

Matroids in terms of Cayley graphs and some related results¹

A. Sinan Çevik, İ. Naci Cangül

Abstract

The point we try to get across is that the generalization of the counterparts of the matroid theory in Cayley graphs since the matroid theory frequently simplify the graphs and so Cayley graphs. We will show that, for a Cayley graph Γ_G , the cutset matroid $M^*(\Gamma_G)$ is the dual of the circuit matroid $M(\Gamma_G)$. We will also deduce that if Γ_G^* is an abstract-dual of a Cayley graph Γ , then $M(\Gamma_G^*)$ is isomorphic to $(M(\Gamma_G))^*$.

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1 Introduction and Preliminaries

1.1 Cayley graphs

Let G be finite group, and let S be a generating set of G . Let $V(G, S)$ be the vertices and let $E(G, S)$ be the edges sets which are defined by

$V(G, S)$: The elements of G ,

$E(G, S)$: The elements of the set $G \times S = \{(g, s) : g \in G, s \in S\}$

and their inverses.

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Then the graph obtained by the above sets is called *Cayley graph* of G and denoted by Γ_G . The initial vertex of the edge (g, s) is g and the terminal is gs . Also the inverse of the edge (g, s) is given by (gs, s^{-1}) . In other words,

$$\iota(g, s) = g, \quad \tau(g, s) = gs \quad \text{and} \quad (g, s)^{-1} = (gs, s^{-1}).$$

Therefore the equalities

$$\iota(g, s) = \tau((g, s)^{-1}) \quad \tau(g, s) = \iota((g, s)^{-1}) \quad \text{and} \quad ((g, s)^{-1})^{-1} = (g, s)$$

are hold. Since the direction of the edges are different than each other, we have $(g, s) \neq (g, s)^{-1}$. These above give us Γ_G is actually defined a graph. Moreover Γ_G is connected ([1]).

Similar definitions for Cayley graphs can also be found, for instance, in [2, 3].

1.2 Matroid theory

There is a close similarity between certain results in graph theory and their analogues in transversal theory ([4]). In order to do this it is convenient to introduce the idea of a matroid (see, for instance, [6], [8]) and hence we can get a close connection between results of graph theory and matroid theory.

A *matroid* M is a pair (E, \mathcal{B}) , where E is a non-empty finite set and \mathcal{B} is a non-empty collection of subsets of E (called *bases*) satisfying the following properties:

- 1) no base properly contains another base;
- 2) if B_1 and B_2 are bases and if e is any element of B_1 , then there is an element f of B_2 with the property that $(B_1 - \{e\}) \cup \{f\}$ is also a base.

By repeatedly using the second property, it is easy to see that any two bases of a matroid M contain the same number of elements; this number is called the *rank* of M .

A matroid can be associated in a natural way with any graph Γ (of course with a Cayley graph Γ_G) by letting E be the set of edges of Γ and taking as bases the edges of the spanning forest of Γ . This matroid is called the *circuit matroid* of Γ and is denoted by $M(\Gamma)$.

A subset of E will be called *independent* if it is contained in some base of the matroid M . Then the bases of M are the maximal independent sets, that is those independent sets which are contained in no larger independent set, and hence that any matroid is uniquely defined by specifying its independent sets. For any graph Γ (or Γ_G), the independent sets of $M(\Gamma)$ are simply the edges of Γ which contained no circuit (or the edge-sets of the forests contained in Γ).

By [6], there is a simple definition of a matroid as follows: A *matroid* M is a pair (E, \mathcal{K}) , where E is a non-empty finite set, and \mathcal{K} is a non-empty collection of subsets of E (called *independent sets*) satisfying the following properties:

- 1)' any subset of an independent set is independent;
- 2)' if I and J are independent sets with $|J| > |I|$, then there is an element e contained in J but not in I , such that $I \cup \{e\}$ is independent.

We note that with this definition, a *base* is defined to be any maximal independent set since, by using the second property repeatedly, one can show that any independent set can be extended to a base.

If $M = (E, \mathcal{K})$ is a matroid defined in terms of its independent sets, then a subset of E is said to be *dependent* if it is not independent. Moreover a minimal dependent set is called a *circuit*. We note that if $M(\Gamma)$ is the circuit matroid of a graph Γ , then the circuits of $M(\Gamma)$ are especially the circuits of Γ . Since a subset of E is independent if and only if it contains no circuits, a definition of a matroid can be given in terms of its circuits (see [8]).

