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On Principal Ideals of Triply-Generated Telescopic Semigroups[1]

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Abstract

In this paper, we investigate principal ideals of triply-generated telescopic numerical semigroups of the form $S =< a, a+2, 2a+1 >$, where $a > 2$ is even integer. We examine relations between these ideals and the Apery sets of S.

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0 Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of integers and non-negative integers,respectively. A numerical semigroup S is a subset of $\mathbb{N}$ that is closed under addition, contains 0, and generates $\mathbb{Z}$ as a group.

This paper consists of three sections. In Section 1, we establish basic definitions, notations and assumptions related to numerical semigroups that will be needed for our investigation. In Section 2, we examine sums, unions, and intersections of certain principal ideals of S. In Section 3, we investigate

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relations between Apery sets and these principal ideals of S.

1 Background, Notations and Assumptions

We begin by establishing the definitions and notations associated with numerical semigroups necessary for this investigation. For more background on the topic of numerical semigroups, the reader is encouraged to see Barucci et al.(1997) and Froberg et al.(1987).

Definition 1.1. *Let* $\mathbb{N}$ *denote the non-negative integers. A numerical semigroup* S *is a subset of* $\mathbb{N}$ *such that*
(1) $0 \in S$,
(2) S *is closed under addition,*
(3) $\mathbb{N} \backslash \mathbb{S}$ *is finite.*

Notation 1.2. The following notations will be used throughout this paper:
$\mathbb{Z}$ = the set of integers;
S = a numerical semigroup as $S =< a, a+2, 2a+1 >$, where $a > 2$ is an even integer;
$g(S) = max(\mathbb{N} \backslash \mathbb{S})$ the Frobenius number of S.

Definition 1.3. *We say that a numerical semigroup* S *is symmetric provided* $(g(S) - z) \in S$ *for all integers* $z \notin S$ (see also Froberg et al.(1987)).

Definition 1.4. *We say* $\{s_1, s_2, ...s_n\} \subset S$ *is a generating set of* S *provided*

$$S = \{k_1 s_1 + k_2 s_2 + ... + k_n s_n : k_1, k_2, ..., k_n \in \mathbb{N}\}.$$

We say that a generating set $\{s_1, s_2, ...s_n\}$ *is the minimal generating set of* S *if no proper subset is a generating set of* S*. When we write* $S =< s_1, s_2, s_3 >$ *we mean that* $\{s_1, s_2, ...s_n\}$ *is the minimal generating set for* S *and* $0 < s_1 < s_2 < ... < s_n$ (See Madero and Herzinger (2005)).

Definition 1.5. *A numerical semigroup $S =< s_1, s_2, s_3 >$ is called a triply-generated telescopic semigroup if $s_3 \in< s_1/d, s_2/d >$, where $d = gcd(s_1, s_2)$.* (See Matthews 2001). It known that if a is even and $a > 2$ then $S =< a, a + 2, 2a + 1 >$ is triply-generated telescopic and symmetric numerical semigroup and $g(S) = a^2/2 + a - 1$ (See Ilhan 2006).

Example 1.6. *Let $S =< 6, 8, 13 >= \{6k_1 + 8k_2 + 13k_3 : k_1, k_2, k_3 \in \mathbb{N}\}$. Then*

$$S = \{0, 6, 8, 12, 13, 14, 16, 18, 19, 20, 21, 22, 24, \rightarrow\}$$

where " $\rightarrow$ " indicates that all integers greater then 24 are in S. Thus, S is triply-generated telescopic numerical semigroup since $d = gcd(6, 8) = 2$ and $13 \in< 6/2, 8/2 >$. We see that $g(S) = 23$, and S is symmetric since $23 - z \in S$ for all integers $z \notin S$.

Definition 1.7. *A subset I of S is an ideal of S if $I + S = \{i + s : i \in I, s \in S\} \subseteq I$. An ideal I is generated by $A \subseteq S$ if $I = A + S$. Finally, we say I is principal if it can be generated by a single element. That is there exists $x_0 \in S$ such that $I = x_0 + S = \{x_0 + s : s \in S\}$; in this case, we usually write $I = [x_0]$ instead of $I = x_0 + S$* (See Rosales et al. (2004)).

