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On the unbounded divergence of some projections operators in the discrete best approximation [1]

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Abstract

The main results of this paper establish the superdense unbounded divergence of some discrete best approximation operators.

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1 Introduction

Denote by C the Banach space of all continuous real functions defined on the interval $[-1, 1]$, endowed with the uniform norm $\| \cdot \|$ and let C^0 be the

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subspace of all even functions of C; more generally, if B is a set of real functions defined on an interval $I \subseteq \mathbb{R}$, which is symmetric with respect to the origin, we put B^0 for the subset of all even functions in B. Let us consider, too, a sequence $(E_n)_{n\geq 0}$ of finite subsets of $[-1,1]$ so that each E_n contains at least $n+1$ points.

Given an integer $n \geq 0$, denote by $\mathcal{P}_n$ the space of all polynomials of real coefficients having the degree at most n and let us introduce the operator $U_n : C \to \mathcal{P}_n$, which associates to each f in C the unique polynomial $U_n f \in \mathcal{P}_n$ for which the infimum of the set

$$\{\max\{|f(x) - P(x)| : \ x \in E_n\} : \ P \in \mathcal{P}_n\}$$

is attained. The polynomial $U_n f$ is said to be the *E_n-projection* of $f \in C$ on the space $\mathcal{P}_n$ and the operators U_n, $n \geq 0$, will be referred to as *E_n-polynomial projections*. Remark that $U_n f$ is the best approximation polynomial in $\mathcal{P}_n$ of a function $f \in C$, with respect to the discrete set E_n.

It is known that, in the case when each E_n contains $n+1$ points or each E_n contains $n+2$ points, the corresponding operators U_n, $n \geq 0$, are linear and continuous polynomial projections and there exists $g \in C$ so that the sequence $(U_n g)_{n\geq 0}$ is not uniformly convergent to g, [4]. Our aim is to prove the unboundedness of the set $\{\|U_n\| : \ n \geq 0\}$, if each E_n contains at most $n+3$ points, then to describe the topological structure of *the set of unbounded divergence* of E_n-polynomial projections, namely $\{f \in C : \ \limsup\limits_{n\to\infty} \|U_n f\| = \infty\}$. To this purpose, we need the following *principle of condensation of the singularities*, established by I. Muntean and S. Cobzaş.

Theorem 1 *[2], [3]. If X is a Banach space, Y is a normed space and $A_n : X \to Y$, $n \geq 1$, are linear continuous operators so that the set $\{\|A_n\| : \ n \geq 1\}$ is unbounded, then the set of singularities of the family $\{A_n : \ n \geq 1\}$, i.e.*

$$\{x \in X : \ \limsup \|A_n x\| = \infty\}$$

is superdense in X.

We recall that a subset S of a topological space T is named *superdense* in X if it is residual (i.e. its complement is of first Baire category), uncountable and dense in X.

In this paper, we use the following notations. Given a positive integer m and a subset A of the interval $[-1, 1]$ which has $m+1$ points t_k, $1 \leq k \leq m+1$, let $L_m(A; f)$, $f \in C$, be the Lagrange polynomial of degree at most m which interpolates f at the points of A and let $a_m(f)$ be the leading coefficient of $L_n(A; f)$. Particularly, denoting by σ_m a function of C which satisfies the equalities $\sigma_m(t_k) = (-1)^k$, $1 \leq k \leq m+1$, it is easily seen that $a_m(\sigma_m) \neq 0$.

2 The unboundedness of the norms of E_n-polynomial projections

Firstly, remark that if the sets E_n, $n \geq 1$, have $n+1$ or $n+2$ points, then the corresponding operator U_n are linear and continuous polynomial projection of C into $\mathcal{P}_n$, [4], [6]; more exactly, if E_n has $n+1$ points, then

$U_n f = L_n(E_n; f)$. Consequently, in these situations, according to [7], the following inequalities

$$\|U_n\| \geq \frac{4}{\pi^2} \ln n + O(1), \quad n \geq 1 \tag{1}$$

are satisfied.

In what follows, we assume in this section that the sets E_n have $n+3$ points which are symmetric with respect to the origin. Let us examine the operators $U_{2n} : C \to \mathcal{P}_{2n}$, associated to the corresponding sets $E_{2n} = \{x_{2n}^k : 1 \leq k \leq 2n+3\}$, $n \geq 1$.

