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Properties of a class of multivalent functions starlike with respect to symmetric and conjugate points [1]

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Abstract

Let S be the class of functions analytic and multivalent in the open unit disc given by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad \text{and} \quad a_n \in \mathbb{C}.$$

In this paper we have studied three subclasses

$$S_s^* M(\alpha, \beta, \delta), S_c^* M(\alpha, \beta, \delta) \, \text{and} \, S_{sc}^* M(\alpha, \beta, \delta)$$

consisting of analytic functions with negative coefficients and starlike with respect to symmetric points, starlike with respect to conjugate points and starlike with respect to symmetric conjugate points, respectively. Here, we discuss coefficient inequality, growth and distortion theorems, extreme points, closure theorems and convolution properties of these classes.

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1 Introduction

Let S be the class of functions analytic and multivalent in the open unit disc given by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad \text{and} \quad a_n \in \mathbb{C}. \tag{1}$$

Let M be the subclass of S consisting of functions f of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{2}$$

where $a_n \geq 0, a_n \in I\!R$.

In [5], Sakaguchi introduced the class of analytic functions which are univalent and starlike with respect to symmetric points. This class is denoted by S_s^* and satisfies $Re\left\{\frac{zf'}{f(z)-f(-z)}\right\} > 0$ for $z \in I\!D$.

This definition has given rise to many generalized and extended classes of functions. The subclasses $S_s^* M(\alpha, \beta, \delta), S_c^* M(\alpha, \beta, \delta)$ and $S_{sc}^* M(\alpha, \beta, \delta)$ consisting of analytic functions with negative coefficients were introduced by Halim and A. Janteng in [1] and are respectively starlike with respect to symmetric points, starlike with respect to conjugate points and starlike with respect to symmetric conjugate points. Here α, β and δ satisfy the conditions $0 \leq \alpha < 1, 0 < \beta < 1, 0 \leq \delta < p$ and $0 < \frac{2(1-\beta)}{1+\alpha\beta} < 1$. This paper extends the result in [2] to other properties namely distortion, convex combination and convolution.

Let S^* be the subclass of S consisting of functions starlike in D. Notice that $f \in S^*$ iff $Re\left(\frac{zf'}{f}\right) > 0$ for $z \in D$.

Consider S^*_s, the subclass of S consisting of function given by (1) consisting of function given by (1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \quad z \in D.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in [5]. The same class is also considered by Robertson [8], Stankiewicz [4], Z. Wu [14], Owa and Z. Wu [13] and Aini Janteng, M. Darus [2]. K. M. El-Ashwah and Thomas in [10], have introduced two other subclasses namely S^*_c and S^*_{sc}.

In [14], Sudarshan et. al and Aini Janteng, M. Darus [2] have discussed the subclasses $S^*_s M(\alpha, \beta, \delta)$ of functions f analytic and univalent in $\mathbb{D}$ and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-(-z)} - (p+\delta)\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-f(-z)} + (p-\delta)\right|$$

for some $0 \le \alpha \le 1, 0 < \beta \le 1, 0 \le \delta < p$ and $z \in \mathbb{D}$.

However, in this paper we consider the subclass M defined by (2).

Definition 1 *A function $f \in S^*_s M(\alpha, \beta, \delta)$ is said to be starlike with respect to symmetric points if it satisfies*

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - (p+\delta)\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-f(-z)} + (p-\delta)\right| \quad \textit{for } p \in N \textit{ and } z \in D.$$

Definition 2 *A function $f \in S^*_c M(\alpha, \beta, \delta)$ is said to be starlike with respect to conjugate points if it satisfies*

$$\left|\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}} - (p+\delta)\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)+\overline{f(\overline{z})}} + (p-\delta)\right| \quad \textit{for } p \in N \textit{ and } z \in D.$$

Definition 3 *A function $f \in S^*_{sc}M(\alpha,\beta,\delta)$ is said to be starlike with respect to symmetric conjugate points if it satisfies*

$$\left|\frac{zf'(z)}{f(z)-\overline{f(-\overline{z})}}-(p+\delta)\right| < \beta\left|\frac{\alpha zf'(z)}{f(z)-\overline{f(-\overline{z})}}+(p-\delta)\right| \quad \textit{for } p\in N \textit{ and } z\in D.$$

Notice that the above conditions imposed on α,β and δ in the introduction are necessary to ensure that these classes form a subclass of S.

