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Characterizations of weak Cauchy *sn*-symmetric spaces [1]

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Abstract

This paper proves that a space X is a weak Cauchy sn-symmetric space iff it is a sequentially-quotient, π-image of a metric space, which answers a question posed by Z. Li.

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1 Introduction

sn-symmetric spaces is an important generalization of symmetric spaces. Recently, Y. Ge and S. Lin [10] investigate sn-symmetric spaces and obtained some interesting results. However, how characterize sn-symmetric spaces as images of metric spaces? This question is still open. As is well known, each

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weak *Cauchy* symmetric space can be characterized as a quotient, π-image of a metric space [11]. By viewing this result, Z. Li posed the following question [12, Question 3.2].

Question 1 *How characterize weak Cauchy sn-symmetric spaces by means of certain π-images of metric spaces?*

In this paper, we prove that a space X is a weak Cauchy sn-symmetric space iff it is a sequentially-quotient, π-image of a metric space, which answers Question 1 affirmatively.

Throughout this paper, all spaces are assumed to be Hausdorff, and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers. Let P be a subset of a space X and $\{x_n\}$ be a sequence in X converging to x. $\{x_n\}$ is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of a space X and $x \in X$. $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the intersection $\bigcap\{P : P \in \mathcal{P}\}$, respectively. $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$ and $st(x, \mathcal{P}) = \bigcup(\mathcal{P})_x$. A sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X is abbreviated to $\{P_n\}$. A point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n).

2 Definitions and Remarks

Definition 1 *([4]) Let X be a space and $x \in X$. P is called a sequential neighborhood of x, if each sequence $\{x_n\}$ converging to x is eventually in P.*

Remark 1 *([5]) P is a sequential neighborhood of x iff each sequence $\{x_n\}$ converging to x is frequently in P.*

Definition 2 *([6]) Let $\mathcal{P}$ be a family of subsets of a space X and $x \in X$. $\mathcal{P}$ is called a network at x in X, if $x \in \bigcap \mathcal{P}$ and for each neighborhood U of x, there exists $P \in \mathcal{P}$ such that $P \subset U$. Moreover, $\mathcal{P}$ is called an sn-network at x in X if in addition each element of $\mathcal{P}$ is also a sequential neighborhood of x.*

Definition 3 *Let X be a set. A non-negative real valued function d defined on $X \times X$ is called a d-function on X if $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for any $x, y \in X$.*

Let d be a d-function on a space X. For $x \in X$ and $n \in N$, put $S_n(x) = \{y \in X : d(x,y) < 1/n\}$.

Definition 4 *([10]) (X,d) is called an sn-symmetric space and d is called an sn-symmetric on X, if $\{S_n(x) : n \in \mathbb{N}\}$ is an sn-network at x in X for each $x \in X$.*

For subsets A and B of an *sn*-symmetric space (X,d), we write $d(A) = sup\{d(x,y) : x, y \in A\}$ and $d(A,B) = inf\{d(x,y) : x \in A \text{ and } y \in B\}$.

Definition 5 *([1]) Let (X,d) be an sn-symmetric space.*

(1) A sequence $\{x_n\}$ in X is called d-Cauchy if for each $\varepsilon > 0$, there exists $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > k$.

(2) (X,d) is called satisfying weak Cauchy condition if each convergent sequence has a d-Cauchy subsequence.

(3) An sn-symmetric space satisfying weak Cauchy condition is called a weak Cauchy sn-symmetric space.

Remark 2 *([13]) (X,d) satisfies weak Cauchy condition iff for each convergent sequence L in X and for each $\varepsilon > 0$, there exists a subsequence L' of L such that $d(L') < \varepsilon$.*

Definition 6 *([8]) Let $\mathcal{P}$ be a cover of a space X. $\mathcal{P}$ is called a cs^*-cover if for each convergent sequence L, there exists $P \in \mathcal{P}$ such that L is frequently in P.*

Definition 7 *([14]) Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$. $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a σ-strong network of X, if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for each $x \in X$. Moreover, if in addition $\mathcal{P}_n$ is also a cs^*-cover of X for each $n \in \mathbb{N}$, then $\mathcal{P}$ is called a σ-strong network consisting of cs^*-covers.*

Definition 8 *([7]). Let $f : X \longrightarrow Y$ be a mapping. f is called a sequentially-quotient mapping if for each convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.*

Remark 3 *Sequentially-quotient mappings are namely presequential mappings in the sense of J. R. Boone (see [2, 3, 9]).*

Definition 9 *([10]) Let (X, d) be an sn-symmetric and let $f : X \longrightarrow Y$ be a mapping. f is called a π-mapping, if for each $y \in Y$ and each neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$.*

