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Mapping Φ^P in Normed Linear Spaces and Characterization of Orthogonality Problem of Best Approximations in 2-norm[1]

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Abstract

In order to characterizations of best approximations have been given in 2-norm space $(X, \| .,. \|)$. Some generalization of the function Φ^p of Dragomir type have been given in the context where the said generalization help to formulate the characterizations what have been proposed in this article.

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1 Introduction

In a 2-normed linear space $(X, \| .,. \|)$ our present aim is to characterize the set of best approximations and related generalized orthogonality of a pair of elements in 2-normed space with reference to the 2-norm ([7] and [11]). We introduce below the Φ^p function and their properties as were done

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by Dragomir in an earlier reference [6]. We also study the boundedness, monotonocity and convexicity properties of the generalized Φ^p functions.

Let $(X, \| .,. \|)$ be real 2 - normed linear space. Consider the 2- norm derivative

$$(y, x/z)_i = \lim_{t \to 0^-} \frac{\| x + ty, z \|^2 - \| x, z \|^2}{2t}$$

and

$$(y, x/z)_s = \lim_{t \to 0^+} \frac{\| x + ty, z \|^2 - \| x, z \|^2}{2t}$$

which are well defined for every pair $x, y \in X$ and $z \in X/L\{x, y\}, G)$ (where L({x,y}, G) stands for the linear manifolds - spaned by x and y).

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel([2],[3],[4],[5] and [6]),assuming that $p, z \in \{s, i\}$ and $p \neq 2$.

(i) $(x, x/z)_p = \| x, z \|^2$

(ii) $(\alpha x, \beta y/z)_p = \alpha\beta(x, y/z)_p$ if $\alpha, \beta \geq 0$

(iii) $| (x, y/z)_p | \leq \| x, z \| \quad \| y, z \|$

(iv) $(\alpha x + y, x/z)_p = \alpha(x, x/z)_p + (y, x/z)_p$ where $\alpha \in R$

(v) $(-x, y/z)_p = -(x, y/z)_q$

(vi) $(x + y, w/z)_p \leq \| x, z \| \quad \| w, z \| + (y, w/z)_p$

(vii) The mapping $(., ./z)_p$ is continuous and subadditive in the first variable for $p = s(or\ p = i)$.

(viii)The element $x \in X$ is Birkhoff orthogonal to the element $y \in X$ (i.e. $\| x + ty, z \| \geq \| x, z \|\ t$ for all $t \in R$ and $z \in X/L(\{x, y\}, G)$ if and only if

$$(y, x/z)_i \leq 0 \leq (y, x/z)_s$$

(ix) The 2 - normed linear space $(X, \| .,. \|)$ is smooth at the point $x_0 \in X \backslash \{0\}$ if and only if the mapping $y \to (y, x_0/z)_p$ is linear, or if and only if $(y, x_0/z)_s = (y, x_0/z)_i$ for all $y \in X$ and $z \in X/L(\{x, y\}, G)$. (x) If the 2-norm $\| .,. \|$ is induced by an 2 - inner product $(., ./z)$ then $(y, x/z)_i = (y, x/z) = (y, x/z)_s$ for all $x, y \in X$ and $z \in X/L(\{x, y\}, G)$.

2 Properties of the mapping $\Phi^p_{x,y/z}$

For three fixed linearly independent vectors x, y in X and
$z \in X/L(\{x,y\},G)$ we consider the mapping

$$\Phi^p_{x,y/z}(t) = \frac{(y, x+ty/z)_p}{\| x+ty, z \|}, \quad p = s \;\; or \;\; p = i$$

which is well defined for all $t \in R$.

Theorem 2.1. *Let $(X, \| .,. \|)$ be a real 2- normed liner space and x,y, z two linearly independent vectors in X and $z \in X/L(\{x,y\},G)$. Then*
(i) The mapping $\Phi^p_{x,y/z}$ is bounded on R with

$$(2.1) \qquad | \Phi^p_{x,y/z}(t) | \leq \| y, z \| \quad for\ all\ t \in R$$

(ii) We have the inequality

$$(2.2) \qquad \frac{\| x+2uy, z \| - \| x+uy, z \|}{u} \leq \Phi^i_{x,y/z}(u) \leq \Phi^s_{x,y/z}$$
$$\leq \frac{\| x+uy, z \| - \| x, z \|}{u} \quad for\ all \quad u < 0$$

and

$$(2.3) \qquad \frac{\| x+2ty, z \| - \| x+ty, z \|}{t} \geq \Phi^s_{x,y/z}(t) \geq \Phi^i_{x,y/z}(t)$$
$$\geq \frac{\| x+ty, z \| - \| x, z \|}{t}$$

