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A contractive method in the study of a nonlinear perturbation of the Laplacian [1]

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Abstract

In this paper we use a contractive method in the study of a boundary value problem for a semilinear equation of the form $-\Delta u+\lambda u+f(u)=g$, when the nonlinearity f satisfies a Lipschitz condition.

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1 Introduction

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $g \in L^2(\Omega)$. We consider the boundary value problem

$$-\Delta u(x)+\lambda u(x)+f(u(x))=g(x); \qquad x\in\Omega; \tag{1}$$

$$u=0 \text{ on } \partial\Omega, \tag{2}$$

where $f:\mathbf{R}\longrightarrow\mathbf{R}$ satisfies the Lipshitz condition $|f(u)-f(v)|\leqslant\alpha\,|u-v|$ for all $u,v\in\mathbf{R}$ ($\alpha>0$), $f(0)=0$ and Δ is the Laplacian operator. We assume that the positive parameter λ satisfies the condition $\lambda>\alpha$.

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The problems of the type $(1),(2)$ are motivated by the stationary diffusion phenomenon and have been investigated by many authors(we refer for example to [1], [2], [3], [4]).

In [1], a problem of the type $(1),(2)$ is studied when $\lambda = 0$, g is continuous and the nonlinear term f satisfies the following conditions:

(A1) f is a function of C^1 and $f(0) = f'(0) = 0$,

(A2) $h(u) := f(u)/u$ is strictly increasing ($h(0) := 0$).

In a great number of papers the problem $(1),(2)$ is studied when the nonlinearity f satisfies an inequality of the type $|f(u)| \leq a\,|u|^p + b$, or in the case $f(u) = |u|^p$ for some $p \in \mathbf{R} - \{1\}$.

In this paper we investigate the existence and the uniqueness of the solution of the problem $(1),(2)$, when the nonlinearity f satisfies only a Lipschitz condition.

We will prove using a contractive method, that the problem $(1),(2)$, in the said conditions for f, g and the positive parameters λ and α, has a unique solution. The proof of the principal result of this paper (Theorem 1) is a direct proof, which uses essentially the monotonicity properties of the linear differential operator generated by the term $-\Delta u$ of the equation (1). The proof of the Theorem 1 helps us to obtain easily a result of continuous dependence on the free term, result which is presented in the final part of the paper and which implies an estimation result for the solution of the considered problem.

Theorem 1 *Let $f : \mathbf{R} \longrightarrow \mathbf{R}$, $g : \Omega \longrightarrow \mathbf{R}$ and λ, α positive parameters so that:*

(i) $|f(x) - f(y)| \leqslant \alpha\,|x - y|$ *for all* $x, y \in \mathbf{R}$*;*

(ii) $f(0) = 0$*;*

(iii) $g \in L^2(\Omega)$*;*

(iv) $\lambda > \alpha$.

Then the problem $(1),(2)$ *has a unique weak solution.*

2 Proof of Theorem 1.

We denote by E the real Hilbert space $L^2(\Omega)$. The inner product and the corespondent norm in E will be denoted by $\langle \cdot, \cdot \rangle_2$ and $\|\cdot\|_2$.

Let $A : D(A) \subset E \longrightarrow E$ defined by $Au(x) = -\Delta u(x)$ for $x \in \Omega$, where $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is known that the linear operator A is a maximal monotone operator.

Let $L : D(L) = D(A) \longrightarrow E$ defined by $Lu(x) = -\Delta u(x) + \lambda u(x)$ for $x \in \Omega$, i.e. $L = A + \lambda I$ where I is the identity of E.

$Rg(I + \theta A) = \{u + \theta Au/u \in D(A)\} = E$ for all $\theta > 0$, because A is a maximal monotone operator. It results that $Rg(A + \lambda I) = Rg(L) = E$, because A is linear.

Also we have

$$(3) \qquad \langle Lu, u\rangle_2 = \langle Au, u\rangle_2 + \lambda \langle u, u\rangle_2 \geqslant \lambda \langle u, u\rangle_2 = \lambda \|u\|_2^2$$

for all $u \in D(L)$, i.e. L is a strongly positive linear operator.

