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On a subclass of analytic functions[1]

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Abstract

In the present paper, the authors study the differential inequality

$$\Re\left[\alpha\sqrt[m]{q(z)}+\beta z\left(\sqrt[m]{q(z)}\right)'\right]>\gamma$$

where $q(z)$ is an analytic function such that $q(0)=1$ and α,β,γ,m be real numbers and its applications to analytic functions in the open unit disc $E=\{z:|z|<1\}$.

Key Words: Univalent function, Starlike function, Convex function, Multiplier transformation.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k$, $p\in\mathbb{N}=\{1,2,....\}$, which are analytic in the open unit disc $E=\{z:|z|<1\}$. We write $\mathcal{A}_1=\mathcal{A}$. A function $f\in\mathcal{A}_p$ is said to be p-valent starlike of order $\alpha(0\leq\alpha<p)$ in E if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in E.$$

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We denote by $S_p^*(\alpha)$, the class of all such functions. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order $\alpha(0 \leq \alpha < p)$ in E if

$$\Re\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) > \alpha,\ z \in E.$$

Let $K_p(\alpha)$ denote the class of all those functions $f \in \mathcal{A}_p$ which are multivalently convex of order α in E. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order $\alpha, 0 \leq \alpha < 1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$(1) \qquad I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et.al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Srivastava [3] and Cho and Kim [4], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [10]. $I_1(n, 0)$ is the well-known Sălăgean [9] derivative operator D^*, defined as: $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$.

A function $f \in \mathcal{A}$ is said to belong to the class R if it satisfies the condition

$$\Re\left[f^{'}(z) + zf^{''}(z)\right] > 0, \quad z \in E$$

And it is well known that $R \subset S^*$.

A function $f \in \mathcal{A}_p$ is said to be in the class $S_n(p, \lambda, \alpha)$ for all z in E if it satisfies

$$\Re\left[\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] > \frac{\alpha}{p},$$

for some $\alpha\,(0 \leq \alpha < p, p \in \mathbb{N})$. We note that $S_0(1,0,\alpha)$ and $S_1(1,0,\alpha)$ are the usual classes $S^*(\alpha)$ and $K(\alpha)$ of starlike functions of order α and convex functions of order α, respectively.

In 1989, Owa, Shen and Obradovič [7] obtained a sufficient condition for a function $f \in \mathcal{A}$ to belong to the class $S_n(1,0,\alpha) = S_n(\alpha)$.

Recently, Li and Owa [5] studied the operator $I_1(n,0)$.

In this paper, the authors study the differential inequality

$$\Re\left[\alpha \sqrt[m]{q(z)} + \beta z \left(\sqrt[m]{q(z)}\right)'\right] > \gamma$$

where $q(z)$ is an analytic function such that $q(0) = 1$ and α, β, γ, m be real numbers. And the authors also discuss the applications of the above mentioned differential inequality to multiplier transformation defined by (1) in the open unit disc $E = \{z : |z| < 1\}$.

2 Preliminaries

We shall make use of the following lemma of Miller and Mocanu to prove our result.

Lemma 2.1.([6])*Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\Phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let Φ satisfy the following conditions:*
(i) Φ is continuous in $\mathbb{D}$;
(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\Phi(1,0) > 0$; and
(iii) $\Re\Phi(iu_2, v_1) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1+u_2^2)/2$.

Let $r(z) = 1 + r_1 z + r_2 z^2 + \ldots$ be regular in the unit disc E, such that $(r(z), zr'(z)) \in \mathbb{D}$ for all $z \in E$. If

$$\Re[\Phi(r(z), zr'(z))] > 0,\ z \in E,$$

then $\Re r(z) > 0,\ z \in E$.

We, now, state and prove our main results.

3 Main Results

Lemma 3.1. *Let* $q(z) = 1 + q_1 z + q_2 z^2 + ...$ *be an analytic function in* E *which satisfies the condition*

$$\Re\left[\alpha \sqrt[m]{q(z)} + \beta z \left(\sqrt[m]{q(z)}\right)'\right] > \gamma \tag{2}$$

where α, β, γ *and* m *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re\left[\sqrt[2m]{q(z)}\right] > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}$$

Proof. Let us define $q(z)$ as

$$(q(z))^{\frac{1}{2m}} = \delta + (1-\delta)r(z), \quad z \in E \tag{3}$$

where δ is the nonnegative root of the quadratic equation

$$(\alpha+\beta)\delta^2 - \beta\delta - \gamma = 0$$

given by

$$\delta = \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)} \tag{4}$$

which satisfies the condition $0 \leq \delta < 1$.