If $M = (E, \mathcal{K})$ is a matroid defined in terms of its independent sets and if A is a subset of E , then the size of the largest independent set contained in A is called the *rank* of A and is denoted by $\rho(A)$. By [6], since a subset A of E

is independent if and only if $\rho(A) = |A|$, a matroid may be defined in terms of its rank function (see [8] for the details) and denoted by $M = (E, \rho)$.

A *loop* of a matroid $M = (E, \rho)$ is an element e of E satisfying $\rho(\{e\}) = 0$. Also a *pair of parallel elements* of M is a pair $\{e, f\}$ of elements of E which are not loops and which satisfy $\rho(\{e, f\}) = 1$. One can see that if M is the circuit matroid of a graph Γ , then the loops and parallel elements of M correspond to loops and multiple edges of Γ .

Let us denote the isomorphism between two matroids. Suppose $M_1 = (E_1, \mathcal{K}_1)$ and $M_2 = (E_2, \mathcal{K}_2)$ are two matroids defined in terms of their independent sets. Then M_1 and M_2 are said to be *isomorphic* if there is a one to one correspondence between the sets E_1 and E_2 which preserves independence. In other words, a set of elements of E_1 is independent in M_1 if and only if the corresponding set of elements of E_2 is independent in M_2 . In Figure 1, it can be seen an example of the isomorphism between circuit matroids of three graphs.

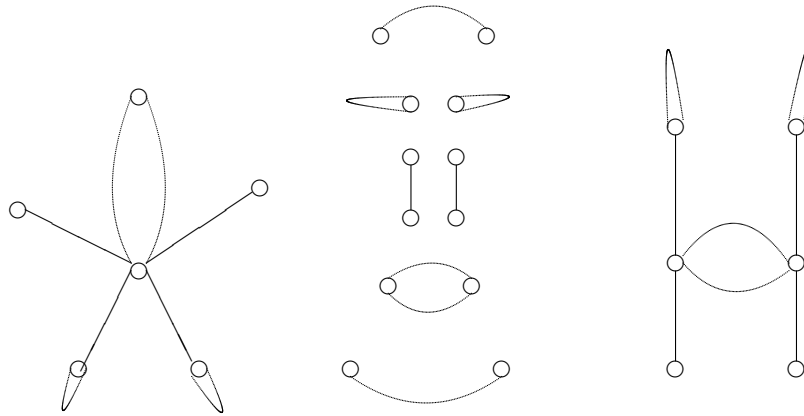


Figure 1

Remark 1.1. By [8], although matroid isomorphisms preserve circuits, cutsets and the number of edges in a graph, it does not in general preserve connectedness, the number of vertices or their degrees.

By [8], for a given any graph Γ , the circuit matroid $M(\Gamma)$ is not the only matroid which can be defined on the set of edges of Γ . Because of the similarity between the properties of circuits and of cutsets in a graph, one can easily say that a matroid can be constructed by taking the cutsets of Γ as circuits of the matroid. This construction does in fact define a matroid and called *cutset matroid* of Γ , written $M^*(\Gamma)$. We note that a set of edges of Γ is independent if and only if it contains no cutset of Γ . A matroid M is called *cographic* if there exists a graph Γ such that M is isomorphic to $M^*(\Gamma)$.

If $M = (E, \rho)$ is a matroid defined in terms of its rank function then we define the *dual matroid* of M , denoted by M^* , to be the matroid on E whose rank function ρ^* is given by the equality

$$(1) \quad \rho^* = |A| + \rho(E - A) - \rho(E),$$

for $A \subseteq E$.

Some more examples and applications of matroids (for all definition of it) can be found, for instance in [6] and [8].

1.3 Main results

Suppose that Γ_G is a Cayley graph. Also let $M(\Gamma_G)$ be the corresponding circuit matroid, $M^*(\Gamma_G)$ be the corresponding cutset matroid of Γ_G , respectively, and $(M(\Gamma_G))^*$ be the dual of the circuit matroid $M(\Gamma_G)$. Then one of the main result of this paper is the following.

Theorem 1.2. $M^*(\Gamma_G) = (M(\Gamma_G))^*$

By [8], a graph Γ^* is said to be an *abstract-dual* of a graph Γ if there is a one to one correspondence between the edges of Γ and the edges of Γ^* with the property that a set of edges of Γ forms a circuit in Γ if and only if the corresponding set of edges of Γ^* forms a cutset in Γ^* . For example, Figure 2 shows graph and its abstract-dual, with corresponding edges sharing the same letter.

