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Central Extension of Mappings on von Neumann Algebras[1]

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Abstract

Let $\mathfrak{M}$ be a von Neumann algebra and $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism. Then ρ is called a centrally extendable $*$-homomorphism (CEH) if there is a maximal abelian subalgebra (masa) $\mathcal{M}$ of the commutant $\mathfrak{M}'$ of $\mathfrak{M}$ and a surjective $*$-homomorphism $\varphi : \mathcal{M} \to \mathcal{M}$ such that $\varphi(Z) = \rho(Z)$ for all Z in the center of $\mathfrak{M}$. A $*$-ρ-derivation $\delta : \mathfrak{M} \to \mathfrak{M}$ is called a centrally extendable $*$-ρ-derivation (CED) if there is a masa $\mathcal{M}$ of $\mathfrak{M}'$ such that δ has a norm preserving extension $\tilde{\delta} : C^*(\mathfrak{M}, \mathcal{M}) \to C^*(\mathfrak{M}, \mathcal{M})$ which is a $*$-$\tilde{\rho}$-derivation for some $*$-homomorphism $\tilde{\rho} : C^*(\mathfrak{M}, \mathcal{M}) \to C^*(\mathfrak{M}, \mathcal{M})$ as an extension of ρ, where $C^*(\mathfrak{M}, \mathcal{M})$ is the C^*-algebra generated by $\mathfrak{M} \cup \mathcal{M}$. In this paper we give some sufficient conditions for a $*$-homomorphism to be a CEH and prove that δ is a CED if and only if ρ is a CEH. Thus the study of ρ-derivations on arbitrary von Neumann algebras is reduced to the case of type I von Neumann algebras.

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Accepted for publication (in revised form) 25 March, 2008 centrally extendable $*$-ρ-derivation (CED), modular conjugation operator, von Neumann algebra, ρ-derivation.

1 Introduction

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras, $\mathfrak{X}$ be a $\mathfrak{B}$-bimodule and $\rho : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{X}$ is called a ρ-derivation if $\delta(ab) = \delta(a)\rho(b) + \rho(a)\delta(b)$ for all $a, b \in \mathfrak{A}$. These maps have been extensively investigated in pure algebra. Recently, they have been treated in the Banach algebra theory (see [1, 2, 3, 6, 7, 9] and references therein). Now suppose that $\mathfrak{M}$ is a von Neumann algebra and $\delta : \mathfrak{M} \to \mathfrak{M}$ is a $*$-ρ-derivation, where $\rho : \mathfrak{M} \to \mathfrak{M}$ is a $*$-homomorphism. Our problem is to find a maximal abelian subalgebra (masa) $\mathcal{M}$ of the commutant $\mathfrak{M}'$ of $\mathfrak{M}$ such that δ has a norm preserving extension $\tilde{\tilde{\delta}} : C^*(\mathfrak{M}, \mathcal{M}) \to C^*(\mathfrak{M}, \mathcal{M})$ which is a $*$-$\tilde{\rho}$-derivation for some $*$-homomorphism $\tilde{\rho} : C^*(\mathfrak{M}, \mathcal{M}) \to C^*(\mathfrak{M}, \mathcal{M})$ as an extension of ρ, where $C^*(\mathfrak{M}, \mathcal{M})$ is the C^*-algebra generated by $\mathfrak{M} \cup \mathcal{M}$. Toward solving the problem, we are naturally interested in finding some sufficient conditions to ensure us that the $*$-homomorphism ρ has the desired extension. Surprisingly, ρ has the property if its restriction to the center $\mathfrak{Z}(\mathfrak{M})$ of $\mathfrak{M}$ is extendable. We therefore deal with the so-called centrally extendable $*$-homomorphisms. We shall find some sufficient conditions on a $*$-homomorphism $\rho : \mathfrak{M} \to \mathfrak{M}$ to be centrally extendable. Our discussion concerning centrally extendable $*$-homomorphisms is interesting on its own right. We also deal with CEH's in the next section by using some ideas from [8] and consider the main problem, i.e. extending a ρ-derivation on $\mathfrak{M}$ to $C^*(\mathfrak{M}, \mathcal{M})$. The importance of our work is to extend a ρ-derivation on an arbitrary von Neumann algebra to a type I von Neumann algebra.