(iii) The mapping $\Phi^p_{x,y/z}$ are strictly increasing on R.
(iv) We have the limits

$$(2.4) \qquad \lim_{u \to -\infty} \Phi^p_{x,y/z}(u) = \| y, z \|, \quad \lim_{t \to +\infty} \Phi^p_{x,y/z}(t) = \| y, z \|$$

and

$$(2.5) \qquad \lim_{t \to 0^+} \Phi^p_{x,y/z}(t) = \frac{(y, x/z)_s}{\| x, z \|}, \quad \lim_{u \to 0^-} \Phi^p_{x,y/z}(u) = \frac{(y, x/z)_i}{\| x, z \|}$$

(v) The mapping Φ^s is right continuous and Φ^i is left continuous at every point of R.

Proof. (i) Follows by the Schwarz inequality.
(ii) Let $u < 0$. By the Schwarz inequality (iii) and by properties (iv) and (ii) of 2 -norm derivatives $(.,./z)_i$, we have

$$\| x+2uy, z \| \quad \| x+uy, z \| \geq (x+2uy, x+uy/z)_s = (x+uy+uy, x+uy/z)_s$$

$$= \| x+uy, z \|^2 - u(-y, x+uy/z)_s$$

$$= \| x+uy, z \|^2 + u(y, x+uy/z)_i.$$

From which we get

$$\| x+2uy, z \| - \| x+uy, z \| \| x+uy, z \| \geq u(y, x+uy/z)_i.$$

This implies

$$\frac{\| x+2uy, z \| - \| x+uy, z \|}{u} \leq \frac{(y, x+uy/z)_i}{\| x+uy, z \|}$$

and the (i) inequality in (2.2) is proved.

Further,

$$\| x, z \| \quad \| x+uy, z \| \geq (x, x+uy/z)_s$$

$$= (x+uy-uy, x+uy/z)_s$$

$$= \| x+uy, z \|^2 + (-uy, x+uy/z)_s.$$

From which we get

$$\frac{\| x+uy, z \|^2 - \| x, z \|^2}{u} \geq \frac{(y, x+uy/z)_s}{\| x+uy, z \|} = \Phi^s_{x,y/z}(u).$$

The (iii) inequality in (2.2) is proved.
Inequality (2.3) is proved similarly.
(iii) Suppose that $p \in \{i, s\}$ and $t_2 > t_1$. Then by Schwarz inequality

$$\| x+t_2y, z \| \| x+t_1y, z \| \geq (x+t_2y, x+t_1y/z)_p$$

for all $x, y \in X$ and $z \in X/L(\{x, y\}, G)$. Using properties of 2-norm derivatives, we obtain

$$(x + t_2 y, x + t_1 y/z)_p \geq (t_2 - t_1/y + x + t_1 y/z)_p$$

$$\| x + t_1 y, z \|^2 + (t_2 - t_1)(y, x + t_1 y/z)_p$$

and the above inequality yields

$$\| x + t_2 y, z \| \| x + t_1 y, z \| \geq \| x + t_1 y, z \|^2 + (t_2 - t_1)(y, x + t_1 y/z)_p$$

Hence

$$\Phi^p_{x,y/z}(t_1) = \frac{(y, x + t_1 y/z)_p}{\| x + t_1 y, z \|} \leq \frac{\| x + t_2 y, z \| - \| x - t_1 y, z \|}{t_2 - t_1}$$

put $t = t_2 - t_1 > 0$ then by (2.3)

$$\frac{\| x + t_2 y, z \| - \| x - t_1 y, z \|}{t_2 - t_1} = \frac{\| x + t_1 y + ty, z \| - \| x + t_1 y, z \|}{t}$$

$$\Phi^p_{x,y/z}(t_1) = \frac{(y, x + t_1 y/z)_p}{\| x + t_1 y, z \|}$$

$$\leq \Phi^p_{x+t_1 y, y/z}(t) = \frac{(y, x + t_1 y + ty/z)_p}{\| x + t_1 y + ty, z \|}$$

$$= \frac{(y, x + t_2 y/z)_p}{\| x + t_2 y, z \|} = \Phi^p_{x,y/z}(t_2)$$

and the statement is proved.

