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On a general class of Beta approximating operators

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Abstract

By using the generalized beta distribution (GB) we obtain a general class of beta operators, which include both the beta operators of the first and second kind (see [5], [6], [9], [10]). We obtain a several positive linear operators, as a special cases of this beta operator.

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1 Introduction

We continue our earlier investigations [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] concerning to use Euler's function for constructing linear positive operators.

We first review the definition of the generalized beta distributions of the first and second kind (GB1 and GB2, respectively) and several important special cases. We then define the generalized beta (GB) family (which includes both GB1 and GB2) and give expressions for the defined moments.

The Euler beta distribution of the first kind is defined for $p, q > 0$ by the following formula

$$\mathcal{B}(t; p, q) = \frac{1}{B(p, q)} t^{p-1}(1-t)^{q-1}, \text{ for } t \in (0, 1) \tag{1}$$

and zero otherwise, where $B(p, q)$ is the beta function.

The generalized beta distribution of the first kind is defined by the probability density function (pdf) (see [2])

$$GB1(y; e, d, p, q) = \frac{|e| y^{ep-1}(1-(y/d)^e)^{q-1}}{d^{ep} B(p, q)} \text{ for } 0 < y^e < d^e \tag{2}$$

and zero otherwise, where the parameters d, p, q are positive and $e \in \mathbb{R}$.

The defined kth-order moments of GB1 random variables are given by [2]

$$E_{GB1}(y^k) = \frac{d^k B(p + k/e, q)}{B(p, q)} \text{ for } p + k/e > 0. \tag{3}$$

This four-parameter pdf is very flexible and includes the modified beta distribution of the first kind for $e = 1$ (see [2])

$$\begin{aligned} MB1(y; d, p, q) &= GB1(y; e = 1, d, p, q) \\ &= \frac{y^{p-1}(d-y)^{q-1}}{d^{p+q-1} B(p, q)}, \quad 0 < y < d. \end{aligned} \tag{4}$$

The distribution $MB1(y; d, p, q)$ with $p, q > 0$ and $0 < d \leq 1$ is said to be the Beta-Stacy (BS) distribution (see [13]).

The standard beta distribution of the first kind (1) corresponds to (4) with $d = 1$.

The Euler beta distribution of the second kind is defined for $p, q > 0$ by the following formula

$$B(u; p, q) = \frac{1}{B(p, q)} \cdot \frac{u^{p-1}}{(1+u)^{p+q}}, \quad u > 0 \tag{5}$$

and zero otherwise.

The generalized beta distribution of the second kind is defined by the pdf (see [2])

$$GB2(v; e, d, p, q) = \frac{|e| v^{ep-1}}{d^{ep} B(p, q)(1 + (v/d)^e)^{p+q}}, \text{ for } v > 0 \tag{6}$$

and zero otherwise.

The defined kth-order moments of the GB2 are given by [2]

$$E_{GB2}(v^k) = \frac{d^k B(p + k/e, q - k/e)}{B(p, q)} \text{ for } -p < k/e < q. \tag{7}$$

Letting $e = 1$ in (6) gives the modified beta distribution of the second kind (MB2)

$$\begin{aligned} MB2(v; d, p, q) &= GB2(v, e = 1, d, p, q) \\ &= \frac{d^q v^{p-1}}{B(p, q)(d + v)^{p+q}}, \quad v > 0. \end{aligned} \tag{8}$$

The standard beta distribution of the second kind (5) is obtained by (8) for $d = 1$.

Since neither the GB1 or GB2 (MB1 or MB2) includes the other as a special case they have often been considered separately, with any comparisons being rather informal. Each of these distributions can be shown to be special cases of a more general distribution, the generalized beta (GB) distribution defined by the pdf [2]

$$GB(y; e, c, d, p, q) = \frac{|e| y^{ep-1}(1 - (1 - c)(y/d)^e)^{q-1}}{d^{ep} B(p, q)(1 + c(y/d)^e)^{p+q}}, \tag{9}$$

for $0 < y^e < \dfrac{d^e}{1-c}$, and zero otherwise with $0 \leq c \leq 1$, and d, p, q positive and $e \in \mathbb{R}$.

The moments of (9) can be shown to be [2]

$$E_{GB}(y^k) = \frac{d^k B(p+k/e, q)}{B(p,q)}\ {}_2F_1\left(\begin{matrix} p+k/e, k/e \\ p+q+k/e \end{matrix}; c\right) \tag{10}$$

where ${}_2F_1$ denotes the hypergeometric series which converges for all k if $c < 1$, or for $k/e < q$ if $c = 1$ (see [14]).

