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Certain Sufficient Conditions For Univalence

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Abstract

We introduce the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,\delta}$ for analytic functions f in the open unit disk $\mathcal{U}$, sufficient conditions for univalence of this integral operator are discussed.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

For $f \in \mathcal{A}$, the integral operator G_α is defined by

$$G_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du \tag{1}$$

for some complex numbers $\alpha (\alpha \neq 0)$.

In [2] Kim-Merkes prove that the integral operator G_α is in the class $\mathcal{S}$ for $\frac{1}{|\alpha|} \leq \frac{1}{4}$ and $f \in \mathcal{S}$.

Also, the integral operator M_γ for $f \in \mathcal{A}$ is given by

$$M_\gamma(z) = \left\{\frac{1}{\gamma}\int_0^z u^{-1}\left(f(u)\right)^{\frac{1}{\gamma}} du\right\}^\gamma \tag{2}$$

γ be a complex number, $\gamma \neq 0$.

Miller and Mocanu [4] have studied that the integral operator M_γ is in the class S for $f \in \mathcal{S}^*, \gamma > 0, \mathcal{S}^*$ is the subclass of $\mathcal{S}$ consisting of all starlike functions f in $\mathcal{U}$.

Pescar in [9] define a general integral operator

$$J_{\gamma_1,\gamma_2\ldots,\gamma_n}(z) =$$

$$= \left\{\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_n}\right)\int_0^z u^{-1}(f_1(u))^{\frac{1}{\gamma_1}}\ldots(f_n(u))^{\frac{1}{\gamma_n}} du\right\}^{\frac{1}{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\ldots+\frac{1}{\gamma_n}}} \tag{3}$$

for $f_j \in \mathcal{A}$ and complex numbers $\gamma_j, (\gamma_j \neq 0), j = \overline{1,n}$, which is a generalization of integral operator M_γ.

For $n = 1, f_1 = f$ and $\gamma_1 = \gamma$, from (3) we obtain the integral operator M_γ.

We introduce the general integral operator

$$H_{\gamma_1,\gamma_2\ldots,\gamma_n,\beta,\delta}(z) = \left\{\delta\beta\int_0^z u^{\delta\beta-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\ldots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}} du\right\}^{\frac{1}{\delta\beta}} \tag{4}$$

for $f_j \in \mathcal{A}$, γ_j complex numbers, $\gamma_j \neq 0$, $j = \overline{1,n}$, δ, β complex numbers, $\delta \neq 0$, $\beta \neq 0$, $n \in \mathbb{N} - \{0\}$.

From (4), for $\beta = \frac{1}{\delta}\sum_{j=1}^{n}\frac{1}{\gamma_j}$ we obtain the integral operator $J_{\gamma_1,\gamma_2...,\gamma_n}$ defined by (3), for $n = 1$, $\delta = 1$, $\beta = 1$, $\gamma_1 = \alpha$ and $f_1 = f$ we have the integral operator G_α given by (1).

If in (4) we take $n = 1$, $\gamma_1 = \gamma$, $\beta = \frac{1}{\gamma}$, $\delta = 1$ and $f_1 = f$ we obtain the integral operator M_γ.

For $n = 1, \gamma_1 = \alpha, \delta = 1$ and $f_1 = f$ from (4) we obtain the integral operator $T_{\alpha,\beta}$ defined in [8] by

$$(5) \qquad T_{\alpha,\beta} = \left[\beta \int_0^z u^{\beta-1}\left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}}$$

for $f \in \mathcal{A}$ and α, β be complex numbers, $\alpha \neq 0$, $\beta \neq 0$.

For $\delta\beta = 1$ we have the integral operator given in [1] and for $\delta = 1$, $\gamma_1 = \gamma_2 = \ldots = \gamma_n = \alpha$ we obtain the integral operator defined in [1].

2 Preliminary results

We need the following lemmas.

Lemma 1 *[7]. Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$(6) \qquad \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \geq Re\ \alpha$ the function

$$(7) \qquad F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du\right]^{\frac{1}{\beta}}$$

is in the class S.