First we state the preliminary results similar to those obtained by Halim et. al. in [1], required for proving our main results.

2 Preliminaries

Theorem 1 $f \in S^*_sM(\alpha,\beta,\delta)$ *if and only if*

$$(3)\quad \sum_{n=1}^{\infty}\frac{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^p]\}+(-1)^p(p+\delta)-\delta}a_{p+n}\le 1 \quad \textit{for } p\in N$$

Corollary 1 *If* $f \in S^*_sM(\alpha,\beta,\delta)$ *then*

$$a_{p+n}\le\frac{\beta\{\alpha_p+(p-\delta)[1-(-1)]\}+(-1)^p(p+\delta)-\delta}{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]},$$

for $p\in N$ *and* $n\ge 1$.

Theorem 2 $f \in S^*_cM(\alpha,\beta,\delta)$ *if and only if*

$$(4)\qquad \sum_{n=1}^{\infty}\frac{(1+\beta\alpha)(p+n)+2[p(\beta-1)-\delta(1+\beta)]}{\beta\{\alpha p+2(p-\delta)\}-(p+2\delta)}a_{p+n}\le 1 \quad \textit{for } p\in N.$$

Corollary 2 *If* $f \in S^*_cM(\alpha,\beta,\delta)$ *then*

$$a_{p+n}\le\frac{\beta\{\alpha p+2(p-\delta)\}-(p+2\delta)}{(1+\beta\alpha)(p+n)+2[p(\beta-1)-\delta(1+\beta)]} \quad \textit{for } p\in N \textit{ and } n\ge 1.$$

Theorem 3 *$f \in S^*_{sc}M(\alpha,\beta,\delta)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^p\}+(-1)^p(p+\delta)-\delta} a_{p+n} \leq 1, \tag{5}$$

for $p \in N$.

Corollary 3 *If $f \in S^*_{sc}M(\alpha,\beta,\delta)$ then*

$$a_{p+n} \leq \frac{\beta\{\alpha p+(p-\delta)[1-(-1)^p]\}+(-1)^p(p+\delta)-\delta}{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]},$$

for $p \in N$ *and* $n \geq 1$.

3 Growth and Distortion Theorems

Theorem 4 *Let the function f be defined by (2) and belong to the class $S^*_sM(\alpha,\beta,\delta)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r-\frac{\beta\{\alpha+2(1-\delta)\}-(1+2\delta)}{2(1+\beta\alpha)}r^2 \leq |f(z)| \leq r+\frac{\beta\{\alpha+2(1-\delta)\}-(1+2\delta)}{2(1+\beta\alpha)}r^2,$$

for $p \in N$.

Proof. Let $f(z) = z^p - \sum\limits_{n=1}^{\infty} a_{p+n}z^{p+n}, \quad p \in N.$

$$\begin{aligned} (6)\quad |f(z)| &\leq |z|^p + \sum_{n=1}^{\infty} a_{p+n}|z|^{p+n} \\ &\leq |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n} \\ &\leq |z|^p + |z|^{p+1}\frac{\beta\{\alpha p+(p-\delta)(1-(-1)^p)\}+(-1)^p(p+\delta)-\delta}{(1+\beta\alpha)(p+1)+[1-\beta p+\delta(1+\beta)(-1)^{p+1}-1]} \\ &= r^p + \frac{\beta\{\alpha p+(p-\delta)(1-(-1)^p)\}+(-1)^p(p+\delta)-\delta}{(1+\beta\alpha)(p+1)+[1-\beta p+\delta(1+\beta)][(-1)^{p+1}-1]}r^{p+1}. \end{aligned}$$

