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Common fixed points of quasi-contractive operators[1]

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Abstract

We establish a general theorem to approximate common fixed points of quasi-contractive operators on a normed space through the modified Ishikawa iteration process with errors in the sense of Liu [8]. Our result generalizes and improves upon, among others, the corresponding result of Berinde [1].

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1 Introduction

Throughout this note, $\mathbb{N}$ will denote the set of all positive integers. Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be a mapping. Let $\{b_n\}$ and $\{b'_n\}$ be two sequences in $[0, 1]$.

The Mann iteration process is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [9]):

$$(1.1) \qquad \begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T x_n, n \in \mathbb{N}. \end{cases}$$

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The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)\, x_n + b_n T y_n, \\ y_n = (1-b'_n)\, x_n + b'_n T x_n,\ n \in \mathbb{N} \end{cases} \tag{1.2}$$

is known as the Ishikawa iteration process [4].

Liu [8] introduced the concept of Ishikawa iteration process with errors by the sequence $\{x_n\}_{n=0}^{\infty}$ defined as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)\, y_n + b_n T y_n + u_n, \\ y_n = (1-b'_n)\, x_n + b'_n T x_n + v_n,\ n \in \mathbb{N} \end{cases} \tag{1.3}$$

where $\{b_n\}$ and $\{b'_n\}$ are sequences in $[0,1]$ and $\{u_n\}$ and $\{v_n\}$ satisfy $\sum\limits_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. This surely contains both (1.1) and (1.2). Also this contains the Mann process with error terms

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)x_n + b_n T x_n + u_n,\ n \in \mathbb{N} \end{cases}. \tag{1.4}$$

For two self mappings S and T of C, the Ishikawa iteration processes have been generalized by Das and Debata [3] as follows

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)\, x_n + b_n S y_n \\ y_n = (1-b'_n)\ x_n + b'_n T x_n,\ n \in \mathbb{N}. \end{cases} \tag{1.5}$$

They used this iteration process to find the common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space. Takahashi and Tamura [15] studied it for the case of two nonexpansive mappings under different conditions in a strictly convex Banach space.

Inspired and motivated by these facts, we suggest the following two-step iterative process with errors in the sense of Liu [8] and define the sequence $\{x_n\}_{n=0}^{\infty}$ as follows

$$
(1.6) \qquad \begin{cases} x_1 = x \in C, \\ x_{n+1} = b_n S y_n + (1 - b_n) x_n + u_n \,, \\ y_n = b_n' T x_n + (1 - b_n') x_n + v_n, \;\; n \in \mathbb{N}, \end{cases}
$$

where $\{u_n\}, \{v_n\}$ are two summable sequences in C.

Clearly, this iteration process contains all the processes (1.1-1.5) as its special cases.

Note that the case of two mappings, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem. See, for example, [14].

We recall the following definitions in a metric space (X, d). A mapping $T : X \to X$ is called an a-contraction if

$$
(1.7) \qquad d(Tx, Ty) \leq a d(x, y) \text{ for all } x, y \in X,
$$

where $a \in (0, 1)$.

The map T is called Kannan mapping [5] if there exists $b \in (0, \frac{1}{2})$ such that

$$
(1.8) \qquad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \;\text{ for all } x, y \in X.
$$

A similar definition is due to Chatterjea [2]: there exists a $c \in (0, \frac{1}{2})$ such that

$$
(1.9) \qquad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X.
$$

Combining these three definitions, Zamfirescu [17] proved the following important result.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b and c satisfying*

$a \in (0,1)$, $b, c \in (0, \frac{1}{2})$ such that for each pair $x, y \in X$, at least one of the following conditions holds:

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

An operator T satisfying the contractive conditions (z_1), (z_2) and (z_3) in the above theorem is called Zamfirescu operator.

In 2004, Berinde [1] introduced a new class of operators on an arbitrary Banach space E_1 satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|Tx - x\| \tag{1.10}$$

for any $x, y \in E_1$, $0 \leq \delta < 1$.

He proved that this class is wider than the class of Zamfiresu operators and used the Ishikawa iteration process (1.2) to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem:

Theorem 1.2. *Let K be a nonempty closed convex subset of an arbitrary Banach space E_1. Let $T : K \to K$ be an operator satisfying* (1.10). *Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process* (1.2). *If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

In this paper, a convergence theorem of Rhoades [12] regarding the approximation of fixed points of some quasi contractive operators in uniformly convex Banach spaces using the Mann iteration process, is extended to the approximation of common fixed points of some quasi contractive operators in normed spaces using the iteration process (1.6).

The following lemma is proved in [8].

Lemma 1.1. *Let* $\{r_n\}, \{s_n\}, \{t_n\}$ *and* $\{k_n\}$ *be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n)r_n + s_n t_n + k_n \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=1}^{\infty} k_n < \infty$ *hold, then* $\lim_{n\to\infty} r_n = 0$.

