_2 \leq 1; \quad z \in \mathcal{U})$$

then

(2.54)

$$-\frac{\pi}{2}\eta_1 < \arg\left\{\frac{(\gamma+p)\int\limits_0^z t^{\gamma-1}H_{p,q,s}(\alpha_1)f(t)dt}{z^{\gamma+p}} - \beta\right\} < \frac{\pi}{2}\eta_2 \quad (\gamma > -p;\ z \in \mathcal{U})$$

where $\eta_1\,(0 \leq \eta_1 \leq 1)$ *and* $\eta_2\,(0 \leq \eta_2 \leq 1)$ *are the solutions of the equations*

$$\delta_1 = \eta_1 + \frac{2}{\pi}\tan^{-1}\left[\frac{(\eta_1+\eta_2)(1-|a|)}{2(\gamma+p)(1+|a|)}\right] \tag{2.55}$$

$$\delta_2 = \eta_2 + \frac{2}{\pi}\tan^{-1}\left[\frac{(\eta_1+\eta_2)(1-|a|)}{2(\gamma+p)(1+|a|)}\right]. \tag{2.56}$$

Proof Consider the function $\phi(z)$ defined in $\mathcal{U}$ by

$$\frac{(\gamma+p)\int\limits_0^z t^{\gamma-1}H_{p,q,s}(\alpha_1)f(t)dt}{z^{\gamma+p}} = \beta + (1-\beta)\phi(z). \tag{2.57}$$

Then $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0) = 1$. Differentiating both sides of (2.57) and simplifying, we get

$$\frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} - \beta = (1-\beta)\left\{\phi(z) + \frac{z\phi'(z)}{\gamma+p}\right\}. \tag{2.58}$$

Now, by using Lemma 2.1 and a similar method in the proof of Theorem 2.3, we get (2.54).

Theorem 2.17. *Let* $f \in \mathcal{A}_p$. *If*

$$-\frac{\pi}{2}\delta_1 < \arg\left\{\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - \frac{\alpha_1-p-\gamma}{\alpha_1}\right\} < \frac{\pi}{2}\delta_2 \tag{2.59}$$

$$(\alpha_1 > 0; \quad p+\gamma > 0;\ 0 < \delta_1, \delta_2 \leq 1; \quad z \in \mathcal{U})$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\left\{\frac{z^\gamma H_{p,q,s}(\alpha_1)f(z)}{\int\limits_0^z t^{\gamma-1}H_{p,q,s}(\alpha_1)f(t)dt}\right\} < \frac{\pi}{2}\eta_2 \quad (\gamma > -p; \quad z \in \mathcal{U}) \tag{2.60}$$

where $\eta_1\,(0 < \eta_1 \leq 1)$ *and* $\eta_2\,(0 < \eta_2 \leq 1)$ *are the solutions of the equations*

$$\delta_1 = \eta_1 + \frac{2}{\pi}\tan^{-1}\left[\frac{(\eta_1+\eta_2)(1-|a|)}{2(\gamma+p)(1+|a|)}\right] \tag{2.61}$$

$$\delta_2 = \eta_2 + \frac{2}{\pi}\tan^{-1}\left[\frac{(\eta_1+\eta_2)(1-|a|)}{2(\gamma+p)(1+|a|)}\right]. \tag{2.62}$$

Proof. The proof of the theorem is much akin to the proof of Theorem 2.16 and hence we omit the details involved.

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