Throughout the paper, $\mathfrak{M}$ and $\mathfrak{N}$ denote von Neumann algebras acting on a Hilbert space $\mathfrak{H}$. we denote by $\mathfrak{M}'$ the commutant of $\mathfrak{M}$, i.e. the set of all T in $\mathcal{B}(\mathfrak{H})$ such that $TA = AT$ for every $A \in \mathfrak{M}$. The double commutant theorem states that $\mathfrak{M}$ is a von Neumann algebra if and only if $\mathfrak{M}'' = \mathfrak{M}$. We denote the center $\mathfrak{M} \cap \mathfrak{M}'$ of $\mathfrak{M}$ by $\mathfrak{Z}(\mathfrak{M})$. The Tomita–Takesaki Theorem states that $\mathfrak{M}' = J\mathfrak{M}J$, where J is a modular conjugation on $\mathfrak{H}$ which satisfies, among many useful properties, $J^2 = I$ and $J^* = J$ and $\langle J\eta, J\xi\rangle = \langle \xi, \eta\rangle$ for each $\eta, \xi \in \mathfrak{H}$. Moreover, we know that $JZJ = Z^*$ for each $Z \in \mathfrak{Z}(\mathfrak{M})$. For more detailed information on von Neumann algebras the reader is referred to [4, 5].

2 Centrally Extendable $*$-Homomorphisms

Definition 2.1. *Let $\mathfrak{M}$ be a von Neumann algebra and $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism. Then ρ is called a centrally extendable $*$-homomorphism (CEH) if there is a maximal abelian subalgebra (masa) $\mathcal{M}$ of $\mathfrak{M}'$ and a surjective $*$-homomorphism $\varphi : \mathcal{M} \to \mathcal{M}$ such that $\varphi(Z) = \rho(Z)$ for each $Z \in \mathfrak{Z}(\mathfrak{M})$. In this case we say that $(\mathcal{M}, \varphi)$ is a central structure corresponding to ρ.*

If ρ is the identity mapping on $\mathfrak{M}$ then it is obviously a CEH. If $\mathfrak{M}$ is a masa in $B(\mathfrak{H})$ then $\mathfrak{M} = \mathfrak{M}'$ and so each surjective $*$-homomorphism on $\mathfrak{M}$ is clearly a CEH. There are also nontrivial situations as the following proposition shows.

Proposition 2.2. *Let $\mathfrak{M}$ be a von Neumann algebra with a modular conjugation J, $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism such that $\rho(\mathfrak{Z}(\mathfrak{M})) \subseteq \mathfrak{Z}(\mathfrak{M})$ and there is a masa $\mathcal{N}$ of $\mathfrak{M}$ with $\rho(\mathcal{N}) = \mathcal{N}$. Then ρ is a CEH.*

Proof. Set $\mathcal{M} = J\mathcal{N}J$. It is easily seen that $\mathcal{M}$ is a masa of $\mathfrak{M}'$. Define $\varphi : \mathcal{M} \to \mathcal{M}$ by $\varphi(M) = J\rho(N)J$ where $M = JNJ$ for some $N \in \mathcal{N}$. Obviously φ is well-defined and for $M_1 = JN_1J, M_2 = JN_2J \in \mathcal{M}$ we have

$$\varphi(M_1M_2) = \varphi(JN_1N_2J) = J\rho(N_1N_2)J = J\rho(N_1)JJ\rho(N_2)J = \varphi(M_1)\varphi(M_2).$$

Hence φ is a homomorphism. Moreover, for $M = JNJ \in \mathcal{M}$ we have

$$\begin{aligned}\langle (JNJ)^* J\eta, \xi\rangle &= \langle J\eta, JNJ\xi\rangle \\ &= \langle NJ\xi, \eta\rangle \\ &= \langle J\xi, N^*\eta\rangle \\ &= \langle JN^*\eta, \xi\rangle,\end{aligned}$$

hence $(JNJ)^*J = JN^*$ and so $(JNJ)^* = JN^*J$. Thus

$$\varphi(M^*) = \varphi(JN^*J) = J\rho(N^*)J = J\rho(N)^*J = (J\rho(N)J)^* = \varphi(M)^*.$$

Therefore φ preserves $*$. Furthermore, φ is onto. To show this, let $M = JNJ \in \mathcal{M}$. Then $N = JMJ \in \mathcal{N} = \rho(\mathcal{N})$ and so there is an $N' \in \mathcal{N}$ with $N = \rho(N')$. Thus $M = J\rho(N')J = \varphi(JN'J)$. Furthermore, for each $Z \in \mathfrak{Z}(\mathfrak{M})$ we have $\rho(Z) \in \mathfrak{Z}(\mathfrak{M})$ and

$$\varphi(Z) = \varphi(JZ^*J) = J\rho(Z)^*J = \rho(Z).$$

Thus $(\mathcal{M}, \varphi)$ is a central structure corresponding to ρ.

