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Some results on sub classes of univalent functions of complex order [1]

T. Ram Reddy, P. Thirupathi Reddy

Abstract

In this paper we introduce the class $R_\lambda^b\ (A,B)$ ($b \neq 0$ complex), of function of the form $f\ (z)$ analytic in the disc $E = \{\ z\colon\ |z|< 1\}$, such that

$$1+\frac{1}{b}\left\{f'(z)-1\right\}\ \prec (1-\lambda)\frac{1+Az}{1+Bz}+\lambda, z\in E.$$

where A and B are fixed numbers, -1 = $B < A = 1$, $0 = \lambda < 1$ and $\prec$ denotes subordination. We determine sharp coefficient estimates, sufficient condition in terms of coefficients, distortion theorem, maximization theorem for the class $R_\lambda^b\ (A,B)$.

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1 Introduction

Let U denote the class of function

$$(1) \qquad w\,(z) \;=\; \sum_{n=1}^{\infty} b_n\, z^n$$

which are analytic in the unit disc $E = \{z\colon\ |z| < 1\}$ and satisfying the condition w(0) = 0 and |w(z)|< 1.

The present paper is devoted to a unified study of various subclasses of univalent functions. For this purpose we introduce the new class R_λ^b (A, B) of functions of the form

$$(2) \qquad f\,(z) \;=\; \mathrm{z} + \sum_{n=2}^{\infty} a_n\, z^n$$

analytic in E and satisfying the condition

$$(3) \qquad \left|\frac{f'\,(z) - 1}{b\,(A-B)\ \ (1{-}\lambda) - B\,(f'\,(z) - 1)}\right| < 1$$

where A and B are fixed numbers with -1 $= B < A = 1$, $0 = \lambda < 1$, b is non-zero complex number. A function f (z) is in R_λ^b (A, B) if and only if there exists a schwarz function w (z) analytic in E and satisfying w (0) = 0 and |w (z)|< 1, such that

$$1 + \frac{1}{b}\,\{f'(z) - 1\} \;= (1-\lambda)\frac{1 + Aw(z)}{1 + Bw(z)} + \lambda, z \in E.$$

By giving specific values of λ, A, B and b in (3) we obtain the following important classes studied by various researchers in their earlier works.

(i) For b = 1, A = δ, B = - δ and λ = 0 we obtain the class of functions f (z) satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \delta, \qquad z \in E.$$

studied by Caplinger and Causey[1] and Padmanabhan[6].

(ii) For b=1, λ=0 we obtain the class of functions f (z) satisfying the condition

$$\left|\frac{f'(z)-1}{Bf'(z)-A}\right| < 1, \qquad z \in E$$

studied by Goel and Mehrok[4].

(iii) For b = e-$i\alpha$ cos α and $\lambda = 0$ we obtain the class of functions f (z) satisfying the condition

$$\left|\frac{e^{i\alpha}\left\{f'(z)-1\right\}}{Be^{i\alpha}f'(z)-(A\cos\alpha+iB\sin\alpha)}\right| < 1, \quad z \in E.$$

studied by Dashrath[3].

In this paper we obtain coefficient estimates, sufficient condition in terms of coefficients, distortion theorem and maximization theorem.

Now we state a lemma due to Keogh and Merks[5].

Lemma 1 *Let w(z)* $=\sum_{k=1}^{\infty} b_k\, z^k$ *be analytic with* $|w\,(z)| < 1$ *in E. If s is any complex number then.*

$$\left|b_2 - sb_1^2\right| \;\le\; \max\ (1,\ |s|\ ).$$

Equality may be attained with functions w(z) = z2 and w (z) = z.

Theorem 1 *If f (z) is in* R_λ^b (A, B) *then*

$$|a_n| \leq \frac{(A-B)(1-\lambda)|b|}{n} \tag{4}$$

The estimates are sharp.