Lemma 2 *(Schwarz [3]). Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ z \in \mathcal{U}_R \tag{8}$$

the equality (in the inequality (8) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3 Main results

Theorem 1 *Let γ_j, α complex numbers, $j = \overline{1,n}$, $a = Re\alpha > 0$ and $f_j \in \mathcal{A}$, $f_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \dots$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2n}|\gamma_j|,\ \ j = \overline{1,n}, \tag{9}$$

for all $z \in \mathcal{U}$, then for any complex numbers β and δ, $Re\ \delta\beta \geq a$, the function

$$H_{\gamma_1,\gamma_2,\dots,\gamma_n,\beta,\delta}(z) = \left\{\delta\beta\int_0^z u^{\delta\beta-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\dots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}}du\right\}^{\frac{1}{\delta\beta}} \tag{10}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z)=\int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\cdots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}}du \tag{11}$$

The function g is regular in $\mathcal{U}$. We define the function $p(z)=\frac{zg''(z)}{g'(z)}$, $z\in\mathcal{U}$ and we obtain

$$p(z)=\frac{zg''(z)}{g'(z)}=\sum_{j=1}^{n}\left[\frac{1}{\gamma_j}\left(\frac{zf_j'(z)}{f_j(z)}-1\right)\right],\quad z\in\mathcal{U} \tag{12}$$

From (9) and (12) we have

$$|p(z)|\le\frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{13}$$

for all $z\in\mathcal{U}$ and applying Lemma 2 we get

$$|p(z)|\le\frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|z|,\quad z\in\mathcal{U} \tag{14}$$

From (12) and (14) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{(1-|z|^{2a})|z|}{a}\cdot\frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{15}$$

for all $z\in\mathcal{U}$.

Because

$$\max_{|z|\le 1}\frac{(1-|z|^{2a})|z|}{a}=\frac{2}{(2a+1)^{\frac{2a+1}{2a}}},$$

from (15) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le 1 \tag{16}$$

for all $z\in\mathcal{U}$. So, by the Lemma 1, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,\delta}$ belongs to class $\mathcal{S}$.

Corollary 1 *Let* γ_j*,* α *complex numbers,* $j=\overline{1,n}$*,* $a = Re\alpha > 0$*,* $\sum\limits_{j=1}^{n} Re\frac{1}{\gamma_j} \geq Re\alpha$ *and* $f_j \in \mathcal{A},\ f_j(z) = z + b_{2j}z^2 + \ldots,\ j=\overline{1,n},\ n \in \mathbb{N} - \{0,1\}$.

If

$$\left|\frac{zf_j'(z)}{f_j} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2n}|\gamma_j|,\quad j=\overline{1,n} \tag{17}$$

for all $z \in \mathcal{U}$*, then the function*

$$J_{\gamma_1,\gamma_2,\ldots\gamma_n}(z) = \left\{\left(\sum_{j=1}^{n}\frac{1}{\gamma_j}\right)\int_0^z u^{-1}\left(f_1(u)\right)^{\frac{1}{\gamma_1}} \ldots \left(f_n(u)\right)^{\frac{1}{\gamma_n}} du\right\}^{\frac{1}{\sum_{j=1}^{n}\frac{1}{\gamma_j}}} \tag{18}$$

is in the class $\mathcal{S}$.

Proof. For $\beta = \frac{1}{\delta}\sum_{j=1}^{n}\frac{1}{\gamma_j}$, from Theorem 1 we obtain Corollary 1.

Corollary 2 *Let* γ*,* α *complex numbers,* $a = Re\alpha > 0$*,* $Re\frac{1}{\gamma} \geq Re\alpha$ *and* $f \in \mathcal{A}, f(z) = z + b_{21}z^2 + b_{31}z^3 + \ldots$.

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\gamma| \tag{19}$$

for all $z \in \mathcal{U}$*, then the integral operator* M_γ *define by (2) belongs to the class* $\mathcal{S}$.

Proof. We take $n=1, \delta=1, \beta=\frac{1}{\gamma}, \gamma_1=\gamma, f_1=f$ in Theorem 1.

