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On some Ostrowski type inequalities [1]

Bogdan Gavrea, Ioan Gavrea

Abstract

In this paper we study Ostrowski type inequalities. We generalize some of the results presented in [1].

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1 Introduction

Let I be a bounded interval of the real axis and $\mathcal{B}(I)$ be the set of all functions which are bounded on $[a, b]$. Let A be a positive linear functional $A : \mathcal{B} \to \mathbb{R}$ such that $A(e_0) = 1$, $e_i(x) = x^i$, $\forall x \in I$, $i \in \mathbb{N}$.

The following inequality is known as the Grüss inequality for the functional A.

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Theorem 1. *Let $f,\ g : I \to \mathbb{R}$ be two bounded functions such that $m_1 \leq f(x) \leq M_1$ and $m_2 \leq g(x) \leq M_2$, for all $x \in I$ with $m_1,\ M_1,\ m_2$ and M_2 constants. Then the inequality*

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1}$$

holds.

In 1928 Ostrowski proved the following result

Theorem 2. *Let $f : I \to \mathbb{R}$ be continuous on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e.,*

$$||f'||_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty.$$

Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)||f'||_\infty, \tag{2}$$

for all $x \in (a,b)$. The constant $\frac{1}{4}$ is the best.

S. S. Dragomir and S. Wang, [4], proved the following version of Ostrowski's inequality.

Theorem 3. *Let $f : I \to \mathbb{R}$ be a differentiable mapping in the interior of I and $a,\ b \in \mathrm{Int}(I)$ with $a < b$. If $f' \in L_1[a,b]$ and $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a,b]$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma) \tag{3}$$

for all $x \in [a,b]$.

In [3], S. S. Dragomir proved the following inequality for mappings with bounded variation.

Theorem 4. *Let $f : I \to \mathbb{R}$ be a mapping of bounded variation. Then, for all $x \in [a,b]$, we have the inequality*

$$\left|\int_a^b f(t)dt - f(x)(b-a)\right| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b f, \tag{4}$$

where $\bigvee_a^b f$ denotes the total variation of f.

In 2005, B. G. Pachpatte, [8], established the following inequality

Theorem 5. *Let $f,\ g : [a,b] \to \mathbb{R}$ be continuous functions on $[a,b]$ and differentiable on (a,b) whose derivatives $f',\ g' : (a,b) \to \mathbb{R}$ are bounded on (a,b). Then*

$$\left|f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt\right]\right|$$
$$\leq \frac{1}{2}\left[|g(x)|||f'||_\infty + |f(x)|||g'||_\infty\right]\frac{(x-a)^2+(b-x)^2}{2(b-a)}, \forall x \in [a,b]. \tag{5}$$

Remark. Inequality (5) follows from (2), since

$$\left|f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt\right]\right|$$
$$= \frac{1}{2}\left|g(x)\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) + f(x)\left(g(x) - \frac{1}{b-a}\int_a^b g(t)dt\right)\right|$$
$$\leq \frac{1}{2}\left[|g(x)|\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| + |f(x)|\left|g(x) - \frac{1}{b-a}\int_a^b g(t)dt\right|\right].$$

In approximation theory it is very useful the so-called *least concave majorant of the modulus of continuity.* More precisely, we have the following definition.

Definition 1. *Let $f \in C[a,b]$. If for $t \in [0,\infty)$, the quantity*

$$\omega(f;t) = \sup\{|f(x)-f(y)|,\ |x-y| \le t\}$$

is the usual modulus of continuity, its least concave majorant is given by

$$\widetilde{\omega}(f;t) = \sup\left\{\frac{(t-x)\omega(f;y)+(y-t)\omega(f;x)}{y-x},\ 0 \le x \le t \le y \le b-a\right\}$$

The following equality is well known

$$\inf_{g\in C(I)}\left(||f-g||_\infty + \frac{t}{2}||g'||_\infty\right) = \frac{1}{2}\widetilde{\omega}(f;t),\ t \ge 0.$$

In 2000, the authors proved the following result, [6].

