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A note on Mathieu′s inequality

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Abstract

In this note we obtain a generalization for Mathieu′s inequality.

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1. Introduction

Mathieu[16] conjectured in 1890 that the inequality

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2} \tag{1}$$

where c is a real number, $c \neq 0$, is valid. It was proved only in 1952 by L. Berg[3]. E. Makai [15] gave a very elegant and elementary proof for (1) and obtained the following lower estimation:

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} > \frac{1}{c^2+\frac{1}{2}}.$$

P. H. Dianada [7] rafined Mathieu′s inequality (1) to

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} > \frac{1}{c^2} - \frac{1}{(2c^2+2c+1)(8c^2+8c+3)}$$

Gh.Costovici [6] has proved the following inequalities of Mathieu type:

$$\sum_{n=1}^{\infty} \frac{4n(n+1)(n+2)}{(n^2+2n+c^2)^4} < \frac{1}{c^4} \tag{2}$$

and

$$\sum_{n=1}^{\infty} \frac{6n(n+1)(n+2)(n+3)(n+4)}{(n^2+5n+c^2)^6} < \frac{1}{c^6}. \tag{3}$$

H. Alzer, J.L. Brenner and O. G. Ruchr [2] showed that the best constants a and b in

$$\frac{1}{c^2+a} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2+b}, \quad c \neq 0$$

are $a = \frac{1}{2}\xi(3)$ and $b = \frac{1}{6}$, where $\xi(\cdot)$ denotes the Riemann Zeta function defined by

$$\xi(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Using the integral expression for Mathieu′s series, many authors obtained interesting refinements and extensions of Mathieu′s inequality ([1], [12], [13], [14], [18], [19]).

In [1] D. Acu proved the inequality

$$\sum_{n=1}^{\infty} \frac{(p+1)n_{[p]}}{(n_{[p]}(n+\frac{p-1}{2})+c^2)^2} < \frac{1}{c^2}$$

$c \neq 0, p \geq 1$, where $n_{[p]} = n(n+1)\ldots(n+p-1)$.

P.Dicu and M. Acu in [8] obtained

$$\frac{1}{a_1^2+c^2+\frac{r^2}{2}-a_1 r} < \sum_{n=1}^{\infty} \frac{2a_n r}{(a_n^2+c^2)^2} < \frac{1}{a_1^2+c^2-a_1 r},$$

$c \neq 0$, where $(a_n)_{n \geq 1}$ is an arithmetic progression with $a_1 > 0$ and the ration $r > 0$.

In this note, we present the proofs more simple for the inequalities (2) and (3), and give new inequalities of type (2)-(3).

2.A simple proof of the inequality (2)

We have

$$(n^2 + 2n + c^2)^4 > (n^2 + 2n)^4 + 6n^2(n+2)^2 \cdot c^4 + c^8$$

and

$$(n^2+2n)^4 = n^4(n+2)^4 = n^2 \cdot n^2(n+2)^2(n+2)^2 > n^2(n^2-1)(n+2)^2(n^2+4n+3) =$$

$$= (n-1)n(n+1)(n+2)n(n+1)(n+2)(n+3).$$

But $6n^2(n+2)^2 > (n-1)n(n+1)(n+2) + n(n+1)(n+2)(n+3)$ because it is equivalent to

$4n^2 + 8n - 2 > 0$, which is true for $n \geq 1$.

Now, it results

$$(n^2+2n+c^2)^4 > [(n-1)n(n+1)(n+2)+c^4][n(n+1)(n+2)(n+3)+c^4]. \tag{4}$$

Form (4), it follows that

$$\sum_{n=1}^{\infty} \frac{4n(n+1)(n+2)}{(n^2+2n+c^2)^4} < \sum_{n=1}^{\infty} \frac{4n(n+1)(n+2)}{[(n-1)n(n+1)(n+2)+c^4][n(n+1)(n+2)(n+3)+c^4]}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)n(n+1)(n+2)+c^4} - \frac{1}{n(n+1)(n+2)(n+3)+c^4} \right) < \frac{1}{c^4}, \text{ q.e.d.}$$

3.A simple proof of (3)

Observe that

$$(5)\qquad (n^2+5n+c^2)^6 > (n^2+5n)^6 + 20(n^2+5n)^3\cdot c^6 + c^{12}.$$

Since $n^2 > n^2-1$ and $n^2(n+5)^5 > (n+1)(n+2)^2(n+3)^2(n+4)^2$ which is equivalent to

$$6n^6+99n^5+631n^4+1771n^3+1489n^2-1344n-576>0$$

for $n\in\mathbb{N}, n\geq 1$, we obtain

$$(6)\qquad (n^2+5n)^6 = n^2n^2n^2(n+5)^5(n+5) > n^2(n^2-1)(n+1)(n+2)^2\cdot$$

$\cdot(n+3)^2(n+4)^5(n+5) = [(n-1)n(n+1)(n+2)(n+3)(n+4)][n(n+1)(n+2)(n+3)(n+4)(n+5)]$.

