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On some fixed point theorems and error estimates involving integral type contractive conditions [1]

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Abstract

In this paper, we shall establish error estimates for two important fixed point theorems involving contractive conditions of integral type. Apart from obtaining the error estimates and the rate of convergence for a recent result of Rhoades [18] by imposing an additional condition, a generalization of Berinde [4] is also established in Theorem 2 by introducing a weak contraction condition of integral type in the present paper. Our results are generalizations and extensions of several well-known results as contained in the reference section of this paper. In particular, our results generalize, extend and improve corresponding results in [4, 5, 6, 8].

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1 Introduction

Let (X,d) be a complete metric space and $f : X \to X$ a selfmap of X. Suppose that $F_f = \{x \in X \mid f(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem was established in Banach [2] by using the following contractive condition: there exists $c \in [0,1)$ such that $\forall\ x,\ y \in X$, we have

$$d(fx, fy) \leq cd(x,y). \tag{1}$$

In a recent paper of Branciari [8], a generalization of [2] was established. In that paper, Branciari [8] employed the following contractive integral inequality condition: there exists $c \in [0,1)$ such that $\forall\ x,\ y \in X$, we have

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt, \tag{2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$.
One of the contractive definitions used in Rhoades [18] is the following: there exists $k \in [0,1)$ such that $\forall\ x,\ y \in X$, we have

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt,\ \forall\ x,\ y \in X, \tag{3}$$

where $m(x,y) = \max\left\{\mathrm{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}}\right\}$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ in (3) is as defined in (2). See Rhoades [18] for a contractive condition which is the integral form of Ciric's condition [11].

Literature abounds with several generalizations of the classical Banach's fixed point theorem since 1922. For some of these generalizations of the classical Banach's fixed point theorem and various contractive definitions that have been employed, the readers can consult [1, 3, 4, 10, 11, 17] and other references listed in the reference section of this paper.

In this paper, we shall establish error estimates for two important fixed point theorems involving contractive conditions of integral type. Our results are generalizations and extensions of several well-known results as contained in the reference section of this paper. In particular, our results generalize, extend and improve corresponding results in [4, 5, 6, 8].

2 Preliminaries

We shall require the following definitions in the sequel.

Definition 1 *[4]A single-valued mapping $f : X \to X$ is called a weak contraction or $(\delta,\ L)-$weak contraction if and only if there exist two constants, $\delta \in [0, 1)$ and $L \geq 0$, such that*

$$d(f(x), f(y)) \leq \delta d(x, y) + Ld(y, f(x)),\ \forall\ x,\ y \in X. \tag{4}$$

For the extension of the Banach's fixed point theorem in the sense of multivalued mapping, the reader is referred to Berinde and Berinde [7]. We shall employ the following definition to obtain our result:

Definition 2 *We shall say that a single-valued mapping $f : X \to X$ is a weak contraction of integral type or $(\delta,\ L)-$weak contraction of integral type if and only if there exist constants $\delta \in [0, 1)$ and $L \geq 0$ such that $\forall\ x,\ y \in X$,*

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq \delta \int_0^{d(x,y)} \varphi(t)dt + L[d(x, f(x))]^r \int_0^{d(y,f(x))} \varphi(t)dt, \tag{5}$$

where $r \in I\!R^+$ and *$\varphi : I\!R^+ \to I\!R^+$ is a Lebesgue-integrable mapping which is summable, nonnegative such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$.*

Remark 1 *(i) The contractive condition (5) reduces to (4) if* $r = 0$ and $\varphi(\mathrm{t}) = 1, \ \forall \ \mathrm{t} \in I\!R^+$.
(ii) The contractive condition (5) extends those of Branciari [8], Chatterjea [9], Kannan [14] and Zamfirescu [21].

3 Main Results

We shall employ the contractive definitions (3) and (5) to obtain our results.