Therefore $r(z)$ is an analytic function in E and $r(z) = 1 + r_1 z + r_2 z^2 + ...$ In view of (2), we have

$$(5)\Re\frac{1}{1-\gamma}\left[\alpha \sqrt[m]{q(z)} + \beta z \left(\sqrt[m]{q(z)}\right)' - \gamma\right] = \Re\frac{1}{1-\gamma}[\alpha[\delta + (1-\delta)r(z)]^2$$

$$+2\beta(1-\delta)(\delta + (1-\delta)r(z))zr'(z) - \gamma].$$

If $\mathbb{D} = \mathbb{C} \times \mathbb{C}$, define $\Phi(u,v) : \mathbb{D} \to \mathbb{C}$ as under

$$\Phi(u,v) = \frac{1}{1-\gamma}\left[\alpha\left[\delta + (1-\delta)u\right]^2 + 2\beta(1-\delta)(\delta + (1-\delta)u)v - \gamma\right].$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0) \in \mathbb{D}$ and $\Re\ \Phi(1,0) = \frac{\alpha-\gamma}{1-\gamma} > 0$.

Further, in view of (5), we get $\Re\Phi(r(z), zr'(z)) > 0,\ z \in E$.

Let $u = u_1 + iu_2, v = v_1 + iv_2$ where u_1, u_2, v_1 and v_2 are all reals. Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\frac{1+u_2^2}{2}$, we have

$$\Re\Phi(iu_2, v_1) = \Re\frac{1}{1-\gamma}\left[\alpha\left[\delta + (1-\delta)iu_2\right]^2 + 2\beta(1-\delta)(\delta + (1-\delta)iu_2)v_1 - \gamma\right]$$

$$= \frac{1}{1-\gamma}\left[(\alpha+\beta)\delta^2 - 2\beta\delta - \gamma\right]$$

$$\leq 0.$$

In view of (3), (4) and Lemma 2.1, proof now follows.

Theorem 3.1. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left[\alpha\sqrt[m]{\frac{I_p(n,\lambda)f(z)}{z^p}} + \beta z\left(\sqrt[m]{\frac{I_p(n,\lambda)f(z)}{z^p}}\right)'\right] > \gamma$$

where α, β, γ and m be real numbers such that $\alpha \geq 0,\ \beta \geq 0$ and $\alpha > \gamma$ then

$$\Re\sqrt[2m]{\frac{I_p(n,\lambda)f(z)}{z^p}} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

Proof. Let us write

$$q(z) = \frac{I_p(n,\lambda)f(z)}{z^p}$$

In view of Lemma 3.1, the proof follows.

Theorem 3.2. *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left[\alpha\sqrt[m]{\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}} + \beta z\left(\sqrt[m]{\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}}\right)'\right] > \gamma$$

where α, β, γ and m be real numbers such that $\alpha \geq 0,\ \beta \geq 0$ and $\alpha > \gamma$ then

$$\Re\sqrt[2m]{\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

Proof. Let us write

$$q(z) = \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}$$

In view of Lemma 3.1, the proof follows.

4 Corollaries

By taking $p = 1$ and $\lambda = 0$ in Theorem 3.1. We have the following
Corollary 4.1. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\alpha\sqrt[m]{\frac{D^n f(z)}{z}}+\beta z\left(\sqrt[m]{\frac{D^n f(z)}{z}}\right)'\right] > \gamma$$

where α, β, γ and m be real numbers such that $\alpha \geq 0$, $\beta \geq 0$ and $\alpha > \gamma$ then

$$\Re\sqrt[2m]{\frac{D^n f(z)}{z}} > \frac{\beta+\sqrt{\beta^2+4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, n = 0$ and $\lambda = 0$ in Theorem 3.1. We have the following
Corollary 4.2. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\alpha\sqrt[m]{\frac{f(z)}{z}}+\beta z\left(\sqrt[m]{\frac{f(z)}{z}}\right)'\right] > \gamma$$

where α, β, γ and m be real numbers such that $\alpha \geq 0$, $\beta \geq 0$ and $\alpha > \gamma$ then

$$\Re\sqrt[2m]{\frac{f(z)}{z}} > \frac{\beta+\sqrt{\beta^2+4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, n = 0, \lambda = 0, \alpha = 1$ and $m = 2$ in Theorem 3.1. We have the following result of Aouf, M.K. and Hossen, H.M. [2] for $n = 1$.
Corollary 4.3.*If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\sqrt[2]{\frac{f(z)}{z}}+\beta z\left(\sqrt[2]{\frac{f(z)}{z}}\right)'\right] > \gamma$$

where β and γ be real numbers such that $\beta \geq 0$ and $\gamma < 1$ then

$$\Re\sqrt[4]{\frac{f(z)}{z}} > \frac{\beta+\sqrt{\beta^2+4\gamma(1+\beta)}}{2(1+\beta)}.$$

By taking $p = 1, n = 1$ and $\lambda = 0$ in Theorem 3.1. We have the following

Corollary 4.4. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\alpha \sqrt[m]{f'(z)} + \beta z \left(\sqrt[m]{f'(z)}\right)'\right] > \gamma$$

where α, β, γ and m be real numbers such that $\alpha \geq 0$, $\beta \geq 0$ and $\alpha > \gamma$ then

$$\Re \sqrt[2m]{f'(z)} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, n = 1, \lambda = 0, \alpha = 1$ and $m = 2$ in Theorem 3.1. We have the following result of Owa, S. and Wu, Z. [8].