Proof. Since $f \in R_\lambda^b$ (A, B), we have

$$1+\frac{1}{b}\{f'(z)-1\}=(1-\lambda)\frac{1+Aw(z)}{1+Bw(z)}+\lambda \tag{5}$$

for some schwarz function *w(z)* in E for $z \in E$. From (5) we have

$$\left[(B-A)(1-\lambda)+\left(\frac{B}{b}\right)\sum_{n=2}^{\infty} na_n z^{n-1}\right] w\left(z\right)=-\frac{1}{b}\sum_{n=2}^{\infty} na_n z^{n-1}$$

that is

$$\left[(A-B)(1-\lambda)-\left(\frac{B}{b}\right)\sum_{n=2}^{\infty} na_n z^{n-1}\right]\left[\sum_{n=1}^{\infty} b_n z^n\right]=\frac{1}{b}\sum_{n=2}^{\infty} na_n z^{n-1} \tag{6}$$

Equating corresponding coefficients in (6) we observe that the coefficient *an* on the right of (6) depends only on $a_2, a_3, a_4, \ldots..a_{n-1}$on the left side of (6). Hence for $n \geq 2$ it follows from (6) that

$$\left[(A-B)(1-\lambda)-\left(\frac{B}{b}\right)\sum_{n=2}^{k-1} na_n z^{n-1}\right] w\left(z\right)=\frac{1}{b}\left[\sum_{n=2}^{k} na_n z^{n-1}+\sum_{n=k+1}^{\infty} na_n z^{n-1}\right]$$

This yields

(7)

$$\left|(A-B)(1-\lambda)-\left(\frac{B}{b}\right)\sum_{n=2}^{k-1} na_n z^{n-1}\right| \geq \left|\frac{1}{b}\left[\sum_{n=2}^{k} na_n z^{n-1}+\sum_{n=k+1}^{\infty} na_n z^{n-1}\right]\right|$$

Squaring both sides of (7) and integrating on the circle $|z|=r$, *(0<r<1)* we obtain

$$(A-B)^2(1-\lambda)^2+\frac{B^2}{|b^2|}\sum_{n=2}^{k-1} n^2|a_n|^2 r^{2n-2}$$

$$\geq \frac{1}{|b|^2}\left[\sum_{n=2}^{k} n^2|a_n|^2 r^{2n-2}+\sum_{n=k+1}^{\infty} n^2|a_n|^2 r^{2n-2}\right]$$

and letting r 1 we get

$$(A-B)^2(1-\lambda)^2+\frac{B^2}{|b|^2}\sum_{n=2}^{k-1} n^2|a_n|^2 \geq \frac{1}{|b|^2}\sum_{n=2}^{k} n^2|a_n|^2$$

or

$$(1-B)^2\sum_{n=2}^{\infty} n^2|a_n|^2+n^2|a_n|^2 \ \leq (A-B)^2(1-\lambda)^2|b|^2 \qquad (8)$$

since $-1 \leq B < 1$ we obtain from (8)

$$n^2|a_n|^2 \leq (A-B)^2(1-\lambda)^2|b|^2$$

This gives

$$|a_n| \leq \frac{(A-B)(1-\lambda)|b|}{n}, \qquad \text{n} = 2, 3.........$$

The sharpness of the result follows for the function

$$f(z)=\int_0^z \left[1+\frac{(A-B)(1-\lambda)bz^{n-1}}{1+B}\right] dz$$

for $n \geq 2$ and $z \in E$.

Theorem 2 *Let f (z) be analytic in E. If for some A, B, -1 = B < A = 1,*

$$\sum_{n=2}^{\infty} (1 + |B|)\, n\, |a_n| \leq (A - B)(1 - \lambda)|b| \tag{9}$$

Then $f \in R_\lambda^b$ (A, B). *The result is sharp.*

Proof. Suppose that condition (9) holds then for |z |< 1,

$$|f'(z) - 1| - |b\,(\mathrm{A} - \mathrm{B})\,(1{-}\lambda) - B\,\{f'(z) - 1\}|$$

$$= \left|\sum_{n=2}^{\infty} n\, a_n\, z^{n-1}\right| - \left|b\,(\mathrm{A} - \mathrm{B})\,(1 - \lambda) - B \sum_{n=2}^{\infty} n\, a_n\, z^{n-1}\right|$$

$$\leq \sum_{n=2}^{\infty} n\, |a_n|\, r^{n-1} - |b|\,(\mathrm{A} - \mathrm{B})\,(1{-}\lambda) + |\mathrm{B}| \sum_{n=2}^{\infty} n\, |a_n|\, r^{n-1}$$

$$< \sum_{n=2}^{\infty} n\, |a_n| - |b|\,(\mathrm{A} - \mathrm{B})\,(1{-}\lambda) + |\mathrm{B}| \sum_{n=2}^{\infty} n\, |a_n|$$

$$= \sum_{n=2}^{\infty} n\,(1 + |\mathrm{B}|)\,|a_n| - (\mathrm{A} - \mathrm{B})\,(1{-}\lambda)\,|b| = 0, \text{by (9)}.$$

Hence it follows that

$$\left|\frac{f'(z) - 1}{b\,(\mathrm{A} - \mathrm{B})\,(1{-}\lambda) - \mathrm{B}\,(f'(z) - 1)}\right| < 1, \ \mathrm{z} \in E.$$

Therefore $f \in \mathrm{R}_\lambda^b$ (A, B). The result is sharp for the function

$f(z) = z + \frac{(\mathrm{A}-\mathrm{B})\,(1-\lambda) b}{(1+|\mathrm{B}|)\, n} z^n$, for n = 2 and $z \in E$.