Similarly,

$$(7)|F(z)| \geq |z|^p - \sum_{n=1}^{\infty} a_{p+n}|z|^{p+n}$$

$$\geq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$

$$\geq |z|^p - |z|^{p+1} \frac{\beta\{\alpha p + (p-\delta)(1-(-1)^p)\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1} - 1]}$$

$$= r^p - \frac{\beta\{\alpha p + (p-\delta)(1-(-1)^p)\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1} - 1]} r^{p+1}.$$

Hence the result.

The result is sharp for

$$(8) \qquad f(z) = z^p - \frac{\beta\{\alpha p + (-\delta)(1-(-1)^p)\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1} - 1]} z^{p+1}$$

at $z = \pm r$.

Next we state similar results for functions belonging to $S_c^* M(\alpha, \beta, \delta)$ and $S_{sc}^* M(\alpha, \beta, \delta)$. Method of proof is same as in Theorem 4.

Theorem 5 *Let the function f be defined by (2) and belong to the class $S_c^* M(\alpha, \beta, \delta)$, then for $\{z : 0 < |z| = r < 1\}$,*

$$r^p - \frac{\beta\{\alpha p + 2(p-\delta)\} - (p+2\delta)}{(1+\beta\alpha)(p+1) + 2[p(\beta-1) - \delta(1+\beta)]} r^{p+1} \leq |f(z)|$$

$$\leq r^p + \frac{\beta\{\alpha p + 2(p-\delta)\} - (p+2\delta)}{(1+\beta\alpha)(p+1) + 2[p(\beta-1) - \delta(1+\beta)]} r^{p+1}.$$

The result is sharp for

$$(9) \qquad f(z) = z^p - \frac{\beta\{\alpha p + 2(p-\delta)\} - (p+2\delta)}{(1+\beta\alpha)(p+1) + 2[p(\beta-1) - \delta(1+\beta)]} z^{p+1}.$$

Theorem 6 *Let the function f be defined by (2) and belong to the class $S_{sc}^* M(\alpha, \beta, \delta)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r^p - \frac{\beta\{\alpha p + 2(p-\delta)\}[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1} - 1]} r^{p+1} \leq |f(z)|$$

$$\leq r^p + \frac{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1} - 1]} r^{p+1}.$$

The result sharp for

$$f(z) = z^p - \frac{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1}-1]} z^{p+1} \tag{10}$$

at $z = \pm r$.

Theorem 7 *Let* $f \in S_s^* M(\alpha, \beta, \delta)$, *then for* $\{z : 0 < |z| = r < 1\}$

$$pr^{p-1} - \frac{(p+1)\beta\{\alpha p + (p-\delta)(1-(-1)^p)\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + (1-\beta p + \delta(1+\beta))[(-1)^{p+1}-1]} r^p \leq |f'(z)| \leq$$

$$pr^{p-1} + \frac{(p+1)\beta\{\alpha p + (p-\delta)(1-(-1)^p)\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1}-1]} r^p.$$

The result is sharp for function given by (8).

Theorem 8 *Let* f *be the function defined by (2) and belonging to the class* $S_c^* M(\alpha, \beta, \delta)$, *then for* $\{z : 0 < |z| = r < 1\}$

$$pr^{p-1} - \frac{(p+1)\beta\{\alpha p + 2(p-\delta)\} - (p+2\delta)}{(1+\beta\alpha)(p+1) + 2[p(\beta-1) - \delta(1+\beta)]} r^p \leq |f'(z)|$$
$$\leq pr^{p-1} + \frac{\beta(p+1)\{\alpha p + 2(p-\delta)\} - (p+2\delta)}{(1+\beta\alpha)(p+1) + 2[p(\beta-1) - \delta(1+\beta)]} r^p.$$

The result is sharp for the function given by (9).