(iv)We have

$$\lim_{t \to +\infty} \frac{\| x + ty, z \| - \| x, z \|}{t} = \lim_{\alpha \to 0^+} \frac{\| x + \frac{y}{\alpha}, z \| - \| x, z \|}{\frac{1}{\alpha}}$$

$$\lim_{\alpha \to 0^+} \frac{\| \alpha x + y, z \| - \alpha \| x, z \|}{t} = \quad \| y, z \|$$

and

$$\lim_{t \to +\infty} \frac{\| x + 2ty, z \| - \| x + ty, z \|}{t} = \lim_{\alpha \to +\infty} | t | \| \frac{x}{t} + 2y, z \| - \| \frac{x}{t} + y, z \|$$

$$= \lim_{t\to+\infty}(\| 2y + \frac{x}{t}, z \| - \| y + \frac{x}{t}, z \|)$$
$$= \lim_{\alpha\to 0^+}(\| 2y + \alpha x, z \| - \| y + \alpha x, z \|)$$
$$= 2 \| y, z \| - \| y, z \| - \| y, z \| .$$

Applying the inequality (2.3) we get the second limit in (2.4) the first limit is obtained similarly.

Further

$$\lim_{t\to 0^+} \frac{\| x + ty, z \| - \| x, z \|}{t} =$$
$$\lim_{t\to 0^+} \frac{\| x + ty, z \|^2 - \| x, z \|^2}{2t} \times \lim_{t\to 0^+} \frac{2}{\| x + ty, z \| + \| x, z \|}$$
$$= \frac{(y, x/z)_s}{\| x, z \|}$$

and

$$\lim_{t\to 0^+} \frac{\| x + 2ty, z \| - \| x + ty, z \|}{t}$$
$$= \lim_{t\to 0^+} \frac{\| x + 2ty, z \| - \| x, z \| -(\| x + ty, z \| - \| x, z \|)}{t}$$
$$= 2 \lim_{t\to 0^+} \frac{\| x + 2ty, z \| - \| x, z \|}{2t} - \lim_{t\to 0^+} \frac{\| x + ty, z \| - \| x, z \|}{t}$$
$$\frac{2(y, x/z)_s}{\| x, z \|} - \frac{(y, x/z)_s}{\| x, z \|} = \frac{(y, x/z)_s}{\| x, z \|)}.$$

Inequality (2.3) applied to these limit yields the first in (2.3); the second limit is obtained similarly.

(v) Let $t_0 \in R$

$$\lim_{\alpha\to t_0^+} \Phi^p_{x,y/z}(\alpha) = \lim_{t\to 0^+} \Phi^p_{x,y/z}(t_0 + t) = \lim_{t\to 0^+} \frac{(y, x + t_0 y + ty/z)_p}{\| x + t_0 y + ty, z \|}$$

$$\lim_{t\to 0^+} \Phi^p_{x,y/z}(t) = \frac{(y, x + t_0 y/z)_s}{\| x + t_0 y, z \|} = \Phi^s_{x,y/z}(t_0)$$

in the statement above the right continuity is proved. The statement about the left continuity is proved similarly.

3 New Characterizations of Birkhoff Orthogonality and Smoothness

The mapping $\Phi^p_{x,y/z}$ can be used to give a characterization of Birkhoff Orthogonality.

Theorem 3.1. *Let $(X, \| .,. \|)$ be a real normed linear space, and let x,y be a two elements of X and $\in X/L(\{x,y\}, G)$. The following statement are equivalent*
(i) $x \perp_z y(B)$
(ii) If $p, q \in \{i, s\}$ and $u < 0 < t$ then the following inequality holds:

$$\Phi^p_{x,y/z}(u) \leq 0 \leq \Phi^q_{x,y/z}(t) \tag{3.1}$$

Proof. We know that Birkhoff Orthogonality $x \perp_z y(B)$ is equivalent to the inequality