As a simple result of the arguments stated in Theorem 2.3.2 of [8] we have

Lemma 2.3. *Let $\mathcal{M}$ be an abelian subalgebra of $\mathfrak{M}'$. If $\mu_{\mathfrak{M},\ \mathcal{M}} : \mathfrak{M}\otimes\mathcal{M} \to \mathcal{B}(\mathfrak{H})$ is defined by $\mu_{\mathfrak{M},\ \mathcal{M}}(\sum_{i=1}^n A_i\otimes M_i) = \sum_{i=1}^n A_iM_i$, then there is an $*$-isomorphism $\tilde{\mu}_{\mathfrak{M},\ \mathcal{M}} : (\mathfrak{M} \otimes_{\min} \mathcal{M})/\ker\mu \to C^*(\mathfrak{M}, \mathcal{M})$.*

Proposition 2.4. *Let $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism with $\rho(\mathfrak{Z}(\mathfrak{M})) \subseteq \mathfrak{Z}(\mathfrak{M})$. Suppose that $\mathcal{F}$ is a family of projections of $\mathfrak{M}$ whose closed linear span is $\mathfrak{M}$. If $\rho(P) \preceq P$ for each projection $P \in \mathcal{F}$, then ρ is a CEH.*

Proof. Let $\mathcal{M}$ be a masa of $\mathfrak{M}' \subseteq \mathfrak{Z}(\mathfrak{M})'$. Then $\mathcal{M}$ is an abelian subalgebra of $\mathfrak{Z}(\mathfrak{M})'$ and so one may consider $\mu = \mu_{\mathfrak{Z}(\mathfrak{M}),\ \mathcal{M}} : \mathfrak{Z}(\mathfrak{M}) \otimes \mathcal{M} \to \mathcal{B}(\mathfrak{H})$. We show that $\rho \otimes \iota_{\mathcal{M}} : \mathfrak{Z}(\mathfrak{M}) \otimes_{\min} \mathcal{M} \to \mathfrak{Z}(\mathfrak{M}) \otimes_{\min} \mathcal{M}$ leaves $\ker\mu$ invariant.

Let $K = \sum_{i=1}^{n} P_i \otimes M_i \in \ker \mu$, where $P_i \in \mathcal{F}$ and $M_i = N_i N_i^*$'s are positive elements of $\mathcal{M}$. Then we have

$$\begin{aligned} \mu((\rho \otimes \iota_{\mathcal{M}})K) &= \sum_{i=1}^{n} \rho(P_i) M_i = \sum_{i=1}^{n} N_i \rho(P_i) N_i^* \\ &\preceq \sum_{i=1}^{n} N_i P_i N_i^* = \sum_{i=1}^{n} P_i M_i = \mu(K) = 0. \end{aligned}$$

Hence the mapping ρ_1 defined on $(\mathfrak{Z}(\mathfrak{M}) \otimes_{\min} \mathcal{M})/\ker \mu$ by

$$\rho_1(Z \otimes M + \ker \mu) = (\rho \otimes \iota_{\mathcal{M}})(Z \otimes M) + \ker \mu,$$

is well-defined. Define φ on $\mathcal{M} = C^*(\mathfrak{Z}(\mathfrak{M}), \mathcal{M})$ by $\varphi = \tilde{\mu} \rho_1 \tilde{\mu}^{-1}$. Since ρ is surjective on $\mathfrak{Z}(\mathfrak{M})$, so is ρ_1 and hence φ is a $*$-homomorphism on $\mathcal{M}$ onto $\mathcal{M}$. For each $Z \in \mathfrak{Z}(\mathfrak{M})$ and $M \in \mathcal{M}$ we have

$$\begin{aligned} \varphi(ZM) &= \tilde{\mu} \rho_1(Z \otimes M + \ker \mu) \\ &= \tilde{\mu}(\rho(Z) \otimes M + \ker \mu) \\ &= \rho(Z) M. \end{aligned}$$

Taking $M = I$ we have $\varphi(Z) = \rho(Z)$, for all $Z \in \mathfrak{Z}(\mathfrak{M})$. This shows that $(\mathcal{M}, \varphi)$ is a central structure corresponding to ρ.

