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Several inequalities about the number of positive divisors of a natural number m[1]

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Abstract

In this paper we intend to establish several properties of the number of positive divisors of a natural number m. Among these,we remark the inequality $\tau^2(mn) \geq \tau(m^2)\tau(n^2)$, for all $m, n \in \mathbb{N}^\star$.

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1 Introduction

For a positive integer m number, we will note $\tau(m)$ the number of positive divisors of m. We remark that: $\tau(1) = 1$ and if p is a prime number, then

$$\tau(p) = 2\,,\ \tau(p^\alpha) = \alpha + 1\,.$$

In papers [1]–[5], [7] we find the following properties of $\tau(m)$:

For $m = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, $m > 1$ we have the relation:

$$\tau(m) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_r + 1)\,. \tag{1}$$

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If $(m,n)=1$, then

$$\tau(mn)=\tau(m)\tau(n)\,,\ \text{for all}\ \ m,n\in\mathbb{N}^{\star}\,. \tag{2}$$

For $m\geq 2$, we have the relation:

$$\tau(m)=\sum_{k=1}^{m}\left(\left[\frac{m}{k}\right]-\left[\frac{m-1}{k}\right]\right)\,. \tag{3}$$

In [6], for $m\geq 1$, we have

$$\tau(m)\leq 2\sqrt{m}\,. \tag{4}$$

In [8] it is shown that

$$\tau(m)\tau(n)\geq\tau(mn)\,,\quad \text{for all}\ m,n\in\mathbb{N}^{\star}\,. \tag{5}$$

In [9] are establish the following inequalities:

$$\tau(m)<m^{\frac{2}{3}}\,,\ for\ any\ m>12\,, \tag{6}$$

$$ln\tau(m)<1,066\frac{ln\,m}{ln\,ln\,m}\,,\ for\ any\ m\geq 3\,. \tag{7}$$

In this paper, we establish some new inequalities for the function τ.

2 Main results

We can remark several properties of these functions for two natural non-zero numbers, m and n.

Theorem 2.1.

$$a)\ \tau(mn)\leq\tau(m)n\,,\quad for\ all\ m,n\in\mathbb{N}^{\star}\,, \tag{8}$$

$$b)\ n|m\,,\ atunci\ \frac{\tau(m)}{m}\leq\frac{\tau(n)}{n}\,,\quad for\ all\ m,n\in\mathbb{N}^{\star}\,. \tag{9}$$

Proof. We will show that $\tau(m) \leq m$, for all $m \in \mathbb{N}^\star$. From the inequality (4), $\tau(m) \leq 2\sqrt{m}$, but $m \geq 2\sqrt{m}$ for $m \geq 4$, therefore $\tau(m) \leq m$, $m \geq 4$.

For $m \in \{1, 2, 3\}$ it is easy to see that the inequality is true.

From the inequality (5), $\tau(m)\tau(n) \geq \tau(mn)$, for all $m, n \in \mathbb{N}^\star$, but $\tau(n) \leq n$, so $\tau(mn) \leq \tau(m)n$, for all $m, n \in \mathbb{N}^\star$.

Because $n|m$, we have $m = nd$, and from the inequality (8) we obtain $\tau(nd) \leq \tau(n)d$, which is equivalent with $n\tau(m) \leq nd\tau(n) = m\tau(n)$.

Corollary 2.1. *We have*

$$(10) \qquad a)\frac{\tau(mn)}{mn} \leq \frac{\tau(m) + \tau(n)}{m + n}, \ \textit{for all}\, m, n \in \mathbb{N}^\star,$$

$$(11) \qquad b)\tau(mn) \leq \frac{m^2\tau(n) + n^2\tau(m)}{m + n}, \ \textit{for all}\, m, n \in \mathbb{N}^\star.$$

Proof. We apply the inequality (8) and we deduce $(m + n)\tau(mn) = m\tau(mn) + n\tau(mn) \leq mn\tau(m) + mn\tau(n) = mn(\tau(m) + \tau(n))$, which means that the proof is complete.

Similarly, we prove the inequality $(m+n)\tau(mn) = m\tau(mn)+n\tau(mn) \leq m^2\tau(n) + n^2\tau(m)$, consequently the inequality (11).

Theorem 2.2.

$$(12) \qquad \tau((m, n))\tau([m, n]) = \tau(m)\tau(n), \quad \textit{for all } m, n \in \mathbb{N}^\star,$$

where (m, n) is the greatest common divisor of m and n and $[m, n]$ is the least common multiple of m and n.