Example 1.8 *Let $S =< 6, 8, 13 >$ as in (1.6). Then, we find that the principal ideals $I = [6]$ and $J = [8]$ of S respectively, $I = [6] = 6 + S = \{6, 12, 14, 18, 19, 20, 22, 24, \rightarrow\}$ and $J = [8] = 8+S = \{8, 14, 16, 20, 21, 22, 24, 26, 27, 28, 29, 30, 32, \rightarrow\}$.*

2 Sum, Union and Intersection of Principal Ideals of S

Definition 2.1. *Let be I and J are ideals of S. Then,we define their ideals sum by $I + J = \{i + j : i \in I, j \in J\}$* (See Barucci et al.(1997), and Madero and Herzinger (2005)).

Note 2.2. For the principal ideals I and J of S, we write that $I + J \subseteq I$ and $I + J \subseteq J$.

Note 2.3. Let be S a numerical semigroup such that $S =< a, a+2, 2a+1 >$, where $a > 2$ is an even integer. Recall from above that S is telescopic and symmetric. Moreover, we will let $I = [a]$ and $J = [a + 2]$. We note that $I + J = [2a + 2]$.

Lemma 2.4. *$(2a + 1) \notin I$ and $2a + 1 \notin J$.*

Proof. If $2a + 1 \notin I$, then there exists $s \in S$ such that $a + s = 2a + 1$. It follows that $a + 1 = s \in S$, a contradiction. By a similar argument, if $2a + 1 \notin J$, we conclude that $a - 1 \in S$, also a contradiction.

Example 2.5. *Let $S =< 4, 6, 9 >= \{0, 4, 6, 8, 9, 10, 12, 13, \rightarrow\}$. Then, we find that the Frobenius number of S as $g(S) = 11$ and the principal ideals $I = [4]$ and $J = [6]$ of S, respectively. $I = [4] = 4 + S = \{4, 8, 10, 12, 13, 14, 16, 17, \rightarrow\}$ and $J = [6] = 6+S = \{6, 10, 12, 14, 15, 16, 18, 19, \rightarrow\}$. In this case, we write $9 \notin I$ and $9 \notin J$. Therefore, we obtain that $I + J = [4 + 6] = [10] = 10 + S = \{10, 14, 16, 18, 19, 20, 22, 23, \rightarrow\} \subseteq I, J$.*

Theorem 2.6. $(I \cup J) = S\backslash\{0, 2a + 1\}$.

Proof. If $x \in (I \cup J)$ then $x \in S$ but $x \neq 0$ by the definitions of I and J. Further, $x \neq 2a + 1$ by lemma 2.4. We conclude $(I \cup J) \subseteq S\backslash\{0, 2a + 1\}$.

To prove the reverse containment, assume $y \in S$. Then we can write $y = k_1a + k_2(a + 2) + k_3(2a + 1)$ where $k_1, k_2, k_3 \geq 0$. Note that if $k_1 > 0$, then $y \in I$ and if $k_2 > 0$, then $y \in J$. Therefore, if we assume that $y \in (S\backslash(I \cup J))$, then $y = k_3(2a + 1)$ where $k_3 \geq 0$. If $k_3 \geq 2$, then $y = 2(2a + 1) + (k_3 - 2)(2a + 1) = 3a + (a + 2) + (k_3 - 2)(2a + 1)$ which is an element of I (and J), a contradiction. Thus we must have $k_3 = 0$ or $k_3 = 1$. Stated differently, we have $y = 0$ or $y = 2a + 1$. Thus completes the proof.

The following lemma is clear from the definitions of the ideal sum and intersection. We offer this statement as a reference for the upcoming proofs.

Lemma 2.7. $(I+J) \subseteq (I \cap J)$.

Theorem 2.8. *The family* $\{I+J=[2a+2], \{g(S)+2a+2\}, \{l.c.m.(a,a+2)\}\}$ *is a partition of* $I \cap J$ *where l.c.m. denotes least common multiple.*

Proof. First note that $g(S)+2a+2 \notin [2a+2]$ since $g(S) \notin S$. Next note that $l.c.m.(a,a+2) \notin [2a+2]$. To see this, observe $l.c.m.(a,a+2) = a^2/2+a = g(S)+1$ and $g(S)+1-(2a+2) = g(S)-(2a+1)$ which is not an element of S. Thus, $l.c.m.(a,a+2)$ cannot be generated by $2a+2$. Finally note that $g(S)+2a+2 \neq l.c.m.(a,a+2)$ since $g(S)+1 = l.c.m.(a,a+2)$. This establishes that the three sets are pairwise disjoint.