Theorem 6. *Let f be a continuously differentiable function on $[a,b]$, such that $f(a)=f(b)=0$. Then the inequality*

$$\tag{6} \begin{aligned} &\left|\frac{f(x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{(x-a)^2+(b-x)^2}{8(b-a)}\widetilde{\omega}\left(f';\frac{2}{3}\cdot\frac{(x-a)^3+(b-x)^3}{(x-a)^2+(b-x)^2}\right) \end{aligned}$$

holds, where x is an arbitrary (but fixed) point in (a,b).

In 2001, Cheng, [2], modified Ostrowski's inequality by introducing the functional

$$B_x(f) := \frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right] - \int_a^b f(t)dt.$$

He proved the following result

Theorem 7. *Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of I and let $a,\ b \in \mathrm{Int}(I)$, $a < b$. If f' is* *integrable and* $\gamma \le f'(t) \le \Gamma$ *for all* $t \in [a,b]$ *and some constants* $\gamma,\ \Gamma \in \mathbb{R}$ *then*

$$|B_x(f)| \le \frac{1}{8}\left[(x-a)^2 + (b-x)^2\right](\Gamma - \gamma)$$

for all $x \in [a,b]$.

Ana Maria Acu and Heiner Gonska, [1], proved the following result:

Theorem 8. *If* $f \in C^1[a,b]$, *then*

$$|B_x(f)| \le \frac{(x-a)^2 + (b-x)^2}{8}\widetilde{\omega}\left(f', \frac{2}{3}\cdot\frac{(x-a)^3 + (b-x)^3}{(x-a)^2 + (b-x)^2}\right). \tag{7}$$

Remark. The results from Theorems 7 and 8 follow from (6), if we put instead of f the function

$$f - L(f;a,b) \quad (a, b \in I, \ \ a < b),$$

where $L(f;a,b)$ is the Lagrange interpolation polynomial of degree one associated with the function f on the points a and b.

In [5], we proved the following result of Ostrowski type

Theorem 9. *Let* f *be a continuous function on* $[a,b]$ *and* $w : [a,b] \to \mathbb{R}$ *be an integrable function on* (a,b) *such that* $\int_a^b w(s)ds = 1$. *Then for any continuous function* f, *the following inequality*

$$\begin{aligned}
&\left|f(x) - \int_a^b w(s)f(s)ds\right| \\
&\le \left(\int_a^x w(t)dt\right)\widetilde{\omega}_{[a,x]}\left(f; \frac{\int_a^x w(t)(x-t)dt}{\int_a^x w(t)dt}\right) \\
&\quad + \left(\int_x^b w(t)dt\right)\widetilde{\omega}_{[x,b]}\left(f; \frac{\int_x^b w(t)(t-x)dt}{\int_x^b w(t)dt}\right)
\end{aligned} \tag{8}$$

holds, where x *is a fixed point in* (a,b).

The following generalization of Ostrowski's inequality for arbitrary $f \in C[a,b]$ was given in [1].

Theorem 10. *Let $L : C[a,b] \to C[a,b]$ be non-zero, linear and bounded, such that $L : C^1[a,b] \to C^1[a,b]$ with $||(Lg)'|| \leq C_L||g'||$ for all $g \in C^1[a,b]$. Then for all $f \in C[a,b]$ and $x \in [a,b]$, we have*

$$\left|Lf(x) - \frac{1}{b-a}\int_a^b Lf(t)dt\right| \leq ||L||\widetilde{\omega}\left(f; \frac{C_L}{||L||} \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)}\right). \tag{9}$$

In this paper we will generalize the result of Theorem 10.

2 Auxiliary results

Let S be a subspace of $C(I)$, $I = [a,b]$ and A a linear functional defined on S. The following definition was given by T. Popoviciu, [9].

Definition 2 (2.1)**.** *The linear functional A defined on the subspace S which contains all polynomials, is P_n simple ($n \geq -1$) if*

(i) $A\left(e_{n+1}\right) \neq 0$

(ii) For every $f \in S$, there exist distinct points $t_1,\ t_2,\ ..., t_{n+2}$ in $[a,b]$ such that

$$A(f) = A\left(e_{n+1}\right)\left[t_1, t_2, ..., t_{n+2}; f\right], \tag{10}$$

where$[t_1, t_2, ..., t_{n+2}; f]$ is the divided difference of the function f on the points $t_1, t_2, ..., t_{n+2}$.