Now, we deduce

$$(7)\qquad 20n^3(n+5) > (n-1)n(n+1)(n+2)(n+3)(n+4)$$

$+n(n+1)(n+2)(n+3)(n+4)(n+5)$, because it is equivalent to

$$9n^5+136n^4+695n^3+1130n^2-124n-48>0$$

which is true for $n\in\mathbb{N}, n\geq 1$.

From (5), (6) and (7) we obtain the inequality

$$(8)\qquad (n^2+5n+c^2)^6 > [(n-1)n(n+1)(n+2)(n+3)(n+4)+c^6]$$

$\cdot[n(n+1)(n+2)(n+3)(n+4)(n+5)+c^6]$.

Using (8) we have

$$\sum_{n=1}^{\infty}\frac{6n(n+1)(n+2)(n+3)(n+4)}{(n^2+5n+c^2)^6} <$$

$$\sum_{n=1}^{\infty}\frac{6n(n+1)(n+2)(n+3)(n+4)}{[(n-1)n(n+1)(n+2)(n+3)(n+4)+c^6][n(n+1)(n+2)(n+3)(n+4)(n+5)+c^6]}=$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)n(n+1)(n+2)(n+3)(n+4)+c^6}-\frac{1}{n(n+1)(n+2)(n+3)(n+4)(n+5)+c^6}\right)$$
$$<\frac{1}{c^6}$$

and the inequality (3) is proved.

4. A new inequality by type (2) and (3)

By a reasoning similar to the proof of (2) and (3) we also can prove the following inequality:

$$\sum_{n=1}^{\infty}\frac{8n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(n^2+7n+c^2)^8}<\frac{1}{c^8}.$$

For the proof we observe the following inequalities

$$(n^2+7n+c^2)^8>(n^2+7)^8+70(n^2+7n)^4c^8+c^{10},$$

$$(n^2+7n)^8>(n-1)n^2(n+1)^2(n+2)^2(n+3)^2(n+4)^2(n+5)^2(n+6)^2(n+7)$$

and

$$70(n^2+7n)^4>(n-1)n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)$$
$$+n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)$$

are valid for $n\in\mathbb{N}, n\geq 1$.

5. The open problem

We denote

$$n_{[p]}=n(n+1)\dots(n+p-1), p\in\mathbb{N}^*, n\in\mathbb{N}^*.$$

Is the inequality

$$\sum_{n=1}^{\infty} \frac{(2p+2)n_{[2p+1]}}{(n^2+(2p+1)n+c^2)^{2p+2}} < \frac{1}{c^{2p+2}}, c \neq 0 \text{ true?}$$

6. The other elementary inequalities of Mathieu′s type

6.1 *If $c \neq 0$, then we have*

(9)
$$\sum_{n=1}^{\infty} \frac{n}{[3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1) + c^2]^2} < \frac{1}{2(c^2+1)}.$$

Proof of (9) follows from

$$\frac{n}{[3{\cdot}5{\cdot}7{\cdot}\ldots{\cdot}(2n-1)+c^2]^2} < \frac{n}{[3{\cdot}5{\cdot}7{\cdot}\ldots{\cdot}(2n+1)+c^2][3{\cdot}5{\cdot}7{\cdot}\ldots{\cdot}(2n+1)+c^2]} =$$

$$= \frac{1}{2}\left(\frac{1}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)+c^2} - \frac{1}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)+c^2}\right).$$

6.2 *If $c \neq 0$ and $a > 0$, then we have*

(10)
$$\sum_{n=1}^{\infty} \frac{2an}{(an^2+an+c^2)^2} < \frac{1}{c^2}.$$

Proof of (10) follows from

$$\frac{2an}{(an^2{+}an{+}c^2)^2} < \frac{2an}{(an^2{-}an{+}c^2)(an^2{+}an{+}c^2)} = \frac{1}{an^2{-}an{+}c^2}\frac{1}{an^2{+}an{+}c^2} =$$

$$= \frac{1}{an^2-an+c^2} - \frac{1}{a(n+1)^2-a(n+1)+c^2}.$$

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