Now, we need to prove that $I \cap J = [2a+2] \cup \{g(S)+2a+2\} \cup \{l.c.m.(a,a+2)\}$. First we assume that $x \in [2a+2] \cup \{g(S)+2a+2\} \cup \{l.c.m.(a,a+2)\}$. If $x \in [2a+2] then x \in I \cap J$ by lemma 2.7. If $x = l.c.m.(a,a+2)$, then $x = k_1 a = k_2(a+2)$ where k_1 and k_2 are non-negative integers. Thus $x \in I \cap J$. Finally, if $x = g(S)+2a+2$, then

(i) $x = a+(a+g(S)+2) \in I$ since $a+g(S)+2 \in S$

(ii) $x = (a+2)+(a+g(S)) \in J$ since $a+g(S) \in S$. We conclude $x \in I \cap J$.

For the reverse containment, assume that $y \in I \cap J$ but $y \notin [2a+2] = I+J$. We will show that either $y = g(S)+2a+2$ or $y = l.c.m.(a,a+2)$. Since $y \notin [2a+2]$, we known that $y-(2a+2) \notin S$. Since S is symmetric, we known that $g(S)-(y-(2a+2)) \in S$. Note that $g(S)+1 = a^2/2+a = l.c.m.(a,a+2)$. Thus, since $g.c.d.(a,a+2) = 2$, the statement $g(S)-(y-(2a+2)) \in S$ can be rewritten as $l.c.m.(a,a+2)+2a+1-y \in S$.

Now, suppose $l.c.m.(a,a+2)+2a+1-y \in I$. Since $y \in J$, we see that $l.c.m.(a,a+2)+2a+1 \in I+J = [2a+2]$. Therefore, $l.c.m.(a,a+2)+2a+1-(2a+2) \in S$ which says $l.c.m.(a,a+2)-1 = g(S) \in S$. This is contradiction. We conclude that $l.c.m.(a,a+2)+2a+1-y \notin I$. By a similar argument we can show that $l.c.m.(a,a+2)+2a+1-y \notin J$.

We have shown that $l.c.m.(a, a+2) + 2a + 1 - y \in S\backslash(I \cup J)$. By Theorem 2.6, we know that either $l.c.m.(a, a+2) + 2a + 1 - y = 0$ or $l.c.m.(a, a+2) + 2a + 1 - y = 2a + 1$. In the former case, we have $y = g(S) + 2a + 2$ and in the latter case, we have $y = l.c.m.(a, a+2)$. This completes the proof.

Example 2.9. *Let $S =< 4, 6, 9 >$ as in (2.5.). Then, we write that union and intersection the principal ideals $I = [4]$ and $J = [6]$ of S such that $I \cup J = \{4, 6, 8, 10, 12, 13, 14, \rightarrow\} = S\backslash\{0, 9\}$, and*

$$
\begin{aligned}
I \cap J &= \{10, 12, 14, 16, 18, 19, 20, 21, \rightarrow ...\} \\
&= \{10, 14, 16, 18, 19, 20, 22, \rightarrow ...\} \cup \{12\} \cup \{21\} \\
&= [10] \cup \{l.c.m\{4, 6\}\} \cup \{11 + 4 + 6\}.
\end{aligned}
$$

3 Relations Between Apery sets and Principal Ideals of S

Definition 3.1. *Let $n \in S\backslash\{0\}$, we define the Apery set of the element n as the set $Ap(S, n) = \{s \in S : s - n \notin S\}$. It can easily be proved that $Ap(S, n)$ consists of the smallest elements of S belonging to the different congruence classes mod$n$. Thus, $\sharp(Ap(S, n)) = n$ and $g(S) = max(Ap(S, n)) - n$, where $\sharp(A)$ denotes Cardinality A.* (See Rosales (2000), and Madero and Herzinger (2005)).

Note 3.2. In this section, we investigate relations between the Apery sets and the principal ideals of S that we investigated in Section 2. The following lemma is clear from the definitions.

Lemma 3.3. *If $I = [a]$ and $Ap(S, a) = \{s \in S : s - a \notin S\}$ then $I \cap Ap(S, a) = \emptyset$. Similarly, if $J = [a + 2]$, then $J \cap Ap(S, a + 2) = \emptyset$.*

Theorem 3.4. *$\{[a], Ap(S, a)\}$ and $\{[a+2], Ap(S, a+2)\}$ are both partitions of S.*

Proof. We will prove the result only for $\{[a], Ap(S,a)\}$. The proof for $\{[a+2], Ap(S,a+2)\}$ is similar. According to Lemma 3.3, it is sufficient to show that $S = [a] \cup Ap(S,a)$. Now it is clear that $[a] \cup Ap(S,a) \subseteq S$.