Theorem 1 *Let* (X, d) *be a complete metric space,* $k \in [0, 1)$, *and let* $f : X \to X$ *be a mapping satisfying (3). Let* $\varphi : I\!R^+ \to I\!R^+$ *be a Lebesgue-integrable mapping which is summable, nonnegative, and such that for each* $\epsilon > 0, \ \int_0^\epsilon \varphi(t)dt > 0$. *Suppose that for two sequences* $\{x_n\}_{n=0}^\infty, \ \{y_n\}_{n=0}^\infty \subset X$, *there exists a decreasing sequence* $\{a_{np}\} \subset (0, 1), \ n, \ p = 0, 1, 2, \cdots$, *such that* $|d(x_n, y_n) - \int_0^{d(x_n,y_n)} \varphi(t)dt| \leq a_{np}$, *with* $\lim_{n\to\infty} a_{np} = \lim_{p\to\infty} a_{np} = 0$. *Then,*
(i) f *has a unique fixed point* $x^* \in X$ *such that for each* $x \in X, \ \lim_{n\to\infty} f^n x = x^*$.
(ii) a priori and a posteriori error estimates are respectively given by

$$(6) \qquad \int_0^{d(x_n,x^*)} \varphi(t)dt \leq \frac{k^n}{1-k} \int_0^{d(x_1,x_0)} \varphi(t)dt, \ n = 0, 1, 2, \cdots,$$

and

$$(7) \qquad \int_0^{d(x_n,x^*)} \varphi(t)dt \leq \frac{k}{1-k} \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \ n = 0, 1, 2, \cdots,$$

(iii) rate of convergence is given by

$$(8) \qquad \int_0^{d(x_n,x^*)} \varphi(t)dt \leq k \int_0^{d(x_{n-1},x^*)} \varphi(t)dt, \ n = 0, 1, 2, \cdots.$$

Proof. The proof of the part (i) of this theorem (that is, existence and uniqueness of the fixed point of f) is contained in Rhoades [18]. We shall

establish only parts (ii) and (iii) of the theorem. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = f(x_{n-1}) = f^n(x_0)$,
$n = 1, 2, \cdots$, be the Picard iteration associated to f which converges to $x^* \in X$.
Putting $u = d(x_n, y_n) - \int_0^{d(x_n,y_n)} \varphi(t)dt$ in the above relation, we have that

$$d(x_n, y_n) - a_{np} \leq \int_0^{d(x_n,y_n)} \varphi(t)dt \leq d(x_n, y_n) + a_{np}, \tag{9}$$

where a_{np} is a function of n and p. Starting with an expression of integral form on the left-hand side, the inequality condition (9) enables us to obtain an inequality relation in which the triangle inequality is applicable on the right-hand side and we can also conveniently get expression(s) of integral form(s) on the right-hand side of the inequality by some manipulations with the term a_{np}.

We now prove the error estimates in part (ii) of the theorem:
Using condition (3), we have as in Rhoades [18] that
$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \int_0^{d(f(x_{n-1}),f(x_n))} \varphi(t)dt \leq k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$

$$\leq k^2 \int_0^{d(x_{n-2},x_{n-1})} \varphi(t)dt \leq \cdots \leq k^n \int_0^{d(x_0,x_1)} \varphi(t)dt. \tag{10}$$

By inequality condition (9), the triangle inequality and the fact that $\{a_{np}\}$ is a decreasing sequence in $(0, 1)$, we have that
$\int_0^{d(x_n,x_{n+p})} \varphi(t)dt \leq d(x_n, x_{n+p}) + a_{np}$

$$\begin{aligned} &\leq \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt + \int_0^{d(x_{n+2},x_{n+3})} \varphi(t)dt \\ &\quad + \cdots + \int_0^{d(x_{n+p-1},x_{n+p})} \varphi(t)dt + (p+1)a_{np}. \end{aligned} \tag{11}$$

Using (10) inductively in (11) yields
$\int_0^{d(x_n,x_{n+p})} \varphi(t)dt \leq k^n[1 + k + k^2 + \cdots + k^{p-1}] \int_0^{d(x_0,x_1)} \varphi(t)dt + (p+1)a_{np}$

$$= \frac{k^n(1 - k^p)}{1 - k} \int_0^{d(x_0,x_1)} \varphi(t)dt + (p+1)a_{np}. \tag{12}$$

By taking limits of both sides in (12) as $p \to \infty$, we have that

$$\begin{aligned}\int_0^{d(x_n,x^*)}\varphi(t)dt = \int_0^{d(x^*,x_n)}\varphi(t)dt &= \lim_{p\to\infty}\int_0^{d(x_{n+p},x_n)}\varphi(t)dt\\ &\leq \tfrac{k^n}{1-k}\int_0^{d(x_0,x_1)}\varphi(t)dt,\end{aligned}$$

since $\lim_{p\to\infty}(p+1)a_{np} = \lim_{p\to\infty}(p+1).\lim_{p\to\infty}a_{np} = \infty.0 = 0$, thus proving the a priori error estimate in (6).