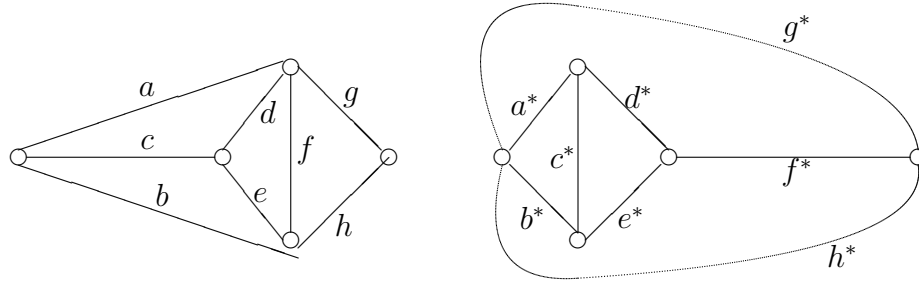


Figure 2

We then have another result as follows.

Theorem 1.3. *If Γ_G^* is an abstract-dual of the Cayley graph Γ_G , then*

$$(M(\Gamma_G))^* \cong M(\Gamma_G^*).$$

2 Proofs

In this part of the paper we use the same notation as in the previous sections.

By [8], the following lemma shows that the rank function ρ^* , as defined in (1), is actually the rank function of a matroid on E .

Lemma 2.1. *$M^* = (E, \rho^*)$ is matroid on E .*

Let us denote the bases of a dual matroid of M . In fact the following proposition asserts that the bases of M^* can be described quite simply in terms of the bases of M .

Proposition 2.2. *The bases of M^* are precisely the complements of the bases of M .*

Proof. Suppose that B^* is a base of M^* . Then our aim is to show that $E - B^*$ is a base of M . In fact the converse result is obtained by simply reversing the argument. Recall that, by Lemma 2.1, ρ^* is actually the rank function of a matroid on E .

By the definition of a base, since B^* is independent in M^* , we then get $|B^*| = \rho^*(B^*)$. Hence we have

$$\rho(E - B^*) = \rho(E).$$

Therefore we just need to show that $E - B^*$ is independent in M . But, by the equation (1) and the fact $\rho^*(B^*) = \rho^*(E)$, independency of the base $E - B^*$ is clear. This gives the result.

Remark 2.3. 1) *As a consequence of the above proposition we can say that every matroid has a dual and this dual is unique. Also the double-dual M^{**} , which will not be needed here, is equal to M .*

2) *We also note that the term “base of M^* ” will be needed in the proof of Theorem 1.2.*

To proof of the first main result in this paper, we further need to state the following lemmas which proofs can be found in [5] and [8].

Lemma 2.4. *If T is any spanning forest of a Cayley graph Γ_G then*

- (i) *every cutset of Γ_G has an edge in common with T ;*
- (ii) *every circuit of Γ_G has an edge in common with the complement of T .*

Lemma 2.5. *Let E be a set of edges of a Cayley graph Γ_G . If E has an edge in common with any spanning forest of Γ_G , then E contains a cutset.*

Proof.1.2. We will show that the cutset matroid $M^*(\Gamma_G)$ of a graph Γ_G is the dual of the circuit matroid $M(\Gamma_G)$.

Since the circuits of $M^*(\Gamma_G)$ are the cutsets of Γ_G , we must check that C^* is a circuit of $(M(\Gamma))^*$ if and only if C^* is a cutset of Γ_G .

Suppose first that C^* is a cutset of Γ_G . If C^* is independent in $(M(\Gamma))^*$, then C^* can be extended to a base B^* of $(M(\Gamma_G))^*$. We get that $C^* \cap (E - B^*)$ is empty which is a contradicting the result of Lemma 2.4 since $E - B^*$ is a spanning forest of Γ_G . It follows that C^* is a dependent set $(M(\Gamma_G))^*$. Thus, by the definition, C^* contains a circuit of $(M(\Gamma_G))^*$.

On the other hand, if D^* is a circuit of $(M(\Gamma_G))^*$, then D^* is not contained in any base of $(M(\Gamma_G))^*$. So that D^* intersects every base of $M(\Gamma_G)$. This means that D^* intersects every spanning forest of Γ_G . Thus, by Lemma 2.5, D^* contains a cutset.

Hence the result.

Before giving the proof of the other theorem, let us define some more terminology.

We say that a set of elements of a matroid M form a *cocircuit* of M if they form a circuit of M^* . We note that, by the result of Theorem 1.2, we can say that *the cocircuits of the circuit matroid of a Cayley graph Γ_G are precisely the cutsets of Γ_G* . We also note the reader can find the definitions of *cobase*, *corank*, *co-independent* of M in [5], [6] and [8].

