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Order of approximation of functions of two variables by new type gamma operators[1]

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Abstract

The theorems on weighetd approximation and order of approximation of continuous functions of two variables by new type Gamma operators on all positive square region are established.

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1 Introduction

Lupaş and Müller[11] introduced the sequence of linear positive operators $\{G_n\}, G_n : C(0,\infty) \to C(0,\infty)$ defined by

$$G_n(f;x) = \int_0^\infty g_n(x,u) f(\frac{n}{u}) du$$

G_n is called Gamma operator, where $g_n(x,u) = \dfrac{x^{n+1}}{n!} e^{-xu} u^n$, $x > 0$.

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Mazhar [12] used same $g_n(x,u)$ of Gamma operator and introduced the following sequence of linear positive operators:

$$\begin{aligned} F_n(f;x) &= \int\limits_0^\infty g_n(x,u)du \int\limits_0^\infty g_{n-1}(u,t)f(t)dt \\ &= \frac{(2n)!x^{n+1}}{n!(n-1)!}\int\limits_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}}f(t)dt,\ n>1,\ x>0 \end{aligned}$$

for any f which the last integral is convergent. Now we will modify the operators $F_n(f;x)$ as the following operators $A_n(f;x)$ (see [9]) which confirm $A_n(t^2;x) = x^2$. Many linear operators $L_n(f;\, x)$ confirm, $L_n(1;\, x) = 1$, $L_n(t;\, x) = x$ but don't confirm $L_n(t^2;\, x) = x^2$ (see [2],[10]).

$$\begin{aligned} A_n(f;x) &= \int\limits_0^\infty g_{n+2}(x,u)du \int\limits_0^\infty g_n(u,t)f(t)dt \\ &= \frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int\limits_0^\infty \frac{t^n}{(x+t)^{2n+4}}f(t)dt,\ x>0 \end{aligned}$$

If we choose

$$K_n(x,t) = \frac{(2n+3)!}{n!(n+2)!}\frac{x^{n+3}t^n}{(x+t)^{2n+4}},\ \ x,t\in(0,\infty),$$

we can show $A_n(f;x)$ as the following form:

$$A_n(f;x) = \int\limits_0^\infty K_n(x,t)f(t)dt. \tag{1}$$

In this study we will investigate approximation and order of approximation of the following operators which defined for two variables functions.

$$A_{n,m}(f;x,y) = \int\limits_0^\infty K_{n,m}(x,t;y,u)f(t,u)dtdu \tag{2}$$

where $K_{n,m}(x,t;y,u) = K_n(x,t) \times K_m(y,u)$.

It can be easily to see that:

$$A_{n,m}(f;x,y) = A_n(f_1;x) + A_m(f_2;y) \ \text{;if } f(t,u) = f_1(t) + f_2(u)$$
$$A_{n,m}(f;x,y) = A_n(f_1;x) \times A_m(f_2;y) \ \text{;if } f(t,u) = f_1(t) \times f_2(u)$$

Thus for any $p, q \in \mathbb{N}$, $p \leq n$ and $q \leq m$, we can see the following equalities (see [9]):

$$A_{n,m}(1;x,y) = 1 \tag{3}$$

$$A_{n,m}(t+u;x,y) = x + y - \frac{x}{n+2} - \frac{y}{m+2} \tag{4}$$

$$A_{n,m}(t^2+u^2;x,y) = x^2 + y^2 \tag{5}$$

$$A_{n,m}(t^3+u^3;x) = x^3 + y^3 + \frac{3}{n}x^3 + \frac{3}{m}y^3 \tag{6}$$

$$A_{n,m}(t^4+u^4;x) = x^4+y^4+\frac{4(2n+3)}{n(n-1)}x^4+\frac{4(2m+3)}{m(m-1)}y^4, \ n>1, \ m>1. \tag{7}$$

$$A_{n,m}(\{t+u\} - \{x+y\});x,y) = -\frac{x}{n+2} - \frac{y}{m+2} \tag{8}$$

$$A_{n,m}(\{(t-x)^2 + (u-y)^2\};x,y) = \frac{2x^2}{n+2} + \frac{2y^2}{m+2} \tag{9}$$

$$A_{n,m}(\{(t-x)^4 + (u-y)^4\};x,y) = \frac{12(n+4)}{(n+2)n(n-1)}x^4 + \frac{12(m+4)y^4}{(m+2)m(m-1)}, \ n>1, \ m>1 \tag{10}$$

Let $C(\mathbb{R}^2)$ be the set of all real-valued functions of two variables continuous on $\mathbb{R}^2 := \{(x,y)\,:\, x \geq 0,\, y \geq 0\}, \sigma(x,y) = 1 + x^2 + y^2,\ -\infty < x, y < \infty$ and B_σ be sets of all functions f defined on $\mathbb{R}^2$ satisfying the condition

$$|\,f(x,y)| \leq M_f \sigma(x,y) \tag{11}$$

where M_f is a constant depending only on f and the norm is defined by

$$\|f\|_\sigma = \sup_{(x,y)\in\mathbb{R}^2} \frac{|\,f(x,y)|}{\sigma(x,y)}.$$

$C_\sigma$denotes the subspaces of all continuous functions which belonging to B_σ and C_σ^k denotes the subspaces of all functions belonging to C_σ with

$$\lim_{x,y\to\infty} \frac{f(x,y)}{\sigma(x,y)} = k < \infty,$$

where k is a constant depending only on f.