For a posteriori error estimate, we proceed as follows: From the first part of eqn. (10), we get

$$\left.\begin{aligned}&\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leq k\int_0^{d(x_{n-1},x_n)}\varphi(t)dt,\\ &\int_0^{d(x_{n+1},x_{n+2})}\varphi(t)dt \leq k^2\int_0^{d(x_{n-1},x_n)}\varphi(t)dt,\end{aligned}\right\} \tag{13}$$

and in general, (13) leads to

$$\int_0^{d(x_{n+l-1},x_{n+l})}\varphi(t)dt \leq k^l\int_0^{d(x_{n-1},x_n)}\varphi(t)dt,\ \ l \geq 1. \tag{14}$$

Using (14) inductively in (11) yields

$$\int_0^{d(x_n,x_{n+p})}\varphi(t)dt \leq \frac{k(1-k^p)}{1-k}\int_0^{d(x_{n-1},x_n)}\varphi(t)dt + (p+1)a_{np}. \tag{15}$$

Taking limits in (15) as $p \to \infty$ yields

$$\begin{aligned}\int_0^{d(x_n,x^*)}\varphi(t)dt = \int_0^{d(x^*,x_n)}\varphi(t)dt &= \lim_{p\to\infty}\int_0^{d(x_{n+p},x_n)}\varphi(t)dt\\ &\leq \tfrac{k}{1-k}\int_0^{d(x_{n-1},x_n)}\varphi(t)dt,\end{aligned}$$

since $\lim_{p\to\infty}(p+1)a_{np} = 0$, which proves the a posteriori error estimate in (7).

We now prove the rate of convergence in (iii): By using (3) again, we obtain

$$\int_0^{d(x_n,x^*)}\varphi(t)dt = \int_0^{d(f(x_{n-1}),f(x^*))}\varphi(t)dt \leq k\int_0^{d(x_{n-1},x^*)}\varphi(t)dt,$$

thus proving the rate of convergence in (8).

Remark 2 *Theorem 1 is a generalization and extension of the classical Banach's fixed point Theorem. Theorem 1 also generalizes and extends Theorem 1 of Berinde [5] and the theorem of Chatterjea [9]. See Agarwal et al [1], Banach [2], Berinde [3, 5, 6], Ciric [10, 11], Zeidler [22] and a host of other references.*

Theorem 2 *Let* (X, d) *be a complete metric space and* $f : X \to X$ *a* $(\delta,\ L)-$*weak contraction of integral type. Let* $\varphi : I\!R^+ \to I\!R^+$ *be a Lebesgue-integrable mapping which is summable, nonnegative, and such that for each* $\epsilon > 0,\ \int_0^\epsilon \varphi(t)dt > 0$. *Suppose that for two sequences* $\{x_n\}_{n=0}^\infty,\ \{y_n\}_{n=0}^\infty \subset X$, *there exists a decreasing sequence* $\{a_{np}\} \subset (0,1),\ n,\ p = 0,1,2,\cdots$, *such that* $|d(x_n, y_n) - \int_0^{d(x_n,y_n)} \varphi(t)dt| \le a_{np}$, *with* $\lim\limits_{n\to\infty} a_{np} = \lim\limits_{p\to\infty} a_{np} = 0$. *Then,*
(i) f *has a unique fixed point* $x^* \in X$ *such that for each* $x \in X$, $\lim\limits_{n\to\infty} f^n x = x^*$.
(ii) a priori and a posteriori error estimates are respectively given by

$$\int_0^{d(x_n,x^*)} \varphi(t)dt \le \frac{\delta^n}{1-\delta} \int_0^{d(x_1,x_0)} \varphi(t)dt,\ n = 0,1,2,\cdots, \tag{16}$$

and

$$\int_0^{d(x_n,x^*)} \varphi(t)dt \le \frac{\delta}{1-\delta} \int_0^{d(x_{n-1},x_n)} \varphi(t)dt,\ n = 0,1,2,\cdots, \tag{17}$$

(iii) rate of convergence is given by

$$\int_0^{d(x_n,x^*)} \varphi(t)dt \le \delta \int_0^{d(x_{n-1},x^*)} \varphi(t)dt,\ n = 0,1,2,\cdots. \tag{18}$$

