General Mathematics Vol. 17, No. 3 (2009), 135–139

On a problem in the theory of univalent functions [1]

Sukhwinder Singh, Sushma Gupta and Sukhjit Singh

Abstract

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0)-1 = 0$. In the present note, we prove that $f \in \mathcal{A}$, satisfying the differential inequality

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < \beta,\ z \in \mathbb{E},$$

implies that $\Re\ f'(z) > 0,\ z \in \mathbb{E}$, for all real numbers α and β satisfying $\alpha \geq \beta > 1$ and hence f is univalent.

2000 Mathematics Subject Classification: 30C45, 30C50.

Key words and phrases: Analytic function, univalent function.

[1] *Received 25 September, 2008*

Accepted for publication (in revised form) 22 October, 2008

1 Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

For real number α, let

$$I(\alpha, f(z)) = (1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

It is well-known that any close-to-convex function is univalent.

In 1934/35, Noshiro [3] and Warchawski [5] obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies $\Re\, f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

Al-Amiri and Reade [1], in 1975, have shown that for $\alpha \leq 0$ and also for $\alpha = 1$, the functions $f \in \mathcal{A}$, satisfying the differential inequality $\Re\,[I(\alpha, f(z))] > 0,\ z \in \mathbb{E}$, are univalent in $\mathbb{E}$.

In 2005, Singh, Singh and Gupta [4] proved that for $0 < \alpha < 1$, functions $f \in \mathcal{A}$, satisfying the differential inequality $\Re\,[I(\alpha, f(z))] > \alpha,\ z \in \mathbb{E}$, are univalent in $\mathbb{E}$.

The univalence of the above problem is still open for $\alpha > 1$.

In the present note, we prove, if $f \in \mathcal{A}$ satisfies the differential inequality $\Re\,[I(\alpha, f(z))] < \beta,\ z \in \mathbb{E}$, then $\Re\, f'(z) > 0,\ z \in \mathbb{E}$, for all real numbers α and β satisfying $\alpha \geq \beta > 1$ and hence f is univalent.

We use the following celebrated lemma of Miller to prove our result.

Lemma 1 *([2]). Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$*

(u_1, u_2, v_1, v_2 are reals), let ϕ satisfy the following conditions:

(i) ϕ is continuous in $\mathbb{D}$;

(ii) $(1,0) \in \mathbb{D}$ and $\Re\ \phi(1,0) > 0$; and

(iii) $\Re\ \phi(iu_2, v_1) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1+u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disc E, such that $(p(z), zp'(z)) \in \mathbb{D}$, for all $z \in \mathbb{E}$. If $\Re\ [\phi(p(z), zp'(z))] > 0,\ z \in \mathbb{E}$, then $\Re\ p(z) > 0,\ z \in \mathbb{E}$.

2 Main Result

Theorem 1 *Let α and β be real numbers such that $\alpha \geq \beta > 1$. Assume that an analytic function $f \in \mathcal{A}$ satisfies*

(1) $$\Re\ [I(\alpha, f(z))] < \beta,\ z \in \mathbb{E}.$$

Then $\Re\ f'(z) > 0$ in $\mathbb{E}$. So, f is close to convex and hence univalent in $\mathbb{E}$.

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in $\mathbb{E}$ such that,

(2) $$f'(z) = p(z),\ z \in \mathbb{E}.$$

Then,

$$I(\alpha, f(z)) = (1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = (1-\alpha)p(z) + \alpha\left(1 + \frac{zp'(z)}{p(z)}\right).$$

Thus, condition (1) is equivalent to

(3) $$\Re\left(\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha-\beta}{1-\beta}\right) > 0,\ z \in \mathbb{E}.$$

If $\mathbb{D} = (\mathbb{C}\backslash\{0\}) \times \mathbb{C}$, define $\phi(u, v) : \mathbb{D} \to \mathbb{C}$ as

$$\phi(u,v) = \frac{1-\alpha}{1-\beta}u + \frac{\alpha}{1-\beta}\frac{v}{u} + \frac{\alpha-\beta}{1-\beta}.$$

Then ϕ is continuous in $\mathbb{D}$, $(1, 0) \in \mathbb{D}$ and $\Re\, \phi(1, 0) = 1 > 0$. Further, in view of (3), we get $\Re\, \phi(p(z), zp'(z)) > 0,\ z \in \mathbb{E}$. Let $u = u_1+iu_2, v = v_1+iv_2$ where u_1, u_2, v_1 and v_2 are all reals. Then, for $(iu_2, v_1) \in \mathbb{D}$, we have

$$\begin{aligned} \Re\, \phi(iu_2, v_1) &= \Re\left(\frac{1-\alpha}{1-\beta}u_2 i + \frac{\alpha}{1-\beta}\frac{v_1}{u_2 i} + \frac{\alpha-\beta}{1-\beta}\right) \\ &= \frac{\alpha-\beta}{1-\beta} \\ &\leq 0. \end{aligned}$$

In view of (2) and Lemma 1, proof now follows.

References

[1] Al-Amiri, H. S. and Reade, M. O., *On a linear combination of some expressions in the theory of univalent functions*, Monatshefto für mathematik, 80(1975), 257-264.

[2] Miller, S. S., *Differential Inequalities and Carathéodory functions*, Bull. Amer. Math. Soc., 81(1975), 79-81.

[3] Noshiro, K., *On the theory of schlicht functions*, J. Fac. Sci., Hokkaido Univ., 2(1934-35), 129-155.

[4] Singh, V., Singh, S. and Gupta, S., *A problem in the theory of univalent functions*, Integral Transforms and Special Functions, 16, 2(2005), 179-186.

[5] Warchawski, S. E., *On the higher derivatives at the boundary in conformal mappings*, Trans. Amer. Math. Soc., 38(1935), 310-340.

Sukhwinder Singh
Deaprtment of Applied Sciences
B.B.S.B.Engineering College
Fatehgarh Sahib-140 407 (Punjab) India
e-mail: ss_billing@yahoo.co.in

Sushma Gupta and Sukhjit Singh
Department of Mathematics
S.L.I.E.T. Longowal-148 106 (Punjab) India
e-mail: sushmagupta1@yahoo.com
e-mail: sukhjit_d@yahoo.com