Corollary 2.5. *If $\rho : \mathfrak{M} \to \mathfrak{M}$ is a CEH then $\rho(\mathfrak{Z}(\mathfrak{M})) \subseteq \mathfrak{Z}(\mathfrak{M})$.*

Proof. Let $(\mathcal{M}, \varphi)$ be a central structure corresponding to ρ. Then using the notations of the above proposition we can define ρ_1 by $\rho_1 = \tilde{\mu}^{-1} \varphi \tilde{\mu}$ which maps $\mathfrak{Z}(\mathfrak{M})$ into $\mathfrak{Z}(\mathfrak{M})$ and is equal to ρ on $\mathfrak{Z}(\mathfrak{M})$. Now $\rho(I) \in \rho(\mathfrak{Z}(\mathfrak{M})) \subseteq \mathfrak{Z}(\mathfrak{M})$ implies that $\rho(I)$ commutes with each member of $\mathfrak{M}$.

3 Centrally Extendable ρ-Derivations

Let $\mathcal{M}$ be an abelian $*$-subalgebra of $\mathfrak{M}'$. By the Gelfand representation, $\mathcal{M}$ is of the form $\mathcal{C}(\Omega)$ for some compact Hausdorff space Ω. It is known that $\mathfrak{M} \otimes_{\min} \mathcal{C}(\Omega)$ is isometrically $*$-isomorphic to the C^*-algebra $\mathcal{C}(\Omega, \mathfrak{M})$ of $\mathfrak{M}$-valued continuous functions on Ω. Let us state the first result.

Proposition 3.1. *Let $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism, $\delta : \mathfrak{M} \to \mathfrak{M}$ be a $*$-ρ-derivation, $\mathcal{M}$ be an abelian subalgebra of $\mathfrak{M}'$ and $\varphi : \mathcal{M} \to \mathcal{M}$ be a surjective $*$-homomorphism. Then $\rho \otimes \varphi$ is a $*$-homomorphism and $\delta \otimes \varphi : \mathfrak{M} \otimes_{\min} \mathcal{M} \to \mathfrak{M} \otimes_{\min} \mathcal{M}$ is a $(\rho \otimes \varphi)$-derivation with $\|\delta \otimes \varphi\| \leq \|\delta\|$.*

Proof. Identifying $\mathcal{M}$ with $\mathcal{C}(\Omega)$, the character space of $\mathcal{C}(\Omega)$ with Ω, and $\mathfrak{M} \otimes_{\min} \mathcal{C}(\Omega)$ with $\mathcal{C}(\Omega, \mathfrak{M})$, we define $\delta_0 : \mathcal{C}(\Omega, \mathfrak{M}) \to \mathcal{C}(\Omega, \mathfrak{M})$ by $\delta_0(f)(\omega) = \delta\big(f(\hat{\omega} \circ \varphi)\big)$ where $f \in \mathcal{C}(\Omega, \mathfrak{M}), \omega \in \Omega$ and $\hat{\omega}$ is the character on $\mathcal{C}(\Omega)$ defined by $\hat{\omega}(h) = h(\omega),\ h \in \mathcal{C}(\Omega)$. Similarly we can define $\rho_0 : \mathcal{C}(\Omega, \mathfrak{M}) \to \mathcal{C}(\Omega, \mathfrak{M})$ by $\rho_0(f)(\omega) = \rho\big(f(\hat{\omega} \circ \varphi)\big),\ f \in \mathcal{C}(\Omega, \mathfrak{M}), \omega \in \Omega$. Since φ is surjective, $\hat{\omega} \circ \varphi$ is a character on $\mathcal{C}(\Omega)$.