Let us prove the equality

$$U_{2n} f = L_{2n+2}(E_{2n}; f) - \frac{a_{2n+2}(f)}{a_{2n+2}(\sigma_{2n+2})} L_{2n+2}(E_n; \sigma_{2n+2}), \tag{2}$$

for each function f in C^0.

Denoting by $P_{2n} f$ the polynomial of the right member in (1) and remarking that $L_{2n+2}(E_{2n}; f)$ and $L_{2n+2}(E_{2n}; \sigma_{2n+2})$ are even polynomials in $\mathcal{P}_{2n+2}$, it is obvious that $P_{2n} f$ is an even polynomial in $\mathcal{P}_{2n}$. Moreover,

$$(P_{2n} f)(x_{2n}^k) - f(x_{2n}^k) = (-1)^{k+1} \frac{a_{2n+2}(f)}{a_{2n+2}(\sigma_{2n+2})}, \quad 1 \leq k \leq 2n+3. \tag{3}$$

The relations (3), together with Theorem of Charles de la Vallée-Poussin [1], [5], [8], lead to the equality $P_{2n} f = U_{2n} f$, so that (2) is true.

Further, let us point out a lower bound for the norms of the operators $U_{2n}^0 : C^0 \to \mathcal{P}_{2n}^0$, $n \geq 1$, where each U_{2n}^0 is the restriction of U_{2n} to C^0.

Theorem 2 *The inequalities* $\|U_{2n}^0\| \geq \frac{2}{\pi^2} \ln(2n)$ *hold for all integers* $n \geq 1$.

Proof. Given $T > 0$, denote by C_T the set of all continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying $g(x+T) = g(x)$, $\forall\, x \in \mathbb{R}$. If $f \in C$, define $\widetilde{f} \in C_{2\pi}$ by $\widetilde{f}(x) = f(\cos x)$, $\forall\, x \in \mathbb{R}$. It is clear that $\widetilde{f} \in C^0_\pi$ for each $f \in C^0$.

The operator $F : C^0 \to C^0_\pi$, $Ff = \widetilde{f}$, is an isomorphism and $\|\widetilde{f}\| = \|Ff\| = \|f\|$, $\forall\, f \in C^0$. For each integer $n \geq 0$, denote by $\mathcal{E}_n$ the space of all trigonometric polynomials of degree $n \geq 0$ and introduce the operator $\widetilde{U}^0_{2n} : C^0_\pi \to \mathcal{E}_{2n}$ by the equality

$$(4) \qquad (\widetilde{U}^0_{2n}\widetilde{f})(x) = (U_{2n}f)(\cos x) = (U_{2n}(F^{-1}\widetilde{f}))(\cos x),\ \forall\, x \in [0,\pi].$$

In order to establish a lower bound for the norm of $\widetilde{U}^0_{2n}$, let τ be a given real number and define the translation-operator $T_\tau : C_{2\pi} \to C_{2\pi}$ as

$$(T_\tau\widetilde{f})(x) = \widetilde{f}(x+\tau), \quad \widetilde{f} \in C_{2\pi},\ x \in \mathbb{R}.$$

Setting $S_\tau = T_\tau + T_{-\tau}$, $\tau \in \mathbb{R}$ and noticing that $S_\tau\widetilde{f} \in C^0_\pi$ for each $\widetilde{f} \in C^0_\pi$, we obtain:

$$(5) \qquad \frac{1}{2\pi}\int_0^\pi S_\tau(\widetilde{U}^0_{2n}(S_\tau\widetilde{f}))(x)d\tau = (\phi^0_{2n} + \phi_0)(\widetilde{f})(x)$$

for each $\widetilde{f} \in C^0_\pi$ and $x \in [0,\pi]$, where $\phi_n : C_{2\pi} \to C_{2\pi}$, $n \geq 0$ are the Fourier projections

$$(6) \qquad (\phi_n\widetilde{f})(x) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(t)D_n(x-t)dt, \quad \widetilde{f} \in C_{2\pi},\ x \in \mathbb{R}$$

with $D_0(t) = 1$, $D_n(t) = 1 + 2\sum_{k=1}^{n}\cos(kt)$, $t \in \mathbb{R}$, $n \geq 1$ and ϕ^0_{2n} is the restriction of ϕ_{2n} to the space C^0_π.