Proof. We shall establish this theorem by an alternative method different from those of Branciari [8] and Rhoades [18]. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^\infty$ defined by $x_n = f(x_{n-1}) = f^n(x_0),\ n = 1,2,\cdots$, be the Picard iteration associated to f. Since f is a $(\delta,\ L)-$weak contraction of integral type, then from condition (5), we have that

$$(19) \qquad \int_0^{d(x_n,x_{n+1})}\varphi(t)dt = \int_0^{d(f(x_{n-1}),f(x_n))}\varphi(t)dt$$

$$\leq \delta\int_0^{d(x_{n-1},x_n)}\varphi(t)dt \leq \delta^2\int_0^{d(x_{n-2},x_{n-1})}\varphi(t)dt \leq \cdots \leq \delta^n\int_0^{d(x_0,x_1)}\varphi(t)dt.$$

We now establish that $\{x_n\}$ is a Cauchy sequence: Using (19) inductively in (11) leads to

$$(20) \qquad \int_0^{d(x_n,x_{n+p})}\varphi(t)dt \leq \frac{\delta^n(1-\delta^p)}{1-\delta}\int_0^{d(x_0,x_1)}\varphi(t)dt + (p+1)a_{np}.$$

By taking limits of both sides in (20) as $n \to \infty$, we have $\lim\limits_{n\to\infty}\frac{\delta^n(1-\delta^p)}{1-\delta}\int_0^{d(x_0,x_1)}\varphi(t)dt = 0$ and $\lim\limits_{n\to\infty}(p+1)a_{np} = (p+1)\lim\limits_{n\to\infty}a_{np} = 0$, leading to $\lim\limits_{n\to\infty}\int_0^{d(x_n,x_{n+p})}\varphi(t)dt = 0$, since $\int_0^{\epsilon}\varphi(t)dt > 0$ for each $\epsilon > 0$. Therefore, it follows that $\lim\limits_{n\to\infty}d(x_n,x_{n+p}) = 0$.

Hence, the sequence $\{x_n\}$ is Cauchy and so it is convergent. Since (X,d) is a complete metric space, $\{x_n\}$ converges to some $x^* \in X$, that is, $\lim\limits_{n\to\infty}x_n = x^*$. Again, by condition (5) and proceeding as in Rhoades [18], we obtain that $d(x^*, f(x^*)) = 0$, or, $x^* = f(x^*)$.

Using condition (5), we see that the proof of the uniqueness of the fixed point of f is similar to those of Branciari [8] and Rhoades [18].

We now prove the error estimates in part (ii) of the theorem:

By taking limits of both sides in (20) as $p \to \infty$, we have that

$$\int_0^{d(x_n,x^*)}\varphi(t)dt \leq \frac{\delta^n}{1-\delta}\int_0^{d(x_0,x_1)}\varphi(t)dt,$$

since $\lim\limits_{p\to\infty}(p+1)a_{np} = 0$, thus proving the a priori error estimate in (16).

For a posteriori error estimate, we proceed as follows: By a similar deduction as in (13), we get

$$(21) \qquad \int_0^{d(x_{n+l-1},x_{n+l})}\varphi(t)dt \leq \delta^l\int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \ l \geq 1.$$

By using (21) inductively in (11), we obtain

$$\int_0^{d(x_n,x_{n+p})} \varphi(t)dt \leq \frac{\delta(1-\delta^p)}{1-\delta} \int_0^{d(x_{n-1},x_n)} \varphi(t)dt + (p+1)a_{np}. \tag{22}$$

Taking limits in (22) as $p \to \infty$ yields

$$\int_0^{d(x_n,x^*)} \varphi(t)dt \leq \frac{\delta}{1-\delta} \int_0^{d(x_{n-1},x_n)} \varphi(t)dt,$$

since $\lim_{p\to\infty}(p+1)a_{np} = 0$, which proves the a posteriori error estimate in (17). We now prove the rate of convergence in (iii): By using (5), we obtain

$$\int_0^{d(x_n,x^*)} \varphi(t)dt = \int_0^{d(f(x^*),f(x_{n-1}))} \varphi(t)dt \leq \delta \int_0^{d(x_{n-1},x^*)} \varphi(t)dt,$$

thus proving the rate of convergence in (18).

Remark 3 *Theorem 2 is a generalization and extension of the celebrated Banach's fixed point [1, 2, 3, 5, 6, 22] as well as an extension of the results of Branciari [8], Chatterjea [9], Kannan [14] and Zamfirescu [21]. Theorem 2 is also a generalization and extension of some results of Berinde [4], both Theorem 1 and Theorem 2 of Berinde [5] as well as Theorem 2 of Berinde and Berinde [7].*

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