In what follows we assume that $\Pi \subset S$. The following result was proved in [6].

Theorem 11. *Let A be a linear functional $A : S \to \mathbb{R}$. If A is bounded, then*

$$|A(f)| = \inf_{g\in C^k(I)} (||A||||f-g|| + |A(g)|) \tag{11}$$

Corollary 1. *Let A be a linear bounded functional $A : C[a,b] \to \mathbb{R}$ with $\left|A(g^{(k)}\right| \le C\left|\left|g^{(k)}\right|\right|$ for all $g \in C^{(k)}[a,b]$. Then for all $f \in C[a,b]$, we have*

$$|A(f)| \le ||A||K\left(f, \frac{C}{||A||}, C(I), C^{(k)}(I)\right).$$

For $k = 1$, we obtain

$$|A(f)| \le ||A||K\left(f, \frac{2C}{||A||}\frac{1}{2}, C(I), C^1(I)\right) = \frac{||A||}{2}\widetilde{\omega}\left(f; \frac{2C}{||A||}\right). \tag{12}$$

Let us consider the following functional

$$A(f) = Lf(x) - \frac{1}{b-a}\int_a^b Lf(t)dt,$$

where $L : C[a,b] \to C[a,b]$ is a non-zero linear and bounded operator, $L : C^1[a,b] \to C^1[a,b]$ with $||(Lg)'|| \le C_L\,||g'||$ for all $g \in C^1[a,b]$. We have

$$||A|| \le 2||L||.$$

Using Ostrowski's inequality, for all $g \in C^1[a,b]$, we get

$$|A(g)| \le \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \le \frac{(x-a)^2 + (b-x)^2}{2(b-a)} C_l||g'||. \tag{13}$$

From (12) and (13) we obtain the result from Theorem 1.10. The following result was proved by H. Gonska and R. Kovacheva in [7].

Theorem 12. *For $f \in C[0,1]$ and $0 < h \le \frac{1}{2}$ fixed, for any $\epsilon > 0$, there are polynomials $p = p(f;h)$ such that*

$$\begin{aligned} ||f-p||_\infty &\le \frac{3}{4}\omega_2(f;h) + \epsilon \\ ||p''|_\infty &\le \frac{3}{2h^2}\omega_2(f;h). \end{aligned}$$

3 Main results

Let V be a linear set of real functions defined on $[a,b]$. We assume that $C[a,b] \subset V$ and that every step function defined on $[a,b]$ belongs to V. Let A be a linear bounded functional, $A : V \to \mathbb{R}$, such that $A(e_0) = 0$.

Lemma 1. *For all $f \in C^1[a,b]$, we have*

$$|A(f)| \leq \left(\int_a^b |A(\sigma(a-x))| dx \right) ||f'|| \,, \tag{14}$$

where $\sigma(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$.

Proof. Inequality (14) follows from the identity

$$f(t) = f(a) + \int_a^b \sigma(t-x) f'(x) dx.$$

Remark. For

$$A(f) = f(x) - \frac{1}{b-a} \int_a^b f(t) dt,$$

where x is fixed, (14) is Ostrowski's inequality.

Theorem 13. *For all $f \in C[a,b]$, we have*

$$|A(f)| \leq \frac{||A||}{2} \widetilde{\omega} \left(f; \frac{2C}{||A||} \right), \tag{15}$$

where $C = \int_a^b |A(\sigma(\cdot - x)| dx$.

Proof. The proof follows from (12) and (14).

Corollary 2. *Let f be a continuous function and x be a fixed number, $x \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \widetilde{\omega} \left(f; \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right). \tag{16}$$

Proof. Inequality (16) follows from (15) for the functional

$$A(f) = f(x) - \frac{1}{b-a}\int_a^b f(t)dt.$$

Corollary 3. *Let $L : C[a,b] \to C[a,b]$ be a non-zero linear and bounded operator. Then for all $f \in C[a,b]$ and $x \in [a,b]$ we have*

$$\left| Lf(x) - \frac{1}{b-a}\int_a^b Lf(t)dt \right| \leq \widetilde{\omega}\left(Lf; \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right). \tag{17}$$

Remark. If $L : C^1[a,b] \to C^1[a,b]$ with $||(Lg)'|| \leq C_l ||g'||$ for all $g \in C^1[a,b]$, then from (17) and from the representation theorem

$$|A(f)| = \inf_{g \in C^1[a,b]} \left(||f-g|| + |A(g)| \right),$$

we obtain Acu and Gonska's result, [1].