The validity of (5) follows from standard arguments: firstly, it is true from $\widetilde{f} \in \widetilde{P}^0_{2n} = span\{c_{2k} : \ 0 \leq k \leq n\}$, with $c_k(x) = \cos(kx)$, $x \in \mathbb{R}$, then we use the relation $\overline{\mathcal{P}^0} = C^0$ and the properties of F, which lead to the equality $\overline{\widetilde{\mathcal{P}}^0} = C^0_\pi$.

Noticing that $\|T_\tau\| = 1$ and $\|S_\tau\| \leq 2$, it follows from (5):

$$\|\phi^0_{2n} + \phi_0\| = \sup\{\|\phi^0_{2n}\widetilde{f} + \phi_0\widetilde{f}\| : \ \widetilde{f} \in C^0_\pi, \ \|\widetilde{f}\| \leq 1\}$$

$$\leq \frac{1}{2\pi}\sup\left\{\|S_\tau\| \cdot \|\widetilde{U}^0_{2n}\| \cdot \|S_\tau\| \cdot \|\widetilde{f}\| \cdot \int_0^\pi d_\tau : \ \widetilde{f} \in C^0_\pi, \ \|\widetilde{f} \leq 1\right\},$$

i.e.

$$\|\phi^0_{2n} + \phi_0\| \leq 2\|\widetilde{U}^0_{2n}\|. \tag{7}$$

On the other hand, according to (5), we obtain:

$$\|\phi^0_{2n} + \phi_0\| = \frac{1}{2\pi}\max\left\{\int_0^{2\pi} |1 + D_{2n}(x-t)|dt : \ 0 \leq x \leq 2\pi\right\}$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} |D_0(t) + D_{2n}(t)|dt = \frac{1}{\pi}\int_0^\pi |D_{2n}(t) + D_0(t)|dt$$

which, combined with (7) and the inequality [8]:

$$\frac{1}{\pi}\int_0^\pi |D_{2n}(t) + D_0(t)|dt \geq \frac{4}{\pi^2}\ln 2n,$$

gives:

$$\|\widetilde{U}^0_{2n}\| \geq \frac{2}{\pi^2}\ln(2n) \tag{8}$$

Now, the relations $\|U^0_{2n}\| = \|\widetilde{U}^0_{2n}\|$, obtained from (4) and (8), complete the proof.

3 Superdense Unbounded Divergence of E_n-polynomial Projections

In this section we describe the topological structure of the set of unbounded divergence of E_n-polynomial projections.

Theorem 3 *If each set E_{2n}, $n \geq 1$, contains $2n+3$ points which are symmetric with respect to the origin, then the set of unbounded divergence of the discrete best approximation operators (i.e. E_n-polynomial projections) U_n, namely $\left\{ f \in C : \limsup_{n\to\infty} \|U_n f\| = \infty \right\}$, is superdense in the Banach space $(C, \|\cdot\|)$.*

Proof. Indeed, according to Theorem 2, we get:

$$\sup\{\|U_n\| : n \geq 1\} \geq \sup\{\|U_{2n}\| : n \geq 1\} \geq \sup \left\{ \frac{2}{\pi^2} \ln(2n) : n \geq 1 \right\} = \infty,$$

which proves the unboundedness of the set $\{\|U_n\| : n \geq 1\}$. Now, let us apply Theorem 1 and remark that the set of singularities of the family $\{U_n : n \geq 1\}$ represents the set of the unbounded divergence of this family.

Theorem 4 *Denote $I_s = \{n \geq 1 : card E_n = n + s\}$, $s \in \{1, 2, 3\}$ and let I_3^0 be the subset of all $n \in I_3$ with the property that the nodes of E_{2n} are symmetric with respect to the origin. If at least one of the sets I_1, I_2 and I_3^0 is unbounded, then the set of unbounded divergence of the discrete best approximation operators U_n is superdense in the Banach space $(C, \|\cdot\|)$.*

Proof. Take into account the inequalities (1) and Theorem 2.

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