Corollary 3 *Let* γ_j*,* α *complex numbers,* $j=\overline{1,n}$*,* $a = Re\alpha$*,* $a \in (0,1]$ *and* $f_j \in \mathcal{A},\ f_j(z) = z + b_{2j}z^2 + \ldots,\ j=\overline{1,n},\ n \in \mathbb{N} - \{0\}$.

If

$$\left| \frac{z f_j'(z)}{f_j(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2n} |\gamma_j|, \quad j = \overline{1,n} \tag{20}$$

for all $z \in \mathcal{U}$, then the function

$$K_{\gamma_1,\gamma_2,\ldots\gamma_n}(z) = \int_0^z \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\gamma_n}} du \tag{21}$$

belongs to class $\mathcal{S}$.

Proof. We take $\beta = \frac{1}{\delta}$ in Theorem 1.

Corollary 4 *Let α, γ complex numbers $a = Re\gamma > 0$, $n \in \mathbb{N} - \{0\}$ and $f_j \in \mathcal{A}$, $f_j(z) = z + b_{2j} z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left| \frac{z f_j'(z)}{f_j(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2n} |\alpha|, \quad j = \overline{1,n} \tag{22}$$

for all $z \in \mathcal{U}$, then for any complex number β with $Re\ \beta \geq Re\gamma$ the function

$$L_{\alpha,\beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\alpha}} \ldots \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\alpha}} du \right\}^{\frac{1}{\beta}} \tag{23}$$

is in the class $\mathcal{S}$.

Proof. For $\delta = 1$ and $\gamma_1 = \gamma_2 = \ldots = \gamma_n = \alpha$ in Theorem 3.1. we have the Corollary 4.

Theorem 2 *Let γ_j, α complex numbers, $j=\overline{1,n}$, $a = Re\alpha > 0$ and $f_j \in \mathcal{S}$, $f_j(z) = z + \sum_{k=2}^{\infty} b_{kj} z^k$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n} \frac{1}{|\gamma_j|} \leq \frac{a}{2}, \quad for \quad 0 < a < \frac{1}{2} \tag{24}$$

or

$$\sum_{j=1}^{n} \frac{1}{|\gamma_j|} \leq \frac{1}{4}, \quad for \quad a \geq \frac{1}{2} \tag{25}$$

then for any complex numbers β, δ, $Re\ \delta\beta \geq a$, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,\delta}$ given by (4) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}} du \tag{26}$$

The function g is regular in $\mathcal{U}$. We have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\sum_{j=1}^{n}\left[\frac{1}{|\gamma_j|}\left|\frac{zf_j'(z)}{f_j(z)} - 1\right|\right] \tag{27}$$

Because $f_j \in \mathcal{S}$, $j = \overline{1,n}$ we have

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathcal{U}, \quad j = \overline{1,n} \tag{28}$$

From (27) and (28) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\frac{2}{1-|z|}\sum_{j=1}^{n}\frac{1}{|\gamma_j|} \tag{29}$$

for all $z \in \mathcal{U}$.

For $0 < a < \frac{1}{2}$ we have

$$\max_{|z|\le 1} \frac{1-|z|^{2a}}{1-|z|} = 1$$

and from (24), (29) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le 1, \quad z \in \mathcal{U} \tag{30}$$

For $a \ge \frac{1}{2}$ we have

$$\max_{|z|\le 1} \frac{1-|z|^{2a}}{1-|z|} = 2a$$

and from (25), (29) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le 1, \quad z \in \mathcal{U} \tag{31}$$

From (30), (31) and Lemma 1 it results that the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,\delta}$ belongs to class $\mathcal{S}$.

Corollary 5 *Let γ_j, α complex numbers, $j=\overline{1,n}$, $a = Re\alpha > 0$, $\sum\limits_{j=1}^{n} Re\frac{1}{\gamma_j} \ge Re\alpha$ and $f_j \in \mathcal{S}$, $f_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n} \frac{1}{|\gamma_j|} \le \frac{a}{2}, \quad \textit{for} \quad 0 < a < \frac{1}{2} \tag{32}$$

or

$$\sum_{j=1}^{\delta} \frac{1}{|\gamma_j|} \le \frac{1}{4}, \quad \textit{for} \quad a \ge \frac{1}{2} \tag{33}$$

then the integral operator $J_{\gamma_1,\gamma_2,\ldots\gamma_n}$ given by (3) is in the class $\mathcal{S}$.