Theorem 3 *If f (z)* $\in \mathrm{R}_\lambda^b$ (A, B) *then*

$$\mathrm{Re}\, f'(z) \geq \frac{1-(1-\lambda)\ \mathrm{AB}r^2\, \mathrm{Re}\,(b) - \mathrm{B}^2 r^2.\mathrm{Re}\,(1-(1-\lambda)\, b) - (\mathrm{A} - \mathrm{B})\ |b|\ r}{1 - \mathrm{B}^2 r^2}$$

and

$$\mathrm{Re}\, f'(z) \leq \frac{1-(1-\lambda)\ \mathrm{AB}r^2\, \mathrm{Re}\,(b) - \mathrm{B}^2 r^2.\mathrm{Re}\,(1-(1-\lambda)\, b) + (\mathrm{A} - \mathrm{B})\ |b|\ r}{1 - \mathrm{B}^2 r^2}$$

The bounds are sharp.

Proof. Since $f \in R^{b}_{\lambda}(A, B)$, we have

$$1 + \frac{1}{b}\{f'(z) - 1\} = (1-\lambda)\frac{1 + Aw(z)}{1 + Bw(z)} + \lambda = p(z) \tag{10}$$

It is known that the images of the closed disk $|z| = r$ under the transformations

$$p(z) = (1-\lambda)\frac{1 + Aw(z)}{1 + Bw(z)} + \lambda$$

are contained in the closed disk with centre C and radius D, where

$$C = \frac{1-(1-\lambda)\ ABbr^2 + ((1-\lambda)b - 1)B^2r^2}{1 - B^2r^2}$$

and

$$D = \frac{(A-B)\ (1-\lambda)\ |b|\ r}{1 - B^2r^2}$$

Thus we have

$$\left|p(z) - \frac{\left(1 - \lambda\, B^2r^2\right) - (1-\lambda)\ ABr^2}{1 - B^2r^2}\right| \leq \frac{(A-B)(1-\lambda)\ r}{1 - B^2r^2} \tag{11}$$

Equation (10) and (11) yields

$$\left|f'(z) - \frac{1-(1-\lambda)\ ABbr^2 + ((1-\lambda)b - 1)B^2r^2}{1 - B^2r^2}\right| \leq \frac{(A-B)\ (1-\lambda)\ |b|\ r}{1 - B^2r^2}$$

Hence

$$\mathrm{Re}f'(z) \geq \frac{1-(1-\lambda)\ ABr^2\ \mathrm{Re}(b) - B^2r^2\ \mathrm{Re}\ (1-(1-\lambda)\ b) - (A-B)\ (1-\lambda)\ |b|\ r}{1-B^2r^2}$$

and

$$\mathrm{Re}f'(z) \leq \frac{1-(1-\lambda)\ ABr^2\,\mathrm{Re}(b) - B^2r^2\,\mathrm{Re}\ (1-(1-\lambda)\ b) + (A-B)\ (1-\lambda)\ |b|\ r}{1-B^2r^2}$$

Equalities is attained for the function

$$f(z) = \frac{B + (A-B)\ (1-\lambda)\ b}{B}\, z - \frac{(A-B)\ (1-\lambda)\ b}{B^2\ e^{i\gamma}} \log\left(1 + Bze^{i\gamma}\right)$$

where

$$e^{i\gamma} = \frac{|b| \ - \mathrm{B}zb}{b - \ \mathrm{B}z\,|b|}$$

Theorem 4 *If $f\,(z) \in \mathrm{R}^b_\lambda\,(\mathrm{A},\mathrm{B})$ and μ is any complex number then*

(12)

$$\left|a_3 - \mu\, a_2^2\right| \ \le \frac{|b| \ (\mathrm{A}-\mathrm{B}) \ (1{-}\lambda)}{3} \max \left\{1, \frac{|4\mathrm{B} \ + \ 3\mu b \ (\mathrm{A}-\mathrm{B}) \ (1-\lambda)|}{4}\right\}$$

The result is sharp.