Substituting $k = 0$ into (10) verifies that (9) integrates to one.

Comparing (9) with (2) and (7), respectively, we can easily verify that

$$GB1(y; e, d, p, q) = GB(y; e, d, c = 0, p, q)$$

and

$$GB2(y; e, d, p, q) = GB(y; e, d, c = 1, p, q)$$

i.e., the GB includes the GB1 and GB2 corresponding to $c = 0$ and $c = 1$.

A modified beta distribution (MB) can be defined in terms of the pdf

$$\begin{aligned} MB(y; c, d, p, q) &= GB(y; e = 1, c, d, p, q) \\ &= \frac{y^{p-1}(1-(1-c)(y/d))^{q-1}}{d^p B(p,q)(1+c(y/d))^{p+q}} \text{ for } 0 < y < d/(1-c) \end{aligned} \tag{11}$$

and zero otherwise.

The modified beta family (MB) includes the MB1 and MB2 distributions as members corresponding to $c = 0$ and $c = 1$, respectively.

The GB distribution nests both GB1 and GB2; hence, the MB1 and MB2 are also special cases of the modified beta distribution (MB) as well as of the generalized beta distribution (GB).

If $d = 1$ in (11) the corresponding pdf is

$$B(y; c, p, q) = MB(y; c, d = 1, p, q) = \frac{y^{p-1}(1-(1-c)y)^{q-1}}{B(p,q)(1+cy)^{p+q}}, \tag{12}$$

$0 < y < 1/(1-c)$, and will be referred to the standard form of the beta distribution. The standard beta distribution of the first kind (1) corresponds to (12) with $c = 0$ and the standard beta distributions of the second kind (5) is obtained by (12) for $c = 1$.

2 The general beta transform

Consider the Euler's beta integral

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p, q > 0. \tag{13}$$

The objective of this section is to consider some extensions of this formula equivalent with (11), and to indicate their applications to construct positive linear operators. The bilinear transformation

$$t = \frac{\alpha y + \beta}{cy + d}, \quad \alpha d - \beta c \neq 0 \tag{14}$$

reduce the Euler's beta integral (13) to the form

$$B(p, q) = \int_{-\frac{b}{a}}^{\frac{d-\beta}{a-c}} \frac{(\alpha y + \beta)^{p-1}((c-\alpha)y + d - \beta)^{q-1}}{(cy+d)^{p+q}}(\alpha d - \beta c)dy. \tag{15}$$

For $\alpha = 1$, $\beta = 0$ we obtain the following formula

$$B(p, q) = \int_0^{\frac{d}{1-c}} \frac{y^{p-1}(d-(1-c)y)^{q-1}}{(d+cy)^{p+q}}dy \tag{16}$$

which is equivalent with (11).

Let us denote by $M[0,\infty)$ the linear space functions defined on $[0,\infty)$, bounded and Lebesgue measurable in each interval $[c,d]$, $0<c<d<\infty$.

By using (16), or equivalent the modified beta distribution (MB), defined by (11), we can define the following general transform

$$B_{p,q}^{(a,b,c)}f=\frac{d}{B(p,q)}\int_0^{\frac{d}{1-c}}\frac{y^{p-1}(d-(1-c)y)^{q-1}}{(d+cy)^{p+q}}f\left(\frac{y^a(d-(1-c)y)^b}{(d+cy)^{a+b}}\right)dy$$

or, equivalent

(17)
$$\mathcal{B}_{p,q}^{(a,b,c)}f=\frac{1}{B(p,q)}\int_0^{\frac{1}{1-c}}\frac{z^{p-1}(1-(1-c)z)^{q-1}}{(1+cz)^{p+q}}f\left(\frac{z^a(1-(1-c)z)^b}{(1+cz)^{a+b}}\right)dz$$

where $f\in M[0,\infty)$ such that $\mathcal{B}_{p,q}^{(a,b,c)}|f|<\infty$.