Proof of Theorem 1.3. Suppose that Γ_G^* is an abstract-dual of the Cayley graph Γ_G . Then, by the definition, there is a one to one correspondence between the edges of Γ_G and the edges of Γ_G^* with the property that circuits in Γ_G correspond to cutsets in Γ_G^* and conversely. It follows from this that the circuits of $M(\Gamma_G)$ correspond to the cocircuits of $M(\Gamma_G^*)$. Thus, by Theorem 1.2, $M(\Gamma_G^*)$ is isomorphic to $M^*(\Gamma_G)$, as required.

3 Applications

The easiest application of the results in Theorems 1.2 and 1.3 is the following.

Corollary 3.1. $M(\Gamma_G^*) \cong M^*(\Gamma_G)$.

In view of Theorem 1.2 and by the definition of the term cocircuit we will prove the analogue for matroids, are obtained from a Cayley graph Γ_G , of Lemma 2.4. Thus, let M be a matroid obtained from Γ_G . We then have

Corollary 3.2. *Every cocircuit of M intersects every base.*

Proof. Let C^* be cocircuit of M , and suppose that there exists a base B of M with the property that $C^* \cap B$ is empty. Then C^* is contained in $E - B$, and so C^* is a circuit of the dual matroid of M which is contained in a base of this dual matroid. But this makes a contradiction the result.

By applying the above result to the dual matroid of M , we get the following corollary.

Corollary 3.3. *Every circuit of a matroid intersects every cobase.*

The details of the definitions, given in the following two paragraphs, can be found, for instance, in [4], [5], [6], [7] and [8].

A Cayley graph Γ_G is called *plane Cayley graph* if it is drawn in the plane in such a way that no two edges (or the curves representing) intersect geometrically except at a vertex to which they are both incident. Moreover a *planar Cayley graph* is one which is isomorphic to a plane Cayley graph. In other words a Cayley graph is planar if it can be embedded in the plane, and that any such embedding is a plane Cayley graph. For example, in Figure 3, all three graphs are planar, but only the second and third are plane.

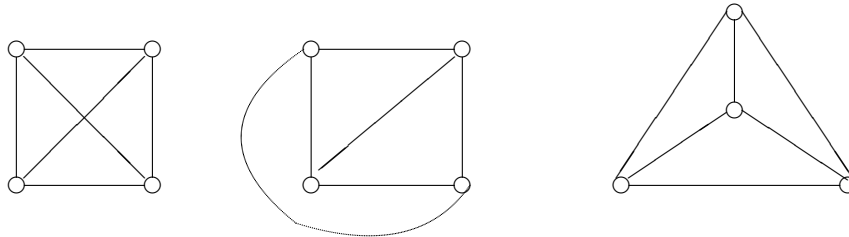


Figure 3

Given a plane Cayley graph Γ_G , we can define another graph $\Gamma_G^{G^*}$ called the *geometric dual* of Γ_G . The construction is in two steps:

- (i) inside each face F_i of Γ_G we choose a point $v_i^{G^*}$. These points are the vertices of $\Gamma_G^{G^*}$;
- (ii) by corresponding to each edge e of Γ_G , we draw a line e^{G^*} which crosses

e but no other edge of Γ_G , and join the vertices $v_i^{G^*}$ which lie in the faces F_i adjoining e . These lines are the edges of $\Gamma_G^{G^*}$.

We remark that, by [1], the Cayley graph Γ_G is connected.

After these above paragraphs, we can get the following result as a quick consequence of Theorem 1.3.

Corollary 3.4. *If $\Gamma_G^{G^*}$ is a geometric-dual of a connected planar Cayley graph Γ_G , then $M(\Gamma_G^{G^*})$ is isomorphic to $(M(\Gamma_G))^{G^*}$.*

We note that a planar Cayley graph can have several different duals, whereas a matroid can have only one. The reason for this is if $\Gamma_G^{G^*1}$ and $\Gamma_G^{G^*2}$ are two duals of Γ_G , which are non-isomorphic, then the circuit matroids of $\Gamma_G^{G^*1}$ and $\Gamma_G^{G^*2}$ are isomorphic matroids.

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Sinan Çevik

Selçuk University, Department of Mathematics,
Faculty of Science, Alaaddin Keykubat Campus, Konya Turkey,
sinan.cevik@yahoo.com

Naci Cangül

Uludag University, Faculty of Science and Art,
Department of Mathematics, Gorukle Campus, Bursa Turkey,
cangul@uludag.edu.tr