3 The Main Results

Lemma 1 *Let (X, d) be an sn-symmetric space, $n \in \mathbb{N}$ and $x \in X$. Put $\mathcal{P}_n = \{P \subset X : d(P) < 1/n\}$, then $st(x, \mathcal{P}_n) = S_n(x)$.*

Proof. If $y \in st(x, \mathcal{P}_n)$, then there exists $P \in \mathcal{P}_n$ such that $x, y \in P$. So $d(x, y) \leq d(P) < 1/n$, and hence $y \in S_n(x)$. On the other hand, if $y \in S_n(x)$, then $d(x, y) < 1/n$. So $\{x, y\} \in \mathcal{P}_n$, thus $y \in st(x, \mathcal{P}_n)$. Consequently, $st(x, \mathcal{P}_n) = S_n(x)$.

Lemma 2 *Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network of X and $x \in X$, If $P_n \in (\mathcal{P}_n)_x$ for each $n \in \mathbb{N}$, then $\{P_n\}$ is a network at x in X.*

Proof. Let $x \in U$ with U open in X. Since $\mathcal{P}$ is a σ-strong network of X, there exists $m \in \mathbb{N}$ such that $st(x, \mathcal{P}_m) \subset U$. Note that $P_m \subset st(x, \mathcal{P}_m)$, so $x \in P_m \subset U$. This proves that $\{P_n\}$ is a network at x in X.

Lemma 3 *Let $\{\mathcal{P}_n\}$ be a sequence of cs^*-covers of a space X, and S be a sequence in X converging to x. Then there is a subsequence S' of S such that for each $n \in \mathbb{N}$, S' is eventually in P_n for some $P_n \in \mathcal{P}_n$.*

Proof. Since $\mathcal{P}_1$ is a cs^*-cover of X and S is a convergent sequence in X, there is a subsequence S_1 of S such that $S_1 \bigcup\{x\} \subset P_1$ for some $P_1 \in \mathcal{P}_1$. Put x_1 is the first term of S_1. Similarly, $\mathcal{P}_2$ is a cs^*-cover of X and S_1 is a convergent sequence in X, there is a subsequence S_2 of S_1 such that $S_2 \bigcup\{x\} \subset P_2$ for some $P_2 \in \mathcal{P}_2$. Put x_2 is the second term of S_2. Assume that $x_1, x_2, \cdots, x_{n-1}$, $S_1, S_2, \cdots, S_{n-1}$, and $P_1, P_2, \cdots, P_{n-1}$ have been constructed as the above method. we construct x_n, S_n and P_n as follows. Since $\mathcal{P}_n$ is a cs^*-cover of X and S_{n-1} is a convergent sequence in X, there is a subsequence S_n of S_{n-1} such that $S_n \bigcup\{x\} \subset P_n$ for some $P_n \in \mathcal{P}_n$. Put x_n is the n-th term of S_n. By the inductive method, we construct x_n, S_n and P_n for each $n \in \mathbb{N}$. Put $S' = \{x_n\}$, then S' is a subsequence of S. For each $n \in \mathbb{N}$, $\{x_k, x\} \in S_k \subset S_n \subset P_n$ for all $k > n$, so S' is eventually in P_n.

Now we give the main theorem in this paper.

Theorem 1 *The following are equivalent for a space X.*

(1) X is a weak Cauchy sn-symmetric space.

(2) X has a σ-strong network consisting of cs^*-covers.

(3) X is a sequentially-quotient, π-image of a metric space.

Proof. (1) $\Longrightarrow$ (2): Let (X, d) be a weak Cauchy sn-symmetric space. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P \subset X : d(P) < 1/n\}$. By Lemma 1, $st(x, \mathcal{P}_n) = S_n(x)$ for each $x \in X$ and each $n \in N$. $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for each $x \in X$ because $\{S_n(x) : n \in \mathbb{N}\}$ is a network at x in X. It is clear that $\mathcal{P}_{n+1} \subset \mathcal{P}_n$, so $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Thus $\{\mathcal{P}_n\}$ is a σ-strong network of X. Let $n \in \mathbb{N}$ and $L = \{x_k\}$ be a sequence in X converging to x. It suffices to prove that L is frequently in P for some $P \in \mathcal{P}_n$. Without loss of generality, we may assume that $d(x, x_k) < 1/n$ for each $k \in N$. Since (X, d) satisfying weak Cauchy condition, by Remake 2.7, there exists a subsequence L' of L such that $d(L') < 1/n$. Put $P = L' \bigcup \{x\}$, then $d(P) < 1/n$, and hence L is frequently in $P \in \mathcal{P}_n$.