Proof. For $\beta = \frac{1}{\delta}\sum_{j=1}^{n}\frac{1}{\gamma_j}$, from Theorem 2 we obtain that the integral operator $J_{\gamma_1,\gamma_2,\ldots\gamma_n}$ is in the class $\mathcal{S}$.

Corollary 6 *Let γ_j, α complex numbers, $j=\overline{1,n}$, $a = Re\alpha$, $a \in (0,1]$ and $f_j \in \mathcal{S},\ f_j(z) = z + b_{2j}z^2 + \ldots,\ j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n}\frac{1}{|\gamma_j|} \le \frac{a}{2},\ \ for\ \ 0 < a < \frac{1}{2} \tag{34}$$

or

$$\sum_{j=1}^{n}\frac{1}{|\gamma_j|} \le \frac{1}{4},\ \ for\ \ \frac{1}{2} \le a \le 1 \tag{35}$$

then the integral operator $K_{\gamma_1,\gamma_2,\ldots\gamma_n}$ given by (21) is in the class $\mathcal{S}$.

Proof. We take $\beta = \frac{1}{\delta}$ and from Theorem 2 it results that $K_{\gamma_1,\gamma_2,\ldots\gamma_n}$ given by (21) belongs to class $\mathcal{S}$.

Corollary 7 *Let α, γ complex numbers, $a = Re\gamma > 0$ and $f_j \in \mathcal{S},\ f_j(z) = z + \sum_{k=2}^{\infty} b_{kj}z^k,\ j = \overline{1,n}$.*

If

$$\frac{1}{|\alpha|} \le \frac{a}{2},\ \ for\ \ 0 < a < \frac{1}{2} \tag{36}$$

or

$$\frac{1}{|\alpha|} \le \frac{1}{4},\ \ for\ \ a \ge \frac{1}{2} \tag{37}$$

then for any complex number β, $Re\ \beta \ge Re\ \gamma$ the integral operator $L_{\alpha,\beta}$ given by (23) is in the class $\mathcal{S}$.

Proof. We take $\delta = 1, \gamma_1 = \gamma_2 = ... = \gamma_n = \alpha$ in Theorem 2.

Corollary 8 *Let* α, γ *complex numbers,* $a = Re\ \alpha > 0$ *and* $f \in \mathcal{S}$, $f(z) = z + b_{21}z^2 + b_{31}z^3 + ...$

If

$$\frac{1}{|\gamma|} \leq \frac{a}{2}, \ \ for \ \ 0 < a < \frac{1}{2} \tag{38}$$

or

$$\frac{1}{|\gamma|} \leq \frac{1}{4}, \ \ for \ \ a \geq \frac{1}{2} \tag{39}$$

then the integral operator M_γ *define by (2) belongs to class* $\mathcal{S}$.

Proof. For $n = 1,\ \delta = 1,\ \beta = \frac{1}{\gamma},\ \gamma_1 = \gamma,\ f_1 = f$ in Theorem 2 we have the Corollary 8.

Corollary 9 *Let* α, γ *complex numbers,* $a = Re\gamma > 0$ *and* $f \in \mathcal{S}$, $f(z) = z + b_{21}z^2 + b_{31}z^3 + ...$

If

$$\frac{1}{|\alpha|} \leq \frac{a}{2}, \ \ for \ \ 0 < a < \frac{1}{2} \tag{40}$$

or

$$\frac{1}{|\alpha|} \leq \frac{1}{4}, \ \ for \ \ a \geq \frac{1}{2} \tag{41}$$

then for any complex number β, $Re\ \beta \geq Re\ \gamma$, *the integral operator* $T_{\alpha,\beta}$ *given by (5) is in the class* $\mathcal{S}$.

Proof. For $n = 1,\ \delta = 1,\ \gamma_1 = \alpha$ and $f_1 = f$ from Theorem 2 we obtain Corollary 9.

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