Theorem 1 *The moment of order k of the transform $\mathcal{B}_{p,q}^{(a,b,c)}$ has the following value*

(18) $$\mathcal{B}_{p,q}^{(a,b,c)}e_k=\frac{B(p+ka,q+kb)}{B(p,q)}.$$

Proof. We have

$$B_{p,q}^{(a,b,c)}e_k=\frac{1}{B(p,q)}\int_0^{\frac{1}{1-c}}\frac{z^{p-1}(1-(1-c)z)^{q-1}}{(1+cz)^{p+q}}\cdot\frac{z^{ka}(1-(1-c)z)^{kb}}{(1+cz)^{k(a+b)}}dz$$

$$=\frac{1}{B(p,q)}\int_0^{\frac{1}{1-c}}\frac{z^{p+ka-1}(1-(1-c)z)^{q+kb-1}}{(1+cz)^{p+q+k(a+b)}}dz=\frac{B(p+ka,q+kb)}{B(p,q)}.$$

□

We impose $B_{p,q}^{(a,b,c)}e_1=e_1$, that is $x=\dfrac{B(p+a,q+b)}{B(p,q)}$ and we obtain

$$B_{p,q}^{(a,b,c)}((t-x)^2;x)=\frac{B(p+2a,q+2b)B(p,q)-B^2(p+a,q+b)}{B^2(p+a,q+b)}x^2.$$

3 Special cases

1. The beta first kind operators

For $c = 0$ we obtain by (17) the (a, b)-beta first kind transform of a function f (see [5], [9])

$$\mathcal{B}_{p,q}^{(a,b)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t^a(1-t)^b)dt \tag{19}$$

where $f \in M[0, \infty)$ such that $\mathcal{B}_{p,q}^{(a,b)}|f| < \infty$.

If we put in (19) $b = 0$ we obtain the generalized beta first kind transform of a function f

$$\mathcal{B}_{p,q}^{(a)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t^a)dt. \tag{20}$$

Clearly $\mathcal{B}_{p,q}^{(a)}$ is a positive linear functional.

If we choose in (20) $a = 1$ we obtain the beta first kind transform $\mathcal{B}_{p,q}$ of a function $f \in C[0, 1]$ defined by

$$\mathcal{B}_{p,q} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t)dt. \tag{21}$$

Corollary 1 *The moment of order k of the functional $\mathcal{B}_{p,q}$ has the following value*

$$\mathcal{B}_{p,q} e_k = \frac{p(p+1)\ldots(p+k-1)}{(p+q)\ldots(p+q+k-1)} = \frac{(p)_k}{(p+q)_k}$$

Consequently we obtain

$$B_{p,q} e_1 = \frac{p}{p+q}, \quad B_{p,q} e_2 = \frac{p(p+1)}{(p+q)(p+q+1)}. \tag{22}$$

Proof. The result follows from Theorem 1 for $a = 1$, $b = 0$. □

We impose $\mathcal{B}_{p,q}e_1 = e_1$, that is $\dfrac{p}{p+q} = x$, or $p = \dfrac{\beta}{\alpha}x$, $q = \dfrac{\beta}{\alpha}(1-x)$, $x \in (0,1)$, $\alpha, \beta > 0$ and we obtain the following linear positive operators

$$(23) \qquad (\mathcal{B}_{p,q}^{(\alpha,\beta)} f)(x) = \frac{1}{B\left(\dfrac{\beta}{\alpha}x, \dfrac{\beta}{\alpha}(1-x)\right)} \int_0^1 t^{\frac{\beta}{\alpha}x-1}(1-t)^{\frac{\beta}{\alpha}(1-x)-1} f(t)dt$$

which for $\beta = \alpha n$, $n \in \mathbb{N}$ has been introduced by A. Lupaş [1].

Corollary 2 *One has*

$$B^{(\alpha,\beta)}((t-x)^2; x) = \frac{\alpha}{\alpha+\beta} x(1-x).$$

Proof. It is obtained from (22) for $p = \dfrac{\beta}{\alpha}x$, $q = \dfrac{\beta}{\alpha}(1-x)$. □

For the special cases of the operator (23) see [4], [8].

If we put $a = -1$ in (20) we obtain

$$(24) \qquad B_{p,q}f = \mathcal{B}_{p,q}^{(-1)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\left(\frac{1}{t}\right) dt,$$

where $f \in M[0,\infty)$ such that $B_{p,q}|f| < \infty$.

