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Solution of a polylocal problem using Tchebychev polynomials

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Abstract

Consider the problem:

$$Ly(x) = r(x),\ -1 \leq x \leq 1,$$
$$y(a) = A,\ y(b) = B$$
$$-1 < a < b < 1,\ a, b, A, B \in \mathbb{R},$$

where

$$Ly(x) := -\frac{d}{dx}(\frac{dy}{dx}) + q(x) \cdot y(x),\ -1 \leq x \leq 1$$

and

$$q(x), r(x) \in C[-1, 1],\ y(x) \in C^2[-1, 1].$$

The aim of this paper is to present an approximate solution of this problem based on Tchebychev polynomials.We construct the approximation using Tchebychev-Gauss-Lobatto interpolation nodes.Also we use Maple 10 to obtain an numerical results.

Dedicated to the memory of *prof. Alexandru Lupas*(1942-2007)

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1 Introduction

The purpose of this paper is to approximate the solution of the following problem:

$$(1)\qquad \begin{cases} Ly(x) = r(x),\ -1 \leq x \leq 1, \\ y(a) = A,\ y(b) = B \\ -1 < a < b < 1,\ a, b, A, B \in \mathbb{R}, \end{cases}$$

where:

$$(2)\qquad Ly(x) := -\frac{d}{dx}(\frac{dy}{dx}) + q(x) \cdot y(x),\ -1 \leq x \leq 1$$

and $q(x), r(x) \in C[-1, 1],\ y(x) \in C^2[-1, 1]$, using *a Collocation method.* This is not a *two boundary value problem* ,since $-1 < a < b < 1$.

We have two initial value problem on $[-1, a]$ and $[b, 1]$, respectively, and on $[a, b]$ a classical boundary value problem, the existence and the uniqueness for (1) assure existence and uniqueness of these problems.

Historical note. In 1966 ,two researchers from Tiberiu Popoviciu Institute of Romanian Academy,Cluj-Napoca ,*Dumitru Ripianu and Oleg Arama* published a paper on a polylocal problem, see([6])

2 Principles of the method

The implementation is inspired from([2]).Our method is based on first kind *Tchebychev polynomials* ([3]) and ([5]).

Definition 1 *The polynomials $T_n(x), n \in \mathbb{N}$ defined by :*

(3) $$T_n(x) := \cos(n \arccos(x)), x \in [-1, 1]$$

are called the Tchebychev polynomials of the first kind*.*

Definition 2 *The polynomials $U_n(x), n \in \mathbb{N}$ defined by:*

(4) $$U_n(x) = \frac{\sin(n+1) \cdot \arccos x}{(n+1) \cdot \sqrt{1-x^2}}$$

are called the Tchebychev polynomials of the second kind.

To describe the basic method in this and later section we choose a nonuniform mesh of the given interval $[-1, 1]$ therefore:

(5) $$\Delta : \, x_j = \cos \frac{j \cdot \pi}{n}, \; j = 1, \ldots, n, \; .$$

The are the zeros of *Tchebychev polynomials of second kind.*

The form of solution is:

(6) $$u(x) = \sum_{k=0}^{n+1} c_k \cdot T_k(x)$$

where $T_k(x)$ is the k-th degree first kind *Tchebychev polynomials* on interval $[-1, 1]$.

We shall choose the basis such that the following conditions hold:

- the solution verifies the differential equation

(7) $$Lu(x_j) = r_j, j = 1, 2, ...n$$

- the solution verifies

$$u(a) = A, u(b) = B. \tag{8}$$

We choose this mesh (3),because in([2],pag30) prove that interpolation at *Tchebychev points* is nearly optimal . Since the mesh(3) has n points, we include the points a, b and suppose that $a, b \neq x_j$ for all $j = 1, 2, ...n$.