Corollary 4.5. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\sqrt[2]{f'(z)} + \beta z \left(\sqrt[2]{f'(z)}\right)'\right] > \gamma$$

where β and γ be real numbers such that $\beta \geq 0$ and $\gamma < 1$ then

$$\Re \sqrt[4]{f'(z)} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(1+\beta)}}{2(1+\beta)}.$$

By taking $p = 1, n = 1, \lambda = 0$ and $m = \frac{1}{2}$ in Theorem 3.1. We have the following

Corollary 4.6. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\alpha \left[f'(z)\right]^2 + \beta z \left(\left[f'(z)\right]^2\right)'\right] > \gamma$$

where α, β and γ be real numbers such that $\alpha \geq 0$, $\beta \geq 0$ and $\alpha > \gamma$ then

$$\Re f'(z) > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}$$

i.e. f is close-to-convex and hence univalent.

By taking $p = 1, n = 2$ and $\lambda = 0$ in Theorem 3.1. We have the following

Corollary 4.7. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\alpha \sqrt[m]{f'(z) + zf''(z)} + \beta z \left(\sqrt[m]{f'(z) + zf''(z)}\right)'\right] > \gamma$$

where α, β, γ *and* m *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re \sqrt[2m]{f'(z) + zf''(z)} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, n = 2, \lambda = 0$ and $m = \dfrac{1}{2}$ in Theorem 3.1. We have the following

Corollary 4.8. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re \left[\alpha \left(f^{'}(z) + zf^{''}(z) \right)^2 + \beta z \left[\left(f^{'}(z) + zf^{''}(z) \right)^2 \right]^{'} \right] > \gamma$$

where α, β *and* γ *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re \left(f^{'}(z) + zf^{''}(z) \right) > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}$$

i.e. $f \in R$ *and hence* $f \in S^*$.

By taking $p = 1$ and $\lambda = 0$ in Theorem 3.2. We have the following

Corollary 4.9. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re \left[\alpha \sqrt[m]{\frac{D^{n+1}f(z)}{D^n f(z)}} + \beta z \left(\sqrt[m]{\frac{D^{n+1}f(z)}{D^n f(z)}} \right)^{'} \right] > \gamma$$

where α, β, γ *and* m *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re \sqrt[2m]{\frac{D^{n+1}f(z)}{D^n f(z)}} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, \lambda = 0$ and $m = \dfrac{1}{2}$ in Theorem 3.2. We have the following

Corollary 4.10. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re \left[\alpha \left(\frac{D^{n+1}f(z)}{D^n f(z)} \right)^2 + \beta z \left(\left(\frac{D^{n+1}f(z)}{D^n f(z)} \right)^2 \right)^{'} \right] > \gamma$$

where α, β *and* γ *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re \frac{D^{n+1} f(z)}{D^n f(z)} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}$$

i.e. $f \in S(\delta)$, *where* $\delta = \dfrac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}, \quad 0 \leq \delta < 1.$

By taking $p = 1, n = 0$ and $\lambda = 0$ in Theorem 3.2. We have the following

Corollary 4.11. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re \left[\alpha \sqrt[m]{\frac{zf'(z)}{f(z)}} + \beta z \left(\sqrt[m]{\frac{zf'(z)}{f(z)}} \right)' \right] > \gamma$$

where α, β, γ *and* m *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re \left[\sqrt[2m]{\frac{zf'(z)}{f(z)}} \right] > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, n = 0, \lambda = 0$ and $m = \frac{1}{2}$ in Theorem 3.2. We have the following

Corollary 4.12. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re \left[\alpha \left(\frac{zf'(z)}{f(z)} \right)^2 + \beta z \left(\left(\frac{zf'(z)}{f(z)} \right)^2 \right)' \right] > \gamma$$

where α, β *and* γ *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re \frac{zf'(z)}{f(z)} > \frac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}$$

i.e. $f \in S^*(\delta)$, *where* $\delta = \dfrac{\beta + \sqrt{\beta^2 + 4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}, \quad 0 \leq \delta < 1.$

By taking $p = 1, n = 1$ and $\lambda = 0$ in Theorem 3.2. We have the following

Corollary 4.13. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re \left[\alpha \sqrt[m]{1 + \frac{zf''(z)}{f'(z)}} + \beta z \left(\sqrt[m]{1 + \frac{zf''(z)}{f'(z)}} \right)' \right] > \gamma$$

where α, β, γ *and* m *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re\left[\sqrt[2m]{\frac{1+zf''(z)}{f'(z)}}\right] > \frac{\beta+\sqrt{\beta^2+4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}.$$

By taking $p = 1, n = 1, \lambda = 0$ and $m = \frac{1}{2}$ in Theorem 3.2. We have the following

Corollary 4.14. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re\left[\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + \beta z\left(\left(1+\frac{zf''(z)}{f'(z)}\right)^2\right)'\right] > \gamma$$

where α, β *and* γ *be real numbers such that* $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha > \gamma$ *then*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{\beta+\sqrt{\beta^2+4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}$$

i.e. $f \in K(\delta)$, *where* $\delta = \dfrac{\beta+\sqrt{\beta^2+4\gamma(\alpha+\beta)}}{2(\alpha+\beta)}, \quad 0 \leq \delta < 1.$

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