Proof. Let m and n be two natural non-zero numbers. We will factorize the numbers m and n in prime factors, thus:

$m = p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k} \cdot q_1^{\beta_1}q_2^{\beta_2}...q_s^{\beta_s}$, $n = p_1^{\gamma_1}p_2^{\gamma_2}...p_k^{\gamma_k} \cdot r_1^{\delta_1}r_2^{\delta_2}...r_t^{\delta_t}$, $q_j \neq r_l$, for all $j \in \{1, ..., s\}\textit{and for all} \ \ l \in \{1, ..., t\}$, therefore $\tau(m) = \prod_{i=1}^{k}(\alpha_i + 1)\prod_{j=1}^{s}(\beta_j + 1)$, $\tau(n) = \prod_{i=1}^{k}(\gamma_i+1)\prod_{l=1}^{t}(\delta_l+1)$, we obtain $\tau((m, n)) = \prod_{i=1}^{k}(min\{\alpha_i\,, \gamma_i\}+1)$

and $\tau([m,n]) = \prod_{i=1}^{k}(max\{\alpha_i\,,\gamma_i\}+1)\prod_{j=1}^{s}(\beta_j+1)\prod_{l=1}^{t}(\delta_l+1)$, which means that $\tau((m,n))\tau([m,n]) = \tau(m)\tau(n)\,,\ \ for\,all\,m,n\in\mathbb{N}^\star$.

Theorem 2.3.

$$\tau^2(mn) \geq \tau(m^2)\tau(n^2)\,,\ \ for\,all\,m,n\in\mathbb{N}^\star\,. \tag{13}$$

Proof. We consider $m=\prod_{i=1}^{k}p_i^{\alpha_i}\prod_{j=1}^{s}q_j^{\beta_j}$, $n=\prod_{i=1}^{k}p_i^{\gamma_i}\prod_{l=1}^{t}r_l^{\delta_l}$, which means that $mn=\prod_{i=1}^{k}p_i^{\alpha_i+\gamma_i}\prod_{j=1}^{s}q_j^{\beta_j}\prod_{l=1}^{t}r_l^{\delta_l}$, hence $\tau(m)=\prod_{i=1}^{k}(\alpha_i+1)\prod_{j=1}^{s}(\beta_j+1)$ and $\tau(n)=\prod_{i=1}^{k}(\gamma_i+1)\prod_{l=1}^{t}(\delta_l+1)$, therefore $\tau(mn)=\prod_{i=1}^{k}(\alpha_i+\gamma_i+1)\prod_{j=1}^{s}(\beta_j+1)\prod_{l=1}^{t}(\delta_l+1)$, so $\tau(m)\tau(n)=\tau(mn)\cdot\prod_{i=1}^{k}\frac{(\alpha_i+1)(\gamma_i+1)}{\alpha_i+\gamma_i+1}=\tau(mn)\cdot\prod_{i=1}^{k}\left(1+\frac{\alpha_i\gamma_i}{\alpha_i+\gamma_i+1}\right)\geq\tau(mn)$. Because $\tau(m^2)=\prod_{i=1}^{k}(2\alpha_i+1)\prod_{j=1}^{s}(2\beta_j+1)$ and $\tau(n^2)=\prod_{i=1}^{k}(2\gamma_i+1)\prod_{l=1}^{t}(2\delta_l+1)$, we obtain the equality:

$$\tau(m^2)\tau(n^2)=\prod_{i=1}^{k}(2\alpha_i+1)\prod_{j=1}^{s}(2\beta_j+1)\prod_{i=1}^{k}(2\gamma_i+1)\prod_{l=1}^{t}(2\delta_l+1)\,,$$

but $\tau^2(mn)=\prod_{i=1}^{k}(\alpha_i+\gamma_i+1)^2\prod_{j=1}^{s}(\beta_j+1)^2\prod_{l=1}^{t}(\delta_l+1)^2$. It is easy to see the equality $\tau^2(mn)=\tau(m^2)\tau(n^2)\cdot\prod_{i=1}^{k}\left(1+\frac{(\alpha_i+\gamma_i)^2}{(2\alpha_i+1)(2\gamma_i+1)}\right)\cdot\prod_{j=1}^{s}\left(1+\frac{\beta_j^2}{2\beta_j+1}\right)\cdot\prod_{l=1}^{t}\left(1+\frac{\delta_l^2}{2\delta_l+1}\right)$.

Since $1+\frac{(\alpha_i+\gamma_i)^2}{(2\alpha_i+1)(2\gamma_i+1)}\geq 1$, for all $i=\overline{1,k}$, $1+\frac{\beta_j^2}{2\beta_j+1}\geq 1$, for all $j=\overline{1,s}$, $1+\frac{\delta_l^2}{2\delta_l+1}\geq 1$, for all $l=\overline{1,t}$, we obtain $\tau^2(mn)\geq\tau(m^2)\tau(n^2)$.

Theorem 2.4. *Let m and n be two natural non-zero numbers, then* $\tau(mn)\leq n\sqrt{m}+m\sqrt{n}$.

Proof. We apply the inequality (4) for m and n, we have $n\tau(m) \le 2n\sqrt{m}$ and $m\tau(n) \le 2m\sqrt{n}$. By adding the inequalities, we obtain

$$n\tau(m) + m\tau(n) \le 2n\sqrt{m} + 2m\sqrt{n}\,, \tag{14}$$

but using the inequality (8), we have $\tau(mn) \le \tau(m)n$ and $\tau(mn) \le \tau(n)m\,,\ for\, all\, m, n \in \mathbb{N}^\star$, we deduce

$$2\tau(mn) \le \tau(m)n + \tau(n)m\,, \tag{15}$$

so, from the inequalities (14) and (15), we obtain the inequality

$$\tau(mn) \le n\sqrt{m} + m\sqrt{n}\,.$$

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