(2) $\Longrightarrow$ (3): Let X have a σ-strong network $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in \mathbb{N}\}$ consisting of cs^*-covers. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\}$, and Λ_n is endowed with discrete topology. Put

$$M = \{b = (\beta_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\beta_n}\} \ \textit{is a network at some } x_b \ \textit{in} X\}.$$

Claim 1. M is a metric space:

In fact, Λ_n, as a discrete space, is a metric space for each $n \in \mathbb{N}$. So M, which is a subspace of the Tychonoff-product space $\prod_{n \in \mathbb{N}} \Lambda_n$, is a metric space.

The metric d on M can be described as follows. Let $b = (\beta_n), c = (\gamma_n) \in M$. If $b = c$, then $d(b, c) = 0$. If $b \neq c$, then $d(b, c) = 1/min\{n \in \mathbb{N} : \beta_n \neq \gamma_n\}$.

Claim 2. Let $b = (\beta_n) \in M$. Then there exists unique $x_b \in X$ such that $\{P_{\beta_n}\}$ is a network at x_b in X:

The existence comes from the construction of M, we only need to prove the uniqueness. Let $\{P_{\beta_n}\}$ be a network at both x_b and x'_b in X, then $\{x_b, x'_b\} \subset P_{\beta_n}$ for each $n \in \mathbb{N}$. If $x_b \neq x'_b$, then there exists an open neighborhood U

of x_b such that $x'_b \notin U$. Because $\{P_{\beta_n}\}$ is a network at x_b in X, there exists $n \in \mathbb{N}$ such that $x_b \in P_{\beta_n} \subset U$, thus $x'_b \notin P_{\beta_n}$, a contradiction. This proves the uniqueness.

We define $f : M \longrightarrow X$ as follows: for each $b = (\beta_n) \in M$, put $f(b) = x_b$, where $\{P_{\beta_n}\}$ is a network at x_b in X. By Claim 2, f is definable.

Claim 3. f is onto:

Let $x \in X$. For each $n \in \mathbb{N}$, there exists $\beta_n \in \Lambda_n$ such that $P_{\beta_n} \in (\mathcal{P}_n)_x$ because $\mathcal{P}_n$ is a cover of X. Since $\mathcal{P}$ is a σ-strong network of X, $\{P_{\beta_n}\}$ is a network at x in X by Lemma 2. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$. This proves that f is onto.

Claim 3. f is continuous:

Let $b = (\beta_n) \in M$ and let $f(b) = x$. If U is an open neighborhood of x, then there exists $k \in \mathbb{N}$ such that $x \in P_{\beta_k} \subset U$ because $\{P_{\beta_n}\}$ is a network at x in X. Put $V = ((\prod\{\Lambda_n : n < k\}) \times \{\beta_k\} \times (\prod\{\Lambda_n : n > k\})) \bigcap M$, then V is an open neighborhood of b. Let $c = (\gamma_n) \in V$, then $\{P_{\gamma_n}\}$ is a network at $f(c)$ in X, so $f(c) \in P_{\gamma_n}$ for each $n \in \mathbb{N}$. Note that $\gamma_k = \beta_k$, $f(c) \in P_{\gamma_k} = P_{\beta_k}$. This proves that $f(V) \subset P_{\beta_k}$, and hence $f(V) \subset U$. So f is continuous.

Claim 4. f is a π-mapping.

Let $x \in U$ with U open in X. Since $\mathcal{P}_n$ is a σ-strong network of X, there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. It suffices to prove that $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. Let $b = (\beta_n) \in M$. If $d(f^{-1}(x), b) < 1/2n$, then there is $c = (\gamma_n) \in f^{-1}(x)$ such that $d(b, c) < 1/n$, so $\beta_k = \gamma_k$ if $k \leq n$. Notice that $x = f(c) \in P_{\gamma_n} \in \mathcal{P}_n$ and $f(b) \in P_{\beta_n} = P_{\gamma_n}$, so $f(b) \in st(x, \mathcal{P}_n) \subset U$, thus $b \in f^{-1}(U)$. This proves that $d(f^{-1}(x), b) \geq 1/2n$ if $b \in M - f^{-1}(U)$, so $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$.

Claim 5. f is a sequentially-quotient mapping.