Theorem 9 *Let* f *be the function defined by (2) and belonging to the class* $S_{sc}^* M(\alpha, \beta, \delta)$, *then for* $\{z : 0 < |z| = r < 1\}$

$$pr^{p-1} - \frac{(p+1)\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)](-1)^{p+1}-1]} r^p \leq |f'(z)|$$
$$\leq pr^{p-1} + \frac{(p+1)\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1^p)(p+\delta) - \delta}{(1+\beta\alpha)(p+1) + [1-\beta p + \delta(1+\beta)][(-1)^{p+1}-1]} r^p.$$

The result is sharp for the function given by (10).

$$1 - \frac{\beta\{\alpha + 2(1-\delta)\} - (1+2\delta)}{(1+\beta\alpha)} r \leq |f'(z) \leq 1 + \frac{\beta\{\alpha + 2(1-\delta)\} - (1+2\delta)}{(1+\beta\alpha)} r.$$

The result is sharp for

$$f(z) = z - \frac{\beta\{\alpha + 2(1-\delta)\} - (1+2\delta)}{2(1+\beta\alpha)} z^2$$

at $z = \pm r$ *and for* $p = 1 \in N$.

4 Closure Theorems

All three subclasses discussed here are closed under convex linear combinations. We prove for the class $S_s^* M(\alpha, \beta, \delta)$. It can be proved similarly for $S_c^* M(\alpha, \beta, \delta)$ and $S_{sc}^* M(\alpha, \beta, \delta)$.

Theorem 10 *Consider*

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in S_s^* M(\alpha, \beta, \delta)$$

for $j = 1, 2, 3, \cdots, \ell$ *and* $p \in N$ *then*

$$g(z) = \sum_{j=1}^{\ell} c_j f_j(z) \in S_s^* M(\alpha, \beta, \delta) \quad \textit{where} \quad \sum_{j=1}^{\ell} c_j = 1.$$

Proof. Let

$$\begin{aligned} g(z) &= \sum_{j=1}^{\ell} c_j \left(z^p - \sum_{n=1}^{\infty} a_{p+n,j} z^{p+n} \right) \\ &= z^p - \sum_{n=1}^{\infty} z^{p+n} \sum_{j=1}^{\ell} c_j a_{p+n,j} \\ &= z^p - \sum_{n=1}^{\infty} e_{p+n} z^{p+n} \end{aligned}$$

where $e_{p+n} = \sum_{j=1}^{\ell} c_j, a_{p+n,j}$.

Now $g(z) \in S_s^*M(\alpha,\beta,\delta)$ since

$$\sum_{n=1}^{\infty} \frac{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^p]+(-1)^p(p+\delta)-\delta} e_{p+n}$$

$$\leq \sum_{n=1}^{\infty}\sum_{j=1}^{\ell} \frac{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^p]\}+(-1)^p(p+\delta)-\delta} c_j a_{p+n,j}$$

$$\leq \sum_{j=1}^{\ell} c_j = 1 \ \text{ since,}$$

$$\sum_{n=1}^{\infty} \frac{(1+\beta\alpha)(p+n)[1-\beta p+\delta(1+\beta)[(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^p]+(-1)^p(p+\delta)-\delta} a_{p+n,j} \leq 1.$$

5 Extreme Points

Theorem 11 *Let* $f_p(z) = z^p$,

$$f_{p+n}(z) = z^p - \frac{\beta\{\alpha p+(p-\delta)[1-(-1)^p]\}+(-1)^p(p+\delta)-\delta}{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]} z^{p+n}$$

for $n \geq 1$ *and* $p \in N$, *then* $f \in S_s^*M(\alpha,\beta,\delta)$ *if and only if it can be expressed as in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z)$$

where $\lambda_{p+n} \geq 0$ *and* $\sum\limits_{n=0}^{\infty} \lambda_{p+n} = 1, p \in N$.