$$(y, x/z)_i \leq 0 \leq (y, x/z)_s \tag{3.2}$$

According to the Theorem 2.1, we have that

$$\Phi^p_{x,y/z}(u) \leq \frac{\| x + uy, z \| - \| x, z \|}{u}, \quad u < 0 \tag{3.3}$$

$$\Phi^p_{x,y/z}(t) \geq \frac{\| x + ty, z \| - \| x, z \|}{t}, \quad t > 0 \tag{3.4}$$

whenever $p \in \{s, i\}$.
$(i) \Rightarrow (ii)$ if $x \perp_z y(B)$, then $\| x + \alpha y, z \| \geq \| x, z \|$ for all $\alpha \in R$. Hence

$$\frac{\| x + uy, z \| - \| x, z \|}{u} \leq 0 \leq \frac{\| x + ty, z \| - \| x, z \|}{t}$$

for $u < 0 < t$. Using inequality (3.3) and (3.4) we get (3.1).

$(ii) \Rightarrow (i)$ According to the Theorem 2.1, we have that

$$\lim_{t \to 0^+} \Phi^p_{x,y/z}(t) = \frac{(y, x/z)_s}{\| x, z \|}, \quad \lim_{u \to 0^-} \Phi^p_{x,y/z}(u) = \frac{(y, x/z)_i}{\| x, z \|}.$$

If (3.1) holds then $(y, x/z)_s \geq 0 \geq (y, x/z)_i$ using (3.2) we deduce that $x \perp_z y(B)$.

Theorem 3.2. *Let $(X, \|\ .,.\ \|)$ be a real 2-normed linear space and let $x \in X \backslash \{0\}$. The following statements are equivalent*
(i) X is smooth at x_0,
(ii) The mapping $\Phi^p_{x,y/z}$ is continuous at 0 for all $y \in X$ and some $p \in \{s, i\}$.

Proof. The space X is smooth at x_0 if and only if the function $x \to \| x, z \|$ is Gateaux differentiable at x_0, this is equivalent to $(y, x_0/z)_i = (y, x_0/z)_s$ for all $y \in X$ and $z \in X/L(\{x, y\}, G)$. The equivalence of (i) and (ii) then follows in view of (2.5).

4 New Characterizations of Elements of Best Approximations in 2 - norm Spaces

Definition 4.1. *Let X be a 2- normed linear space, G a set in X, and $x \in X$. An element $g_0 \in G$ is called an element of best approximation at x, if*

$$\| x - g_0, z \| = \inf_{g \in G} \| x - g, z \| \tag{4.1}$$

where $z \in X/L(\{x, y\}, G)$.

We denote by $P_{G,z}(X)$ the set at all such elements g_0, that is

$$P_{G,z}(x) = \{g_0 \in G \ \| x - g_0, z \| = \inf_{g \in G} \| x - g, z \| \}. \tag{4.2}$$

It is of interest to consider the problem of finding necesary and sufficient conditions such that $g_0 \in P_{g,z}(x)$.

Lemma 4.1. *Let $(X, \|\ .,.\ \|)$ be a 2-normed linear space, G a linear subspace of X, $x \in X \backslash \bar{G}$ and $g_0 \in G$. Then $g_0 \in P_{G,z}(x)$ if and only if $x - g_0 \perp_z G(B)$. The following preposition is true.*

Proposition 4.1. *Let $(X, \| .,. \|)$ be a 2 -normed linear space, G a linear subspace of $X, x \in X \backslash \bar{G}$ and $g_0 \in G$. The following statement are equivalent:*

(i) $g_0 \in P_{G,z}(x)$

(ii) We have the equality

$$\sup_{g \in G}(g + x - g_0, x - g_0/z)_i = \| x - g_0, z \|^2 \tag{4.3}$$

Proof. By Lemma 4.1, $g_0 \in P_{G,z}(x)$ is equivalent to

$$x - g_0 \perp_z G(B)$$

and the property (viii) of the introduction to

$$(g, x - g_0/z)_i \leq 0 \leq (g, x - g_0/z)_s \qquad for\ all\ g \in G \tag{4.4}$$