2 Main Results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a normed space* E. *Let* S, $T : C \to C$ *be two operators satisfying condition* Z. *Let* $\{x_n\}_{n=0}^{\infty}$ *be defined through the iterative process* (1.6). *If* $F(T) \neq \phi$ *and* $\sum_{n=1}^{\infty} b_n = \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ *and* $\lim_{n\to\infty} \|v_n\| = 0$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a common fixed point of* S *and* T.

Proof. Following the approach of Berinde [1], since S, $T : C \to C$ be two operators satisfying condition Z, at least one of the conditions $(z_1), (z_2)$ and (z_3) is satisfied. If (z_2) holds, then for $x, y \,\epsilon\, C$

$$\begin{aligned} \|Sx - Ty\| &\leq b\left[\|x - Sx\| + \|y - Ty\|\right] \\ &\leq b\left[\|x - Sx\| + \|y - x\| + \|x - Sx\| + \|Sx - Ty\|\right], \end{aligned}$$

implies

$$(1 - b)\|Sx - Ty\| \leq b\|x - y\| + 2b\|x - Sx\|,$$

which yields (using the fact that $0 \leq b < 1$)

$$\|Sx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Sx\|. \tag{2.1}$$

If (z_3) holds, then similarly we obtain

$$\|Sx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Sx\|. \tag{2.2}$$

Denote

$$\delta = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}. \tag{2.3}$$

Then we have $0 \leq \delta < 1$ and, in view of (z_1), (2.1-2.3) it results that the inequality

$$\|Sx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Sx\| \tag{2.4}$$

holds for all $x, y \in C$.

In a similar fashion, we can find

$$\|Sx - Ty\| \leq \delta \|x - y\| + 2\delta \|y - Ty\|, \tag{2.5}$$

for all $x, y \in E$, $0 \leq \delta < 1$.

Assume that $F \neq \phi$. Let $w \in F$ and $\{x_n\}_{n=0}^{\infty}$ be the iteration process (1.6). Then

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-b_n) x_n + b_n S y_n + u_n - w\| \\ &= \|(1-b_n)(x_n - w) + b_n (S y_n - w) + u_n\| \end{aligned}$$

$$\leq (1-b_n) \|x_n - w\| + b_n \|S y_n - w\| + \|u_n\|. \tag{2.6}$$

Now for $x = y_n$ and $y = w$, (2.5) gives

$$\|S y_n - w\| \leq \delta \|y_n - w\|. \tag{2.7}$$

In a similar fashion, we can get

$$\begin{aligned} \|y_n - w\| &= \|(1-b_n') x_n + b_n' T x_n + v_n - w\| \\ &= \|(1-b_n')(x_n - w) + b_n' (T x_n - w) + v_n\| \end{aligned}$$

$$\leq (1-b_n') \|x_n - w\| + b_n' \|T x_n - w\| + \|v_n\|. \tag{2.8}$$

Again by (2.4), if $x = w$ and $y = x_n$, we get

$$\|T x_n - w\| \leq \delta \|x_n - w\|, \tag{2.9}$$

and hence, by (2.6-2.9) we obtain

$$\|x_{n+1} - w\| \leq [1 - (1-\delta)(1+\delta b'_n)b_n] \|x_n - w\| + \delta b_n \|v_n\| + \|u_n\|,$$

which, by the inequality

$$1 - (1-\delta^2)b_n \leq 1 - (1-\delta)(1+\delta b'_n)b_n \leq 1 - (1-\delta)b_n,$$

yields

$$\|x_{n+1} - w\| \leq [1 - (1-\delta)b_n] \|x_n - w\| + \delta b_n \|v_n\| + \|u_n\|, \ n = 0, 1, 2, ...$$

With the help of Lemma 1 and using the fact that $0 \leq \delta < 1$, $0 \leq b_n, b'_n \leq 1$, $\sum_{n=1}^{\infty} b_n = \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\lim_{n\to\infty} \|v_n\| = 0$, it results that

$$\lim_{n\to\infty} \|x_{n+1} - w\| = 0.$$

Consequently $x_n \to w \in F$ and this completes the proof.

Corollary 2.1. *Let C be a nonempty closed convex subset of a normed space E. Let S, $T : C \to C$ be two operators satisfying* $(2.4 - 2.5)$. *Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process* (1.5). *If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of S and T.*

Corollary 2.2. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.10). *Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process* (1.3). *If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\lim_{n\to\infty} \|v_n\| = 0$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Corollary 2.3. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.10). *Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.4). *If $F(T) \neq$ phi, $\sum_{n=1}^{\infty} b_n = \infty$ and $\|u_n\| = 0(b_n)$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Corollary 2.4. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.10). *Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.2). *If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Corollary 2.5. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.10). *Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.1). *If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Remark 2.1. *1. The Chatterjea's and the Kannan's contractive conditions* (1.8) *and* (1.9) *are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process are obtained in Corollary 2.4.*

2. Theorem 4 of Rhoades [12] in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary 2.5.

3. In Corollary 2.5, Theorem 8 of Rhoades [13] is generalized to the setting of normed spaces.

4. Our result also generalizes Theorem 5 of Osilike [10] and Theorem 2 of Osilike [11].

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