For the reverse containment assume $x \in S$ and $x \notin [a]$. Then $x - a \notin S$ hence $x \in Ap(S,a)$.

Lemma 3.5. $2a+2 \notin Ap(S,a)$ and $2a+2 \notin Ap(S,a+2)$.

Proof. The result follows from the fact that $2a+2-a = a+2 \in S$ and $2a+2-(a+2) = a \in S$.

Example 3.6. *Let*

$S =< 8, 10, 17 >= \{0, 8, 10, 16, 17, 18, 20, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 37, 38, 40, \rightarrow\}$. *Then, we find that* $g(S) = 39$, $I = [8]$, $J = [10]$, $Ap(S,8) = \{s \in S : s-8 \in S\} = \{0, 10, 17, 20, 27, 29, 30, 47\}$ *and* $Ap(S,10) = \{0, 8, 16, 17, 24, 25, 32, 33, 41, 49\}$.

Thus, we obtain that $[8] \cap Ap(S,8) = \emptyset$ and $[10] \cap Ap(S,10) = \emptyset$ by Lemma 3.3, and we write that $[8] \cup Ap(S,8) = S$ and $[10] \cup Ap(S,6) = S$ by Theorem 3.4. Thus, we find that $2.8+2 = 18 \notin Ap(S,8)$ and $2.8+2 = 18 \notin Ap(S,10)$ by Lemma 3.5.

Theorem 3.7. $S \backslash (I+J) = Ap(S,a) \cup Ap(S,a+2) \cup \{g(S)+2a+2\} \cup \{l.c.m.(a,a+2)\}$.

Proof. Assume $x \notin S \backslash (I+J)$. Then either $x \notin S$ or $x \in I+J$.

If $x \notin S$, then

(i) $x \notin Ap(S,a)$ since $Ap(S,a) \subset S$,

(ii) $x \notin Ap(S,a+2)$ since $Ap(S,a+2) \subset S$,

(iii) $x \neq g(S)+2a+2$ since $g(S)+2a+2 \in S$, and

(iv) $x \neq l.c.m.(a,a+2)$ since $l.c.m.(a,a+2) = g(S)+1 \in S$.

If $x \in I + J$, then

(i)$x \notin Ap(S, a)$ since Lemma 2.7 and Lemma 3.3,

(ii)$x \notin Ap(S, a + 2)$ since Lemma 2.7 and Lemma 3.3,

(iii)$x \neq g(S) + 2a + 2$ since $I + J = [2a + 2]$ and $g(S) \notin S$,

(iv) $x \neq l.c.m.(a, a + 2)$ since $I + J = [2a + 2]$ and $g(S) \notin S$.

In either case we have $x \notin Ap(S, a) \cup Ap(S, a + 2) \cup \{g(S) + 2a + 2\} \cup \{l.c.m.(a, a+2)\}$. We conclude $Ap(S, a) \cup Ap(S, a+2) \cup \{g(S) + 2a + 2\} \cup \{l.c.m.(a, a + 2)\} \subseteq S \backslash (I + J)$.

For the reverse containment, assume $x \in S\backslash(I + J)$. If $x \in I \cap J$, then $x = g(S) + 2a + 2$ or $x = l.c.m.(a, a + 2)$ by Theorem 2.8. On the other hand, if $x \notin I \cap J$, then either $x \notin I$ which implies $x - a \notin S$ and hence $x \in Ap(S, a)$, or $x \notin J$ which implies $x - (a + 2) \notin S$ and hence $x \in Ap(S, a + 2)$.

In either case we conclude that $x \in Ap(S, a) \cup Ap(S, a + 2) \cup \{g(S) + 2a + 2\} \cup \{l.c.m.(a, a + 2)\}$.

Example 3.8. *Let* $S =< 4, 6, 9 >$ *and* $I = [4]$ *and* $J = [6]$ *as in (2.5.). Then, we conclude* $Ap(S, 4) = \{0, 6, 9, 15\}$ *and hence* $Ap(S, 6) = \{0, 4, 8, 9, 13, 17\}$. *Finally, we obtain that* $S\backslash[10] = Ap(S, 4) \cup Ap(S, 6) \cup \{g(S) + 4 + 6\} \cup \{l.c.m.(4, 6)\} = \{0, 4, 6, 8, 9, 12, 13, 15, 17, 21\}$ *by Theorem 3.7.*

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