Proof. Let

$$(11) f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z), \quad p \in N$$

$$= z^p - \sum_{n=1}^{\infty} \lambda_{p+n} \frac{\beta\{\alpha p+(p-\delta)[1-(-1)^p]\}+(-1)^p(p+\delta)-\delta}{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+p)][(-1)^{p+n}-1)} z^{p+n}$$

Now $f(z) \in S_s^* M(\alpha, \beta, \delta)$ since

$$\sum_{n=1}^{\infty} \left[\frac{(1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]}{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta} \right]$$
$$\left[\frac{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]} \right] \lambda_{p+n}$$
$$= \sum_{n=1}^{\infty} \lambda_{p+n} = 1 - \lambda_p \leq 1, \quad p \in N.$$

Conversely, suppose that $f \in S_s^* M(\alpha, \beta, \delta)$. Then by Corollary 1

$$a_{p+n} \leq \frac{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]}, \quad n \geq 1.$$

Set

(12)
$$\lambda_{p+n} = \frac{(1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]}{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta} a_{p+n}, n \geq 1, p \in N$$

and $\lambda_p = 1 - \sum\limits_{n=1}^{\infty} \lambda_{p+n}$ then $f(z) = \sum\limits_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z)$.
Similarly extreme points for functions belonging to $S_c^* M(\alpha, \beta, \delta)$ and $S_{sc}^* M(\alpha, \beta, \delta)$ are found.
Method of proving Theorem 12 and Theorem 13 are similar to that of Theorem 11.

Theorem 12 *Let* $f_p(z) = z^p$,

$$f_{p+n}(z) = z^p - \frac{\beta(\alpha p + 2(p-\delta)) - (p+2\delta)}{(1+\beta\alpha)(p+n) + 2[p(\beta-1) - \delta(1+\beta)]} z^{p+n}$$

for $n \geq 1$ *and* $p \in N$. *Then* $f \in S_c^* M(\alpha, \beta, \delta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z) \quad \textit{where} \quad \lambda_{p+n} \geq 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \lambda_{p+n} = 1.$$

Theorem 13 *Let* $f_p(z) = z^p$

$$f_{p+n}(z) = z^p - \frac{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta}{(1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]} z^{p+n}$$

for $n \geq 1$ *and* $p \in N$. *Then* $f \in S^*_{sc}M(\alpha, \beta, \delta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z) \quad \text{where} \quad \lambda_{p+n} \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \lambda_{p+n} = 1.$$

6 Convolution Theorems

The three subclasses $S^*_s M(\alpha, \beta, \delta), S^*_c M(\alpha, \beta, \delta)$ and $S^*_{sc} M(\alpha, \beta, \delta)$ are closed under convolution. We prove first for the class $S^*_s M(\alpha, \beta, \delta)$.

Theorem 14 *Let* $f, g \in S^*_s M(\alpha, \beta, \delta)$ *where*

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad \text{and} \quad g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$$

then $f * g \in S^*_s M(\alpha, \beta, \nu)$. *For*

$$\frac{[(p+n) + (1+\delta)[(-1)^{p+n} - 1]][N]^2 + [\delta - (-1)^p(p+\delta)][D]^2}{[\alpha p + (p-\delta)[1-(-1)^p]][D]^2 - \{\alpha(p+n) + (\delta - p)[(-1)^{p+n} - 1][N]^2} < \nu$$

where

$$\begin{aligned} N &= \beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta \\ D &= (1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]. \end{aligned}$$

Proof. We have $f \in S^*_s M(\alpha, \beta, \delta)$ if and only if

$$(13) \qquad \sum_{n=1}^{\infty} \frac{(1+\beta\alpha)(p+n) + (1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]}{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta} \; a_{p+n} \leq 1.$$