Let S be a sequence in X converging to $x \in X$. By Lemma 3, there exists a subsequence $S' = \{x_k\}$ of S such that for each $n \in N$, S' is eventually in P_{β_n} for some $\beta_n \in \Lambda_n$. Note that $x \in P_{\beta_n}$ for each $n \in \mathbb{N}$. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$ by Lemma 2. For each $k \in \mathbb{N}$, we pick $b_k \in f^{-1}(x_k)$ as follows. For each $n \in N$, if $x_k \in P_{\beta_n}$, put $\beta_{k_n} = \beta_n$; if $x_k \notin P_{\beta_n}$, pick $\beta_{k_n} \in \Lambda_n$ such that $x_k \in P_{\beta_{k_n}}$. Put $b_k = (\beta_{k_n}) \in \prod_{n \in N} \Lambda_n$, then $b_k \in M$ and $f(b_k) = x_k$ by Lemma 2. Put $L = \{b_k\}$, then L is a sequence in M and $f(L) = S'$. It suffices to prove that L converges to b. Let $b \in U$, where U is an element of base of M. By the definition of Tychonoff-product spaces, we may assume $U = ((\prod\{\{\beta_n\} : n \leq m\}) \times (\prod\{\Lambda_n : n > m\})) \bigcap M$, where $m \in \mathbb{N}$. For each $n \leq m$, S' is eventually in P_{β_n}, so there is $k(n) \in N$ such that $x_k \in P_{\beta_n}$ for all $k > k(n)$, thus $\beta_{k_n} = \beta_n$. Put $k_0 = max\{k(1), k(2), ..., k(m), m\}$, then $b_k \in U$ for all $k > k_0$, so L converge to b.

By the above Claims, X is a sequentially-quotient, π-image of a metric space.

(3) $\Longrightarrow$ (1): Let f be a sequentially-quotient, π-mapping from a metric space (M, d) onto X. Put $d'(x, y) = d(f^{-1}(x), f^{-1}(y))$ for each $x, y \in X$. It is clear that d' is a d-function on X. For $b \in M$, $x \in X$ and $n \in \mathbb{N}$, put $S_n(b) = \{c \in M : d(b, c) < 1/n\}$ and $S'_n(x) = \{y \in X : d'(x, y) < 1/n\}$.

Claim 1. $\{S'_n(x) : n \in \mathbb{N}\}$ is a network at x in X for each $x \in X$:

Let U be an open neighborhood of x in X. Since f is a π-mapping, there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/n$. If $y \notin U$, then $f^{-1}(y) \subset M - f^{-1}(U)$, hence $d'(x, y) = d(f^{-1}(x), f^{-1}(y)) \geq d(f^{-1}(x), M - f^{-1}(U)) \geq 1/n$, so $y \notin S'_n(x)$. This proves that $S'_n(x) \subset U$.

Claim 2. Let $x \in X$ and $n \in \mathbb{N}$. Then $S'_n(x)$ is a sequential neighborhood of x:

Let $\{x_m\}$ be a sequence converging to x. By Remark 1, it suffices to prove that $\{x_m\}$ is frequently in $S'_n(x)$. Since f is sequentially-quotient, there exists a sequence $\{b_k\}$ converging to $b \in f^{-1}(x)$ such that each $f(b_k) = x_{m_k}$. Pick $k_0 \in \mathbb{N}$ such that $d(b, b_k) < 1/n$ for all $k \geq k_0$. So $d'(x, x_{m_k}) = d(f^{-1}(x), f^{-1}(x_{m_k})) \leq d(b, b_k) < 1/n$ for all $k \geq k_0$, and hence $x_{m_k} \in S'_n(x)$ for all $k \geq k_0$. Thus $\{x_{m_k}\}$ is eventually in $S'_n(x)$, that is, $\{x_m\}$ is frequently in $S'_n(x)$.

Claim 3. (X, d') satisfies weak Cauchy condition:

Let $\{x_n\}$ be a convergent sequence in X. Since f is sequentially-quotient, there exists a convergent sequence $L = \{b_k\}$ in M such that $f(b_k) = x_{n_k}$ for each $k \in \mathbb{N}$. It suffices to prove that x_{n_k} is a d-Cauchy subsequence. Let $\varepsilon > 0$. Note that each convergent sequence in metric space (M, d) is a d-Cauchy sequence. So there exists $k_0 \in \mathbb{N}$ such that $d(b_i, b_j) < \varepsilon$ for all $i, j > k_0$. Thus $d'(x_{n_i}, x_{n_j}) = d(f^{-1}(x_{n_i}), f^{-1}(x_{n_j})) \leq d(b_i, b_j) < \varepsilon$ for all $i, j > k_0$. This proves that x_{n_k} is a d-Cauchy subsequence.

By the above Claims, d' is an sn-symmetric on X and (X, d') satisfies weak *Cauchy* condition. So X is a weak *Cauchy* sn-symmetric space.

Remark 4 *"σ-strong network" in Theorem 1 can be replaced by "point-star network", where the concept of "point-star networks" is obtained by omitting "$\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$" in the Definition 7 [13].*

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