But

$$\begin{aligned} (g, x - g_0/z)_i &= (x - g_0 + g - x + g_0, x - g_0/z)_i \\ &= \| x - g_0, z \|^2 + (g + x - g_0, x - g_0/z)_i \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} (g, x - g_0/z)_s &= (x - g_0 + g - x + g_0, x - g_0/z)_s \\ &= \| x - g_0, z \|^2 - (-g + x - g_0, x - g_0/z)_s \\ &= \| x - g_0 \|^2 - (-g + x - g_0, x - g_0/z)_i \end{aligned} \tag{4.6}$$

Then (4.4) is equivalent to

$$(g + x - g_0, x - g_0/z)_i \leq \| x - g_0, z \|^2 \quad for\ all\ g \in G$$

$$(-g + x - g_0, x - g_0/z)_i \leq \| x - g_0, z \|^2 \quad for\ all\ g \in G$$

$g \in G$ if and only if $-g \in G$, we deduce that (4.4) is equivalent to (4.3) and the proposition is proved.

Lemma 4.2. *Let $(X, \| .,. \|)$ be a real 2 -normed space and x, y two elements of X and $z \in X/L(\{x,y\},G)$. The following statements are equivalent:*
(i)$x \perp_z y(B)$
(ii) $(y, x+uy/z)_p \leq 0 \leq (y, x+ty/z)_q$ whenever $u < 0 < t$ and $p, q \in \{i, s\}$.

Using this Lemma, we obtain the following new characterization of best approximants in terms of the 2- norm derivatives.

Theorem 4.1. *Let X, G, x and g be as in Proposition4.1 The following statements are equivalents.*
(i) $g_0 \in P_{G,z}(x)$
(ii) We have the inequality

$$(4.7) \qquad (g, x-g_0+ug/z)_p \leq \| x-g_0+w, z \|^2 \quad if \quad w \in G, p \in \{i, s\}$$

Proof By Lemma 4.2, $g_0 \in P_{G,z}(x)$ is equivalent to

$$(4.8) \quad (g, x-g_0+ug/z)_p \leq 0 \leq (g, x-g_0+tg/z)_q \; if \; u < 0 < t, q \in \{i, s\}$$

But

$$(4.9) \qquad (g, x-g_0+tg/z)_q \leq 0, \qquad t > 0$$

is equivalent to

$$(tg, x-g_0+tg/z)_q \geq 0, \qquad t > 0$$

As

$$(tg, x-g_0+tg/z)_q = (x-g_0+tg-x+g_0, x-g_0+tg/z)_q$$

$$= \| x-g_0+tg, z \|^2 - (x-g_0, x-g_0+tg/z)_r$$

with $r \in \{i, s\}, r \neq q$ (4.9) is equivalent to

$$(4.10) \qquad (x-g_0, x-g_0+tg/z)_q \leq \| x-g_0+tg, z \|^2$$

for all $g \in G, t > 0, g \in \{i, s\}$.
The relation

$$(g, x - g_0 + ug/z)_p \leq 0, u < 0, p \in \{i, s\} \tag{4.11}$$

is equivalent to

$$-u(g, x - g_0 + ug/z)_p \leq 0, p \in \{i, s\}.$$

But

$$-u(g, x - g_0 + ug/z)_p = (-ug, x - g_0 + ug/z)_p = -(ug, x - g_0 + ug/z)_r$$

with $r \in \{i, s\}, r \neq p$; hence (4.11) is equivalent to

$$(ug, x - g_0 + ug/z)_p \geq 0, \quad p \in \{i, s\}, u < 0.$$

On the other hand

$$(ug, x - g_0 + ug/z)_p = (x - g_0 + ug - x + g_0, x - g_0 + ug/z)_p$$

$$= \| x - g_0 + ug, z \|^2 - (x - g_0, x - g_0 + ug/z)_r$$

and (4.11) is equivalent to

$$(x - g_0, x - g_0 + ug/z)_p \leq \| x - g_0 + ug, z \|^2 \tag{4.12}$$

for all $g \in G, u < 0, p \in \{i, s\}$.

Combining (4.10) and (4.12) and observing that (4.10) holds (with equality) also for t $= 0$, we conclude that

$$(x - g_0, x - g_0 + tg/z)_p \leq \| x - g_0 + tg, z \|^2$$

for all $g \in G$ and all $t \in R$.

As $g \in G$ if and only if $t, g \in G$ for $t \neq 0$, we deduce the desired equivalence, and the theorem is proved.

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