Similarly

$$(14) \qquad \sum_{n=1}^{\infty} \frac{(1+\beta\alpha)(p+n) + [1-\beta p + \delta(1+\beta)][(-1)^{p+n} - 1]}{\beta\{\alpha p + (p-\delta)[1-(-1)^p]\} + (-1)^p(p+\delta) - \delta} \; b_{p+n} \leq 1.$$

To find a smallest number ν such that

$$(15)\quad \sum_{n=1}^{\infty}\frac{(1+\nu\alpha)(p+n)+[1-\nu p+\delta(1+\nu)][(-1)^{p+n}-1]}{\nu\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta}a_{p+n}b_{p+n}\leq 1.$$

By Cauchy Schwarz inequality (13) and (14) imply

$$(16)\quad \sum_{n=1}^{\infty}\frac{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta}\sqrt{a_{p+n}b_{p+n}}\leq 1$$

(15) will hold for

$$\frac{(1+\nu\alpha)(p+n)+[1-\nu p+\delta(1+\nu)][(-1)^{p+n}-1]}{\nu\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta}a_{p+n}b_{p+n}$$
$$\leq\frac{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}{\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta}$$
$$\sqrt{a_{p+n}b_{p+n}]}\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]+(-1)^{p}(p+\delta)-\delta\quad\sqrt{a_{p+n}b_{p+n}},$$

that is if

$$(17)\sqrt{a_{p+n}b_{p+n}}\leq\{[\nu\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta][(1+\beta\alpha)(p+n)$$
$$+[1-\beta p+\delta(1+\delta)][(-1)^{p+n}-1]\}/\{[\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]\}$$
$$+(-1)^{p}(p+\delta)-\delta][(1+\nu\alpha)(p+n)+[1-\nu p+\delta(1+\nu)][(-1)^{p+n}-1]\}$$

(16) implies

$$(18)\qquad \sqrt{a_{p+n}b_{p+n}}\leq\frac{\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta}{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}.$$

Thus it is enough to show that

$$\frac{\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta}{(1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]}$$
$$\leq\{[\nu\{\alpha p+(p-\delta)[1-(-1)^{p}]\}+(-1)^{p}(p+\delta)-\delta][(1+\beta\alpha)(p+n)$$
$$+[1-\beta p+\delta(1+\beta)][(-1)^{p+n}-1]\}/\{[\beta\{\alpha p+(p-\delta)[1-(-1)^{p}]\}$$
$$+(-1)^{p}(p+\delta)-\delta][(1+\nu\alpha)(p+n)+[1-\nu p+\delta(1+\nu)][(-1)^{p+n}-1]\}$$

which implies to

$$\frac{(p+n)+(1+\delta)[(-1)^{p+n}-1][N]^2+[\delta-(-1)^p(p+\delta)][D]^2}{[\alpha p+(p-\delta)[1-(-1)^p]][D]^2-\{\alpha(p+n)+(\delta-p)[(-1)^{p+n}-1][N]^2} \leq \nu$$

where

$$N = \beta\{\alpha p+(p-\delta)[1-(-1)^p]\}+(-1)^p(p+\delta)-\delta$$

$$D = (1+\beta\alpha)(p+n)+[1-\beta p+\delta(1+\beta)[(-1)^{p+n}-1]$$

Theorem 15 *Let $f, g \in S_c^*M(\alpha,\beta,\delta)$ then $f*g \in S_c^*M(\alpha,\beta,\nu)$ for*

$$\frac{[(p+n)-2(p+\delta)][N]^2-(p+2\delta)[D]^2}{[\alpha p+2(p-\delta)][D]^2-[\alpha(p+n)+2(p-\delta)][N]^2} \leq \nu$$

where

$$N = \beta\{\alpha p+2(p-\delta)\}+(p+2\delta)$$

$$D = (1=\beta\alpha)(p+n)+2[p(\beta-1)-\delta(1+\beta)].$$

Convolution theorem for subclass $S_{sc}^*M(\alpha,\beta,\delta)$ is similar to Theorem 14.

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