Let $F : E \longrightarrow E$ defined by $Fu(x) = f(u(x))$; $x \in \Omega$ (the definition is correct, because from the properties of f, it results that $Fu \in L^2(\Omega)$ for all $u \in L^2(\Omega)$). We have

$$\|Fu-Fv\|_2^2 = \int\limits_\Omega |f(u(x))-f(v(x))|^2 \, d\mu \leqslant \alpha^2 \int\limits_\Omega |u(x)-v(x)|^2 \, d\mu = \alpha^2 \|u-v\|_2^2$$

for all $u, v \in E$(μ is the Lebesgue measure in $\mathbf{R}^N$). It results that the nonlinear operator F is a Lipschitz operator with the constant α.

Now we can writte the problem $(1), (2)$ in the equivalently operatorial form

$$(4) \qquad Lu + Fu = g.$$

From (3) we obtain

$$\|Lu\|_2 \geq \lambda \|u\|_2 \quad \text{for all } u \in D(L).$$

Consequently there exists $L^{-1} : E \longrightarrow D(L) \subset E$ which is linear and bounded, $L^{-1} \in \mathcal{L}(E)$, the Banach space of all linear and bounded operators from E to E. Moreover we have

$$\left\|L^{-1}\right\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda}.$$

The equation (4) can be written now as

$$(5) \qquad u + L^{-1}Fu = L^{-1}g.$$

We consider the operator $T : E \longrightarrow E$ defined by

$$Tu = -L^{-1}Fu + L^{-1}g.$$

Therefore the equation (5) becomes

$$u = Tu, \tag{6}$$

and our problem is reduced to the study of the fixed points of the operator T. We have

$$\|Tu - Tv\|_2 = \left\|L^{-1}Fu - L^{-1}Fv\right\|_2 = \left\|L^{-1}(Fu - Fv)\right\|_2 \leq$$

$$\left\|L^{-1}\right\|_{\mathcal{L}(E)} \|Fu - Fv\|_2 \leq \frac{\alpha}{\lambda} \|u - v\|_2 \quad \text{for all } u, v \in E.$$

It results that T is a strict contraction from E to E because $\lambda > \alpha$. According to the Banach fixed point theorem, T has a unique fixed point, and thus the proof is complete.

3 The continuous dependence of the solution of the problem (1), (2) on the data g and an estimation

Theorem 2 *Let $i \in \{1, 2\}$and u_i be the unique solution of the problem*

$$-\Delta u(x) + \lambda u(x) + f(u(x)) = g_i(x);\ x \in \Omega;\ i \in \{1, 2\};$$

$$u = 0 \ \text{on}\ \partial\Omega,$$

where $g_1, g_2 \in E$. Then

$$\|u_1 - u_2\|_2 \leq \frac{1}{\lambda - \alpha} \|g_1 - g_2\|_2 . \tag{7}$$

Proof. Using (5) we obtain

$$\|u_1 - u_2\|_2 = \left\|L^{-1}g_1 - L^{-1}Fu_1 - L^{-1}g_2 + L^{-1}Fu_2\right\|_2 \leq$$

$$\left\|L^{-1}(g_1 - g_2)\right\|_2 + \left\|L^{-1}(Fu_1 - Fu_2)\right\|_2 \leq$$

$$\left\|L^{-1}\right\|_{\mathcal{L}(E)} \|g_1 - g_2\|_2 + \left\|L^{-1}\right\|_{\mathcal{L}(E)} \|Fu_1 - Fu_2\|_2 \leq$$

$$\frac{1}{\lambda} \|g_1 - g_2\|_2 + \frac{\alpha}{\lambda} \|u_1 - u_2\|_2 .$$

It results that

$$(\lambda - \alpha) \left\| u_1 - u_2 \right\|_2 \leq \left\| g_1 - g_2 \right\|_2 ,$$

and the proof is complete.

The inequality (7) justifies the continuous dependence of the solution of the problem (1), (2) on the data g.

For fixed $g \in L^2(\Omega)$, let $u(\lambda)$ be the unique weak solution of the problem (1), (2) for all $\lambda > \alpha$. According to the Theorem 2. we obtain the following estimation:

$$\left\| u(\lambda) \right\|_2 \leq \frac{1}{\lambda - \alpha} \left\| g \right\|_2 .$$

It results that $\left\| u(\lambda) \right\|_2 \longrightarrow 0$ when $\lambda \longrightarrow \infty$ and this fact signifies that for large values of λ, the solution $u(\lambda)$ has only very small values.

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