Remark 1
- *If* $a = x_j$,$b = x_j$ *then we increment* n.
- *the method do not depend on conditions on* $q(x)$.
- *The* Tchebychev polynomials *are generated via the* orthopoly *package with the Maple sequence:*

$$> \ S := (x, k, a, b) - > T(k, ((b - a)) * x + a + b)/2) :$$

3 Numerical Results

We shall give two examples.For each example we plot the exact and approximate solution and generate the execution profile with the pair *profile-showprofile* see ([4])

- ***First we approximate a oscillating solution*** **:**

$$-Z''(t) - 243 \cdot Z(t) = t; -1 \leq t \leq 1 \tag{9}$$

$$Z(-1) = Z(1) = 0$$

with conditions:

$$Z\left(-\frac{1}{4}\right)=\frac{-\sin\left(\frac{9\sqrt{3}}{4}\right)}{243\sin\left(9\sqrt{3}\right)}+\frac{1}{972} \tag{10}$$

$$Z\left(\frac{1}{2}\right)=\frac{\sin\left(\frac{9\sqrt{3}}{2}\right)}{243\sin\left(9\sqrt{3}\right)}-\frac{1}{486} \tag{11}$$

The exact solution provided by *dsolve* is:

$$Z(t)=\frac{\sin\left(9\sqrt{3}t\right)-t\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}$$

Since

$$\int_{-1}^{1}|q(x)|\,dx>2$$

using disconjugate criteria given by *Lyapunov* (1893) the problem (9) has an oscillatory solution. We used Maple 8 to solve the problem exactly and to approximate the solution, for $n=17$ and $n=50$.We also plot the error in semilogarithmic scale.

The cod Maple is:

```
> restart; with(orthopoly);with(CodeTools);with(plots):
> S:=(x,k,a,b)->T(k,((b-a)*x+a+b)/2);
S := (x, k, a, b) -> T(k, 1/2 (b - a) x + 1/2 a + 1/2 b)
> genceb:=proc(x,n,q,r,c0,d0,alpha,beta)
> local k, ecY, ecd, C, h, Y, c, a, b;
> global S;
> a:=x[0]; b:=x[n-1];
> Y:=0;
```

```
> for k from 0 to n+1 do
> Y:=Y+c[k]*S(t,k,a,b);
> end do;
> Y:=simplify(Y);
> ecY:=-diff(Y,t$2)+q(t)*Y=r(t):
> ecd:=Array(0..n+1);
> for k from 0 to n-1 do
> ecd[k]:=eval(ecY,t=x[k]):
> end do;
> ecd[n]:=eval(Y,t=c0)=alpha:
> ecd[n+1]:=eval(Y,t=d0)=beta:
> C:=solve({seq(ecd[k],k=0..n+1)},[seq(c[k],k=0..n+1)]);
> assign(C):
> return Y:
> end proc:
>#we define the function from differential equations
> q:=t->-243: r:=t->t:
> ecz:=-diff(Z(t),t$2)+q(t)*Z(t)=r(t):
> dsolve({ecz,Z(-1)=0,Z(1)=0},Z(t)):simplify(%):
> assign(%):
> n:=17: b:=1: a:=-1: c0:=-1/4: d0:=1/2:alpha:=eval(Z(t),t=c0);
>beta=eval(Z(t),t=d0);
> u:=[-1/4,seq((b-a)/2*cos(k*Pi/n)+(a+b)/2,k=1..n),1/2]:
> u:=sort(evalf(u)):
> n:=nops(u);x:=Array(0..n-1,u):
```

```
> eval(x):
>profile(genceb): Y:=genceb(x,n,q,r,-1/4,1/2,alpha,beta)
> plot(Y,t=-1..1,title=" Approx TCHEBYCHEV");
> plot(Z(t),t=-1..1,color=[GREEN],title="Exact Solution");
>p1:=plot([Y,Z(t)],t=-1..1,title="Exact&Approx.solution;n=17"):
> p2:=plots[pointplot]([[-1/4,eval(Z(t),t=-1/4)],[1/2,eval(Z(t),t=1/2)]],
>symbol=circle,symbolsize=30,color=[BLACK]):
> plots[display]({p1,p2});showprofile(genceb);
> plot(log(Y-Z(t)),t=-0.99..0.99,title="Error in Semilog. scale;n=17",
color=[BLUE]);
> #quit
```