Proof. Since $f \in \mathrm{R}^{\mathrm{b}}_\lambda\,(\mathrm{A},\mathrm{B})$, we have

(13) $$1 + \frac{1}{b}\{f'\,(z) \ -1\} = (1-\lambda)\,\frac{1+\mathrm{A}w\,(z)}{1+\mathrm{B}w\,(z)} \ + \lambda$$

where $w\,(z) = \sum\limits_{k=1}^{\infty} \ b_k z^k$ is regular in E and satisfies the condition $w(0) = 0$ $|w \ (z)|{<}1$ for $z \in E$.

From (13) we have

$$w\,(z) = \frac{f'\,(z) - 1}{b\,(\mathrm{A}-\mathrm{B}) \ \ (1{-}\lambda) - \mathrm{B}\,\{f'\,(z) - 1\}}$$

$$= \frac{\sum\limits_{n=2}^{\infty} \ na_n z^{n-1}}{b\,(\mathrm{A}-\mathrm{B}) \ \ (1{-}\lambda) - \mathrm{B} \sum\limits_{n=2}^{\infty} na_n z^{n-1}}$$

$$= \frac{\sum\limits_{n=2}^{\infty} \ na_n z^{n-1}}{b\,(\mathrm{A}-\mathrm{B}) \ \ (1{-}\lambda)}\left[1 + \frac{\mathrm{B}}{b \ (\mathrm{A}-\mathrm{B}) \ \ (1{-}\lambda)} \sum\limits_{n=2}^{\infty} \ na_n z^{n-1} + ...\right]$$

and then comparing the coefficients of z and $z2$ on both sides, we have

$$b_1 = \frac{2a_2}{b \ (\mathrm{A}-\mathrm{B}) \ \ (1{-}\lambda)}$$

$$b_2 = \frac{1}{(1-\lambda)\,(\mathrm{A}-\mathrm{B}) \ b}\left[3a_3 + \frac{4\mathrm{B}\ a_2^2}{(1-\lambda) \ \ (\mathrm{A}-\mathrm{B}) \ b}\right]$$

Thus

$$a_2 = \frac{b\,(\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)\ \ b_1}{2}$$

and

$$a_3 = \frac{b\,(1-\lambda)\ \ (\mathrm{A}-\mathrm{B})\ \ b_2}{3} \ - \frac{4\mathrm{B}\ a_2^2}{3b\,(\mathrm{A}-\mathrm{B})\ \ (1-\lambda)}$$

Hence

$$a_3 - \mu a_2^2 = \frac{b\,(\mathrm{A}-\mathrm{B})\ \ (1-\lambda)}{3}\left[b_2 - \left\{\mathrm{B}\ +\ \frac{3\mu b\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)}{4}\right\} b_1^2\right]$$

Therefore

(14)

$$\left|a_3 - \mu a_2^2\right| = \frac{|b|\,(\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)}{3}\ \left[\left|b_2 - \left\{\ \frac{4\mathrm{B} + 3\mu b\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)}{4}\right\} b_1^2\right|\right]$$

Using Lemma 1 in (14) we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)}{3}\ \max\ \left\{1,\ \frac{|4\mathrm{B} + 3\mu b\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)|}{4}\right\}$$

which is (12) of Theorem 4.

If $\left|\dfrac{4\mathrm{B} + 3b\mu\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)}{4}\right| > 1$ then we choose the function

$$f\,(z) = \frac{\mathrm{B}\ +\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)\,b}{\mathrm{B}}\ z - \frac{(\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)\ b}{\mathrm{B}^2}\log\,(1+\mathrm{B}z)$$

and if $\left|\dfrac{4\mathrm{B}\ + 3b\mu\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)}{4}\right| \ <\ 1$, then we choose the function

$$f\,(z)\ \ = \frac{\mathrm{B}\ +\ \ (\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)\,b}{\mathrm{B}}\ z - \frac{(\mathrm{A}-\mathrm{B})\ \ (1{-}\lambda)\ b}{\mathrm{B}}\ \ \int_{\mathrm{o}}^{\mathrm{z}} \frac{dt}{1+\mathrm{B}t^2}$$

for attaining the equality sign in (12)

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T. Ram Reddy, P. Thirupathi Reddy
Department of Mathematics
Kakatiya University
Warangal–506 009 (A.P) India
e-mail: reddypt2@yahoo.com