Corollary 4. *Let L_n be a discretely defined linear operator, $L_n : C[a,b] \to C[a,b]$,*

$$L_n f(x) = \sum_{k=0}^{n} \phi_{n,k}(x) f(x_{k,n}),$$

where $\phi_{n,k} \in C[a,b]$, $k = \overline{0,n}$, $x_{n,k} \in [a,b]$ are distinct points. If $L_n e_0 = e_0$, then

$$|L_n f(x) - f(x)| \leq \frac{||L||+1}{2} \widetilde{\omega}\left(f; \frac{2C_n(x)}{||L||+1} \right), \tag{18}$$

where $C_n(x) = \int_a^b \left| \sum_{k=0}^n \phi_{n,k}(x)\sigma(x_{k,n} - t) - \sigma(x-t) \right| dt$.

Lemma 2. *Let A be a linear bounded functional, $A : C[a,b] \to \mathbb{R}$, which has the degree of exactness k, $k \geq 1$. Then for all $g \in C^{k+1}[a,b]$ the following inequality holds*

$$|A(f)| \leq \frac{1}{k!} C_k(A) \left| \left| f^{(k+1)} \right| \right|, \tag{19}$$

where

$$C_k(A) = \int_a^b \left|A(\cdot - u)_+^k\right| du, \quad (x-a)_+^k = \begin{cases} 0, & x < a \\ (x-a)^k, & x \geq a \end{cases}.$$

Proof. The inequality (19) follows from the identity

$$f(x) = \sum_{i=0}^{k} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{k!} \int_a^b (x-u)_+^k f^{(k+1)}(u) du.$$

Using Lemma 2, we obtain the following result.

Theorem 14. *Let A be a linear bounded functional, $A : C[a,b] \to \mathbb{R}$, having its degree of exactness k, $k \geq 1$. Then for every continuous function f, we have*

$$|A(f)| \leq ||A|| K\left(f; \frac{C_k(A)}{k!||A||}, C(I), C^{k+1}(I)\right). \tag{20}$$

Remark. In the case when A is a P_n–simple functional, inequality (20) becomes

$$|A(f)| \leq ||A|| K\left(f; \frac{|A(e_{n+1}|}{(n+1)!||A||}, C(I), C^{n+1}(I)\right), \tag{21}$$

(see [6]). The reverse is also true, i.e., if inequality (21) holds for all continuous functions f, then A is a P_n simple functional.

Corollary 5. *Let A be a linear bounded functional, $A : C[a,b] \to \mathbb{R}$ with the degree of exactness 1. Then,*

$$|A(f)| \leq ||A|| K\left(f; \frac{C_1(A)}{||A||}, C(I), C^2(I)\right). \tag{22}$$

Using the result of Gonska and Kovacheva as well as inequality (22), we obtain

Theorem 15. *Let A be a linear bounded functional $A : C[0,1] \to \mathbb{R}$ having its degree of exactness* 1*. Then*

$$|A(f)| \leq ||A||\omega_2\left(f; \sqrt{\frac{6C_1(A)}{||A||}}\right). \tag{23}$$

Remark. If $A = B_x$, $[a,b] = [0,1]$, we get

$$|B_x(f)| \leq 2\omega_2\left(f; \frac{1}{2}\sqrt{x^3 + (1-x)^3}\right), \quad [1].$$

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Bogdan Gavrea
Technical University of Cluj-Napoca
Department of Mathematics,Faculty of Automation and Computer Science
Str. G. Baritiu nr. 26-28, 400027 Cluj-Napoca, Romania
e-mail: Bogdan.Gavrea@math.utcluj.ro

Ioan Gavrea
Technical University of Cluj-Napoca
Department of Mathematics,Faculty of Automation and Computer Science
Str. G. Baritiu nr. 26-28, 400027 Cluj-Napoca, Romania
e-mail: Ioan.Gavrea@math.utcluj.ro