The approximation theorems for two variables are proved by Volkov[13].

He proved the theorem:

Theorem A ([13]). *If $\{T_n\}$ is a sequence of linear positive operators satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(1;x_1,x_2) - 1\|_{C(X)} = 0\ .$$
$$\lim_{n\to\infty} \|T_n(t_i;x_1,x_2) - x_i\|_{C(X)} = 0,\ i = 1,2\ .$$
$$\lim_{n\to\infty} \left\|T_n(t_1^2 + t_2^2;x_1,x_2) - (x_1^2 + x_2^2)\right\|_{C(X)} = 0\ .$$

then for any function $f \in C(X)$, which is bounded in $\mathbb{R}^2$

$$\lim_{n\to\infty} \|T_n(f;x_1,x_2) - f(x_1,x_2)\|_{C(X)} = 0\ .$$

where X is a compact set.

Gadzhiev proved the following theorem for one variable functions.

Theorem B ([3, 4]). *$\{T_n\}$ be the sequence of linear positive operators which mapping from C_ρ into B_ρ satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(t^v;\,x) - x^v\|_\rho = 0,\ v = 0,1,2.$$

Then , for any $f \in C_\rho^k$,

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0.$$

and there exist a function $f \in C_\rho \setminus C_\rho^k$ *such that*

$$\lim_{n\to\infty} \|T_n f - f\|_\rho \geq 1.$$

Analogously as in Theorem B, the theorems on weighted approximation for functions of several variables are proved by Gadzhiev [5].

Applying Theorem B to the operators

$$T_n(f;x) = \begin{cases} V_n(f;x), \ if \ x \in [0,a_n] \\ f(x), \ if \ x > a_n \end{cases}$$

one then also has the following theorem.

Theorem C ([6]). *Let* (a_n) *be a sequence with* $\lim_{n\to\infty} a_n = \infty$ *and* $\{V_n\}$ *be a sequence of linear positive operators taking* $C_\rho[0,a_n]$ *into* $B_\rho[0,a_n]$.

If for $v = 0, 1, 2$

$$\lim_{n\to\infty} \|V_n(t^v;x) - x^v\|_{\rho,[0,a_n]} = 0,$$

then for any $f \in C_\rho^k[0,a_n]$

$$\lim_{n\to\infty} \|V_n f - f\|_{\rho,[0,a_n]} = 0,$$

where $B_\rho[0,a_n], C_\rho[0,a_n]$ *and* $C_\rho^k[0,a_n]$ *denote the same as* B_ρ, C_ρ *and* C_ρ^k *respectively, but the functions taken on* $[0,a_n]$ *instead of* $\mathbb{R}$ *and the norm is taken as*

$$\|f\|_{\rho,[0,a_n]} = \sup_{x\in[0,a_n]} \frac{|f(x)|}{\rho(x)}.$$

2 Approximation of $\mathbf{A}_{n,m}$

Let (b_n) is be a sequence has positive terms, increasing and has the following conditions,

$$\lim_{n\to\infty} b_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n^2}{n} = 0 \tag{12}$$

We will denote the rectangular region $(0,\, b_n] \times (0,\, b_m]$ by $D_{n,m}$ and let $B_\sigma\left(D_{n,m}\right)$ be sets of all functions f defined on $D_{n,m}$ satisfying the condition (11)

By the using (3) and (5), we have

$$A_{n,m}(\sigma(t,\, u); x,\, u) = \sigma(x,\, y).$$

Therefore, $\|A_{n,m}(f;x)\|_{\sigma(D_{n,m})}$ is uniformly bounded on $D_{n,m}$. Hence $\{A_{n,m}\}$ is a sequence of linear positive operators taking $C_\sigma(D_{n,m})$ into $B_\sigma\left(D_{n,m}\right)$.