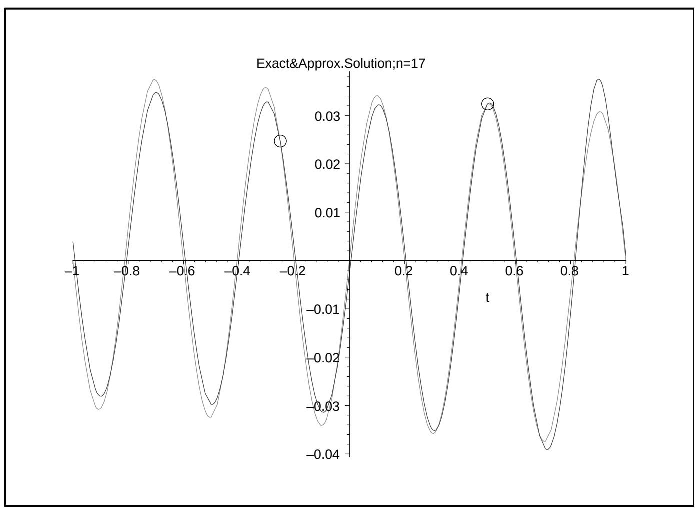

Fig. 1. The graph of exact and approximate solution,oscillating problem, n=17

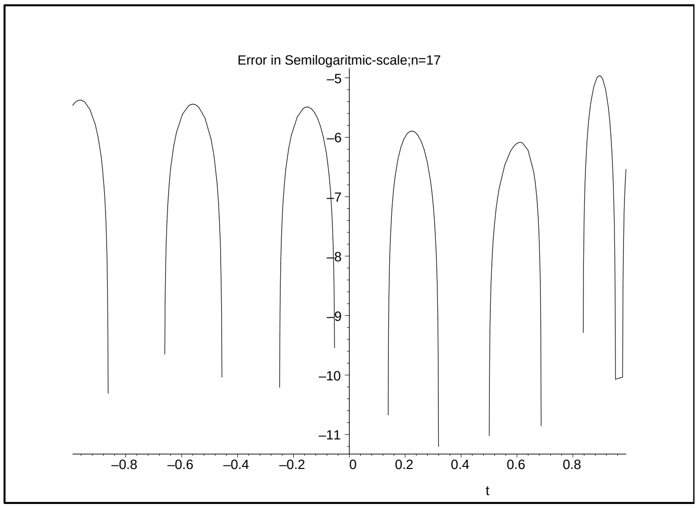

Fig. 2. Error plot, oscillating problem, n=17

Here are the profiles for the procedure in the case of oscillating solution:

function	depth	calls	time	time%	bytes	bytes%
genceb	1	1	1.907	100.00	37992012	100.00
total	1	1	1.907	100.00	37992012	100.00

For $n = 50$, we obtain a very good approximation , but we must increase the number of decimals with Maple command:

$$> Digits := 18;$$

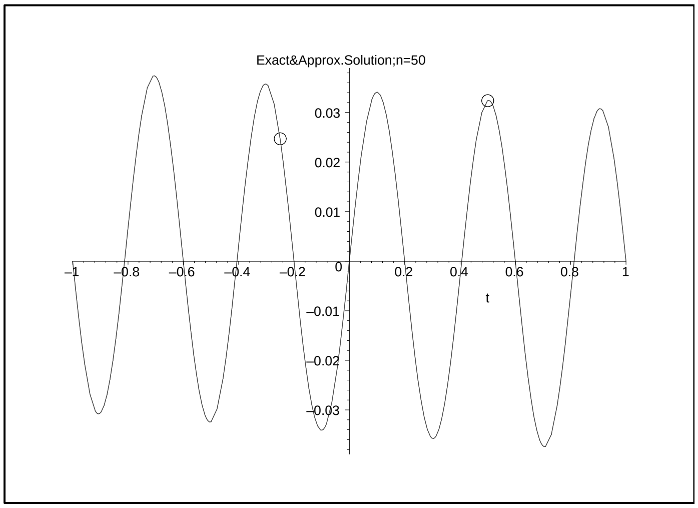

Fig. 3. The graph of exact and approximate solution, oscillating problem, n=50

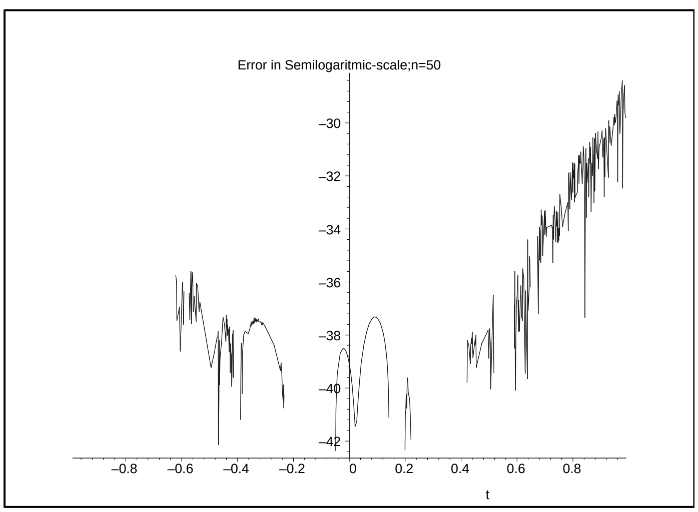

Fig. 4. Error plot, oscillating problem, n=50

Here are the profiles for the procedures in the case of oscillating problem. Here are the profiles for the procedure in the case of oscillating solution:

function	depth	calls	time	time%	bytes	bytes%
genceb	1	1	28.749	100.00	455186448	100.00
total	1	1	28.749	100.00	455186448	100.00

- ***The second example is a nonoscillating solution***.

Example is from ([1, page 560]).

$$-y'' - y = x, \qquad x \in [-1, 1] \tag{12}$$
$$y(-1) = y(1) = 0$$

with conditions:

$$y\left(-\frac{1}{4}\right) = -\frac{\sin\frac{1}{4}}{\sin 1} + \frac{1}{4}$$
$$y\left(\frac{1}{2}\right) = \frac{\sin\frac{1}{2}}{\sin 1} - \frac{1}{2}$$

The exact solution given by $d$solve is $y(t) = -\frac{-\sin(t)+t\sin 1}{\sin 1}$ .Since

$$\int_{-1}^{1} |q(x)|\, dx \leq 2$$

using disconjugate criteria given by *Lyapunov* (1893) the problem (12) has an nonoscillatory solution. For $n = 10$,we plot the graph of exact solution and approximation.

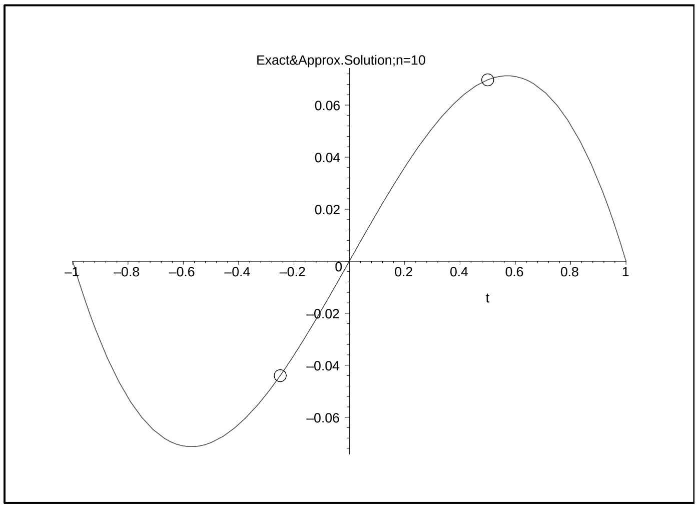

Fig. 5. The graph of exact and approximate solution, non oscillating problem, n=10

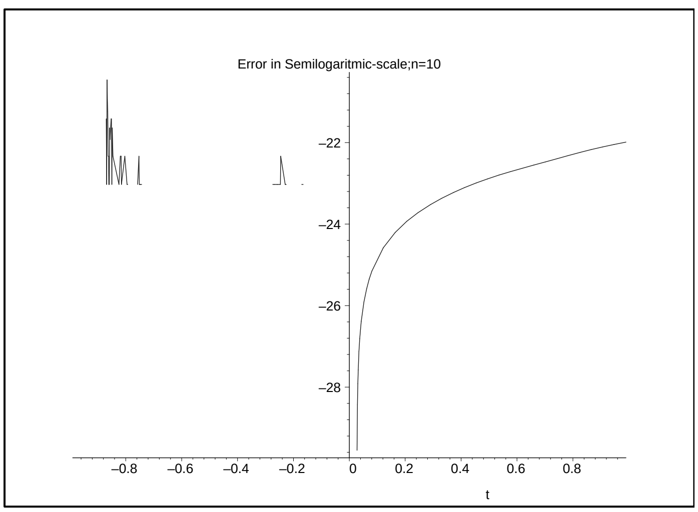

Fig. 6. Error plot, non oscillating problem

The profile in this case is:

function	depth	calls	time	time%	bytes	bytes%
genceb	1	1	0.406	100.00	9541308	100.00
total	1	1	0.406	100.00	9541308	100.00

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