Since ρ is a $*$-homomorphism we easily infer that ρ_0 is also a $*$-homomorphism. δ_0 is a ρ_0-derivation since

$$\begin{aligned}
\delta_0(fg)(\omega) &= \delta\big((fg)(\hat{\omega} \circ \varphi)\big) \\
&= \delta\big(f(\hat{\omega} \circ \varphi)g(\hat{\omega} \circ \varphi)\big) \\
&= \delta\big(f(\hat{\omega} \circ \varphi)\big)\rho\big(g(\hat{\omega} \circ \varphi)\big) \\
&\quad +\rho\big(f(\hat{\omega} \circ \varphi)\big)\delta\big(g(\hat{\omega} \circ \varphi)\big) \\
&= \delta_0(f)(\omega)\rho_0(g)(\omega) + \rho_0(f)(\omega)\delta_0(g)(\omega),
\end{aligned}$$

in which $f, g \in \mathcal{C}(\Omega, \mathfrak{M}), \omega \in \Omega$. Furthermore,

$$\begin{aligned}
\|\delta_0(f)(\omega)\| &= \|\delta\big(f(\hat{\omega} \circ \varphi)\big)\| \\
&\leq \|\delta\| \ \|f(\hat{\omega} \circ \varphi)\| \\
&\leq \|\delta\| \ \|f\| \ \|\hat{\omega} \circ \varphi\|,
\end{aligned}$$

for all $f \in \mathcal{C}(\Omega, \mathfrak{M}), \omega \in \Omega$. Hence

$$\begin{aligned}
\|\delta_0(f)\| &= \sup_{\omega\in\Omega} \|\delta_0(f)(\omega)\| \\
&\leq \|\delta\| \, \|f\| \sup_{\omega\in\Omega} \|\hat{\omega} \circ \varphi\| \\
&\leq \|\delta\| \, \|f\| \sup_{\omega\in\Omega} \sup_{h\in\mathcal{C}(\Omega)} \|(\hat{\omega} \circ \varphi)(h)\| \\
&\leq \|\delta\| \, \|f\| \sup_{h\in\mathcal{C}(\Omega)} \sup_{\omega\in\Omega} \|\varphi(h)(\omega)\| \\
&\leq \|\delta\| \, \|f\|,
\end{aligned}$$

for all $f \in \mathcal{C}(\Omega, \mathfrak{M})$. Thus $\|\delta_0\| \leq \|\delta\|$.

Now we show that under the isomorphism $\pi : \mathfrak{M} \otimes_{\min} \mathcal{C}(\Omega) \simeq \mathcal{C}(\Omega, \mathfrak{M})$, the ρ_0-derivation δ_0 is corresponded to $\delta \otimes \varphi$. By the same argument one can prove that ρ_0 is indeed $\rho \otimes \varphi$. Given $A \in \mathfrak{M}, h \in \mathcal{C}(\Omega), \omega \in \Omega$ we have

$$\pi\big((\delta \otimes \varphi)(A \otimes h)\big)(\omega) = \pi\big(\delta(A) \otimes \varphi(h)\big)(\omega) = \varphi(h)(\omega)\delta(A).$$

On the other hand

$$\begin{aligned}
\delta_0\big(\pi(A \otimes h)\big)(\omega) &= \delta\big(\pi(A \otimes h)(\hat{\omega} \circ \varphi)\big) = \delta\big(h(\hat{\omega} \circ \varphi)A\big) \\
&= h(\hat{\omega} \circ \varphi)\delta(A) = \varphi(h)(\omega)\delta(A)
\end{aligned}$$

Hence $\pi\big((\delta \otimes \varphi)(A \otimes h)\big) = \delta_0(\pi(A \otimes h))$.

The above Proposition shows the importance of the definition of a CEH.

Definition 3.2. *Let $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism and $\delta : \mathfrak{M} \to \mathfrak{M}$ be a $*$-ρ-derivation. δ is called a centrally extendable $*$-ρ-derivation (CED) if there is a masa of $\mathfrak{M}'$ such that δ has a norm preserving extension $\tilde{\delta} : C^*(\mathfrak{M}, \mathcal{M}) \to C^*(\mathfrak{M}, \mathcal{M})$ which is a $*$-$\tilde{\rho}$-derivation for some $*$-homomorphism $\tilde{\rho} : C^*(\mathfrak{M}, \mathcal{M}) \to C^*(\mathfrak{M}, \mathcal{M})$ as an extension of ρ.*

The following Theorem determines our motivation for introducing the notion of CEH.

Theorem 3.3. *Let $\rho : \mathfrak{M} \to \mathfrak{M}$ be a $*$-homomorphism and $\delta : \mathfrak{M} \to \mathfrak{M}$ be a $*$-ρ-derivation. Then δ is a CED if and only if ρ is a CEH.*

Proof. If δ is a CED then ρ is obviously a CEH. Thus let ρ is a CEH and $(\mathcal{M},\varphi)$ be a central structure corresponding to ρ.