Theorem 1. *Let $f \in C_\sigma^k$, then*

$$\lim_{n,m\to\infty} \|A_{n,m}(f;x,y) - f(x,y)\|_{\sigma(D_{n,m})} = 0.$$

Proof.

$$\lim_{n,m\to\infty} \|A_{n,m}(1;x,y) - 1\|_{\sigma(D_{n,m})} = 0.$$

$$\lim_{n,m\to\infty} \|A_{n,m}(t;x,y) - x\|_\sigma = \lim_{n,m\to\infty} \sup_{(x,y)\in(D_{n,m})} \frac{\dfrac{x}{n+2}}{\sigma(x,y)} = 0.$$

$$\lim_{n,m\to\infty} \|A_{n,m}(u;x,y) - y\|_\sigma = \lim_{n,m\to\infty} \sup_{(x,y)\in(D_{n,m})} \frac{\dfrac{y}{m+2}}{\sigma(x,y)} = 0.$$

$$\lim_{n,m\to\infty} \left\|A_{n,m}(t^2+u^2;x,y) - (x^2+y^2)\right\|_{\sigma(D_{n,m})} = 0$$

Similar to Theorem B we obtain the desired results.

3 The Order of Approximation of $\mathbf{A}_{n,m}$

We now want to find the degree of approximation of functions $f \in C_\sigma^k$ by the operators $A_{n,m}$ on $D_{n,m}$. It is well- known that the usual first modulus of continuity

$$\varpi(f;\delta) = \sup\left\{|f(t,u) - f(x,y)| : \sqrt{(t-x)^2 + (u-y)^2} \leq \delta; t,u,x,y \in [a,b]\right\}$$

don't tend to zero, as $\delta \to 0$, on any infinite interval and any infinite area, respectively.

In [8] was defined the weighted modulus of continuity for $f \in C_\sigma^k$ as the following (see also [1]):

$$\Lambda(f;\delta,\eta) = \sup\left\{\frac{|f(x+t,y+u) - f(x,y)|}{\sigma(x,y)\sigma(t,u)} : x,y \in \mathbb{R}^2, |t| \leq \delta, |u| \leq \eta\right\}$$

$\Lambda(f;\delta,\eta)$ is having the following properties:

$$\lim_{\delta,\eta\to 0} \Lambda(f;\delta,\eta) = 0.$$

$$\Lambda(f;\lambda_1\delta,\lambda_2\eta) \leq 4(1+\lambda_1)(1+\lambda_2)\Lambda(f;\delta,\eta), \text{ for } \lambda_1 > 0, \lambda_2 > 0$$

and

(13)

$$|f(t,u) - f(x,y)| \leq 8(1+x^2+y^2)\Lambda(f;\delta_n,\delta_m)(1+\frac{|t-x|}{\delta_n})(1+\frac{|u-y|}{\delta_m})$$

$$\times(1+(t-x)^2)(1+(u-y)^2)$$

Theorem 2. *For every $f \in C_\sigma^k$ the inequality*

$$\sup_{(x,y)\in D_{n,m}} \frac{|A_{n,m}(f;x,y) - f(x,y)|}{(1+x^2+y^2)} \leq 4232\Lambda(f;\sqrt{\frac{2b_n^2}{n+2}},\sqrt{\frac{2b_m^2}{m+2}})$$

is true for all n,m sufficiently large.

Proof. If we use (13) and (3) we have

$$|A_{n,m}(f;x,y)-f(x,y)| \leq 8(1+x^2+y^2)\Lambda(f;\delta_n,\delta_m)$$
$$\times A_{n,m}((1+\frac{|t-x|}{\delta_n})(1+\frac{|u-y|}{\delta_m})(1+(t-x)^2)(1+(u-y)^2);x,y)$$
$$\leq 8(1+x^2+y^2)\Lambda(f;\delta_n,\delta_m)$$
$$\times A_n((1+\frac{|t-x|}{\delta_n})(1+(t-x)^2);x) \times A_m((1+\frac{|u-y|}{\delta_m})(1+(u-y)^2);y)$$

We know that $a.b \leq \frac{a^2+b^2}{2}$ is hold for all positive real numbers a and b. Thus, apply equalities (3), (9), (10) and Cauchy-Schwarz inequalities, we will get

$$\frac{|A_{n,m}(f;x,y)-f(x,y)|}{(1+x^2+y^2)} \leq$$

$$\Lambda(f;\delta_n,\delta_m) \times \left[1+(1+\frac{1}{2\delta_n})\frac{2x^2}{n+2}+\frac{1}{\delta_n}\sqrt{\frac{2x^2}{n+2}}+\frac{1}{2\delta_n}\frac{12(n+4)}{(n+2)n(n-1)}x^4\right]$$
$$\times \left[1+(1+\frac{1}{2\delta_m})\frac{2y^2}{m+2}+\frac{1}{\delta_m}\sqrt{\frac{2y^2}{m+2}}+\frac{1}{2\delta_m}\frac{12(m+4)}{(m+2)m(m-1)}y^4\right]$$

Choosing $\delta_n=\sqrt{\frac{2b_n^2}{n+2}}$ and $\delta_m=\sqrt{\frac{2b_m^2}{m+2}}$ and consider $\delta_n \leq 1$, $\delta_m \leq 1$ for all n, m sufficiently large since $\lim_{n\to\infty}\frac{b_n^2}{n}=0$, we obtain the desired result.

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