Corollary 3 *The moment of order k $(1 \leq k < p)$ of the functional $B_{p,q}$ has the following value*

$$B_{p,q}e_k = \frac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}, \quad 1 \leq k < p.$$

Consequently we obtain

$$(25) \qquad B_{p,q}e_1 = \frac{p+q-1}{p-1}, \quad B_{p,q}e_2 = \frac{(p+q-1)(p+q-2)}{(p-1)(p-2)}, \quad p > 2.$$

Proof. The result follows from Theorem 1 for $a = 1$, $b = 0$. $\square$

We impose $B_{p,q}e_1 = e_1$, that is $\dfrac{p+q-1}{p-1} = x$ or $p = 1 + \dfrac{\beta}{\alpha}$, $q = \dfrac{\beta}{\alpha}(x-1)$, $x > 1$, $\alpha, \beta > 0$, $\beta > \alpha$ and we obtain the following linear positive operators

$$(26)\quad (B^{(\alpha,\beta)}f)(x) = \frac{1}{B\left(1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}(x-1)\right)} \int_0^1 t^{\frac{\beta}{\alpha}}(1-t)^{\frac{\beta}{\alpha}(x-1)-1} f\left(\frac{1}{t}\right) dt.$$

Corollary 4 *One has*

$$B^{(\alpha,\beta)}f((t-x)^2; x) = \frac{\alpha}{\beta-\alpha} x(1-x), \quad \alpha < \beta.$$

Proof. It is obtained from (25) for $p = \dfrac{\beta}{\alpha} + 1$, $q = \dfrac{\beta}{\alpha}(x-1)$. $\square$

For special cases of the operator (26) see [5], [9].

1. The beta second kind transform

For $c = 1$ we obtain by (17) the (a, b)-beta second kind transform of a function f (see [6], [10])

$$(27)\qquad \mathcal{T}_{p,q}^{(a,b)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{u^a}{(1+u)^{a+b}}\right) du,$$

where $f \in M[0, \infty)$ such that $\mathcal{T}_{p,q}^{(a,b)}|f| < \infty$.

If we consider in (27) $a + b = 0$ we obtain the second kind transform of function $f \in M[0, \infty)$

$$(28)\qquad T_{p,q}^{(a)} f = \mathcal{T}_{p,q}^{(a,-a)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u^a) du$$

such that $T_{p,q}^{(a)}|f| < \infty$.

Clearly $T_{p,q}^{(a)}$ is a positive linear functional.

If we put in (28) $a = 1$ we obtain the beta second kind transform

$$T_{p,q}f = T_{p,q}^{(1)}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u)du \tag{29}$$

for $f \in M[0,\infty)$ such that $T_{p,q}|f| < \infty$, considered by D.D. Stancu [15] (see also [6]).

Remark. If $a = -1$ we obtain $T_{p,q}^{(-1)}f = T_{p,q}^{(1)}f = T_{p,q}f$ (see [6]).

Corollary 5 *The moment of order k $(1 \le k < q)$ of the functional $T_{p,q}$ has the following value*

$$T_{p,q}e_k = \frac{p(p+1)\dots(p+k-1)}{(q-1)\dots(q-k)}, \quad 1 \le k < q.$$

Consequently we obtain

$$T_{p,q}e_1 = \frac{p}{q-1}, \quad T_{p,q}e_2 = \frac{p(p+1)}{(q-1)(q-2)}, \quad q > 2. \tag{30}$$

Proof. The result follows from Theorem 1 for $a + b = 0$, $a = 1$. □

We impose $T_{p,q}e_1 = e_1$, that is $\dfrac{p}{q-1} = x$, or $p = \dfrac{\beta}{\alpha}x$, $q = 1 + \dfrac{\beta}{\alpha}$, $x > 0$, $\alpha, \beta > 0$ and we obtain the following linear positive operators

$$(T^{(\alpha,\beta)}f)(x) = \frac{1}{B\left(\frac{\beta}{\alpha}x, 1+\frac{\beta}{\alpha}\right)} \int_0^\infty \frac{u^{\frac{\beta}{\alpha}-1}}{(1+u)^{1+\frac{\beta}{\alpha}(x+1)}} f(u)du. \tag{31}$$

Corollary 6 *One has*

$$T^{(\alpha,\beta)}((t-x)^2; x) = \frac{\alpha}{\beta-\alpha}x(1+x), \quad \beta > \alpha > 0.$$

Proof. It is obtained from (30) for $p = \dfrac{\beta}{\alpha}x$, $q = 1 + \dfrac{\beta}{\alpha}$. □

For special cases of the operator (31) see [6], [10].

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