We show that $\rho\otimes\varphi:\mathfrak{M}\otimes_{\min}\mathcal{M}\to\mathfrak{M}\otimes_{\min}\mathcal{M}$ leaves $\ker\mu$ invariant. Let $K=\sum_{i=1}^{n}A_i\otimes M_i\in\ker\mu$, where $A_i\in\mathfrak{M}$ and $M_i\in\mathcal{M}$. Then $\sum_{i=1}^{n}A_iM_i=0$. By Theorem 5.5.4 of [4], there are operators $Z_{ik},\ \ 1\leq i,k\leq n$ in $\mathfrak{Z}(\mathfrak{M})$ such that $\sum_{i=1}^{n}A_iZ_{ik}=0$ for $1\leq k\leq n$, and $\sum_{k=1}^{n}Z_{ik}M_k=M_i$ for $1\leq i\leq n$. Since $\rho|_{\mathfrak{Z}(\mathfrak{M})}=\varphi|_{\mathfrak{Z}(\mathfrak{M})}$ we have

$$\sum_{i=1}^{n}\rho(A_i)\varphi(Z_{ik})=0$$

and

$$\sum_{k=1}^{n}\varphi(Z_{ik})\varphi(M_k)=\varphi(M_i)$$

Using again Theorem 5.5.4 of [4] and noting $\varphi(Z_{ik})\in\mathfrak{Z}(\mathfrak{M})$, we conclude that

$$\sum_{i=1}^{n}\rho(A_i)\varphi(M_i)=0$$

Thus $\mu(\rho\otimes\varphi)(K)=\sum_{i=1}^{n}\rho(A_i)\varphi(M_i)=0$.

Moreover, $\delta\otimes\varphi$ leaves $\ker\mu$ invariant. To see this, let K be a positive element of $\ker\mu$. Then there is an $S\in\ker\mu$ such that $K=S^2$ and we have

$$\begin{aligned}\mu((\delta\otimes\varphi)S^2) &= \mu((\delta\otimes\varphi)S(\rho\otimes\varphi)S+(\rho\otimes\varphi)S(\delta\otimes\varphi)S)\\ &= \mu((\delta\otimes\varphi)S)\mu((\rho\otimes\varphi)S)+\mu((\rho\otimes\varphi)S)\mu((\delta\otimes\varphi)S)\\ &= 0.\end{aligned}$$

Hence the mappings ρ_1 and δ_1 defined on $(\mathfrak{M}\otimes_{\min}\mathcal{M})/\ker\mu$ by

$$\begin{aligned}\delta_1(A\otimes M+\ker\mu) &= (\rho\otimes\varphi)(A\otimes M)+\ker\mu\\ \delta_1(A\otimes M+\ker\mu) &= (\delta\otimes\varphi)(A\otimes M)+\ker\mu\end{aligned}$$

are well-defined. Note that $\|\delta_1\| \leq \|\delta \otimes \varphi\| \leq \|\delta\|$.
Define $\tilde{\rho}$ and $\tilde{\delta}$ on $C^*(\mathfrak{M}, \mathcal{M})$ by $\tilde{\rho} = \tilde{\mu}\rho_1\tilde{\mu}^{-1}$ and $\tilde{\delta} = \tilde{\mu}\delta_1\tilde{\mu}^{-1}$, respectively. Then $\tilde{\delta}$ is a $\tilde{\rho}$-derivation and for $A \in \mathfrak{M}, M \in \mathcal{M}$ we have

$$\begin{aligned}\tilde{\delta}(AM) &= \tilde{\mu}\delta_1(A \otimes M + \ker\mu) \\ &= \tilde{\mu}(\delta(A) \otimes \varphi(M) + \ker\mu) \\ &= \delta(A)\varphi(M).\end{aligned}$$

Taking $M = I$ we have $\tilde{\delta}(A) = \delta(A)\rho(I) = \delta(A)$. This shows that $\tilde{\delta}$ extends δ. Similarly one can prove that $\tilde{\rho}$ is an extension of ρ. Since $\rho \otimes \varphi$ has a norm dense range, $\tilde{\rho}$ has also a norm dense range. Furthermore, $\|\tilde{\delta}\| \leq \|\tilde{\mu}\| \, \|\delta_1\| \, \|\tilde{\mu}^{-1}\| \leq \|\delta\|$. Thus $\|\tilde{\delta}\| = \|\delta\|$.

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