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On certain classes of p-valent functions defined by Sălăgean operator [1]

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Abstract

Let $\mathcal{A}_p$ be the class of analytic functions f which are of the form $f(z) = z^p + \sum_{m=p+1}^{\infty} a_m\, z^m$, $(p \in \mathbb{N} = \{1, 2, 3, ...\})$, defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We introduce the class $\mathcal{M}(\alpha, \beta, n, p)$ and also the subclass $\mathcal{M}^\star(\alpha, \beta, n, p)$. The aim of the present paper is to derive some convolution properties for functions belonging to the class $\mathcal{M}^\star(\alpha, \beta, n, p)$.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form

(1) $$f(z) = z + \sum_{m=2}^{\infty} a_m\, z^m.$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Sălăgean [5] has introduced the following operator $D^n : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

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$D^0 f(z) = f(z), D^1 f(z) = D(f(z)) = zf^{'}(z)$ and $D^n f(z) = D(D^{n-1} f(z))$, for $n \in \mathbb{N} = \{1, 2, 3, ...\}$. We note that if f is of the form (1), then

$$D^n f(z) = z + \sum_{m=2}^{\infty} m^n \, a_m \, z^m \tag{2}$$

Let $\mathcal{A}_p$ be the class of functions f of the form

$$f(z) = z^p + \sum_{m=p+1}^{\infty} a_m \, z^m, \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{3}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
We define $D^n : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ by $D^0 f(z) = f(z), D^1 f(z) = D(f(z)) = \dfrac{1}{p} zf^{'}(z)$ and $D^n f(z) = D(D^{n-1} f(z))$ for $n \in \mathbb{N} = \{1, 2, 3, ...\}$.
We note that if f is of the form (3), then

$$D^n f(z) = z^p + \sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \, a_m \, z^m \tag{4}$$

We introduce the subclass $\mathcal{M}(\alpha, \beta, n, p)$ consisting of functions $f \in \mathcal{A}_p$ which satisfy

$$\Re\left\{\frac{z(D^n f(z))^{'}}{D^n f(z)}\right\} < \alpha \left|\frac{z(D^n f(z))^{'}}{D^n f(z)} - 1\right| + \beta, \qquad (z \in \mathcal{U}) \tag{5}$$

for some $\alpha \leq 0, \beta > 1$ and $n \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$.
We note that $\mathcal{M}(\alpha, \beta, 0, 1) \equiv \mathcal{M}D(\alpha, \beta)$ and $\mathcal{M}(\alpha, \beta, 1, 1) \equiv \mathcal{ND}(\alpha, \beta)$. The classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ were studied by Owa [3] and $\mathcal{M}(\alpha, \beta, n, 1) \equiv \mathcal{M}(\alpha, \beta, n)$, the class introduced by Mahzoon and Latha [2].

2 Coefficient inequalities

We derive sufficient conditions for f to be in the class $\mathcal{M}(\alpha, \beta, n, p)$ in terms of coefficient inequalities.

Theorem 1 *If $f \in \mathcal{A}_p$ satisfies*

$$\sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \{|m - \beta + 1| + |m - \beta - 1| - 2\alpha(m - 1)\} \, |a_m|$$

$$\leq \{|(1+\beta-p)| - |(p-\beta+1)| + 2\alpha(p-1)\}$$

for some $\alpha \leq 0$, $\beta > 1$ *and* $n \in \mathbb{N}_0$, *then* $f \in \mathcal{M}(\alpha, \beta, n, p)$.

Proof. We suppose that

$$\sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m|$$
$$\leq \{|(1+\beta-p)| - |(p-\beta+1)| + 2\alpha(p-1)\}$$

for $f \in \mathcal{A}_p$. It suffices to show that

$$\left| \frac{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) + 1}{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) - 1} \right| < 1, \quad (z \in \mathcal{U}).$$

We have

$$\left| \frac{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) + 1}{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) - 1} \right|$$

$$= \left| \frac{z(D^n f(z))' - \alpha e^{i\theta}|z(D^n f(z))' - (D^n f(z))| - \beta D^n f(z) + D^n f(z)}{z(D^n f(z))' - \alpha e^{i\theta}|z(D^n f(z))' - (D^n f(z))| - \beta D^n f(z) - D^n f(z)} \right| =$$

$$\left| \frac{p^n(p-\beta+1) + \sum_{m=p+1}^{\infty} m^n(m-\beta+1)a_m z^{m-p} - \alpha e^{i\theta}|p^n(p-1) + \sum_{m=p+1}^{\infty} m^n(m-1)a_m z^{m-p}|}{-\{p^n(1+\beta-p) - \sum_{m=p+1}^{\infty} m^n(m-\beta-1)a_m z^{m-p} + \alpha e^{i\theta}|p^n(p-1) + \sum_{m=p+1}^{\infty} m^n(m-1)a_m z^{m-p}|\}} \right|$$

$$< \frac{|p^n(p-\beta+1| + \sum_{m=p+1}^{\infty} m^n|m-\beta+1||a_m| - \alpha p^n(p-1) - \alpha \sum_{m=p+1}^{\infty} m^n(m-1)|a_m|}{|p^n(1+\beta-p)| - \sum_{m=p+1}^{\infty} m^n|m-\beta-1||a_m| + \alpha p^n(p-1) + \alpha \sum_{m=p+1}^{\infty} m^n(m-1)|a_m|}.$$

The last expression is bounded above by 1 if

$$|p^n(p-\beta+1| + \sum_{m=p+1}^{\infty} m^n|m-\beta+1||a_m| - \alpha p^n(p-1) - \alpha \sum_{m=p+1}^{\infty} m^n(m-1)|a_m|$$

$$\leq |p^n(1+\beta-p)| - \sum_{m=p+1}^{\infty} m^n|m-\beta+1||a_m| + \alpha p^n(p-1) + \alpha \sum_{m=p+1}^{\infty} m^n(m-1)|a_m|$$

which is equivalent to our condition

$$\sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m|$$
$$\leq \{|(1+\beta-p)| - |(p-\beta+1)| + 2\alpha(p-1)\}$$

of the Theorem.

3 Relation for $\mathcal{M}^\star(\alpha, \beta, n, p)$

By Theorem 1, the class $\mathcal{M}^\star(\alpha, \beta, n, p)$ is considered as the subclass of $\mathcal{M}(\alpha, \beta, n, p)$ consisting of f satisfying

$$\sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m|$$
$$\leq \{|(1+\beta-p)| - |(p-\beta+1)| + 2\alpha(p-1)\} \tag{6}$$

for some $\alpha \leq 0$, $\beta > 1$ and $n \in \mathbb{N}_0$.
We note that $\mathcal{M}^\star(\alpha, \beta, 0, 1) \equiv \mathcal{MD}^\star(\alpha, \beta)$ and $\mathcal{M}^\star(\alpha, \beta, 1, 1) \equiv \mathcal{N}D^\star(\alpha, \beta)$. The classes $\mathcal{M}D^\star(\alpha, \beta)$ and $\mathcal{ND}^\star(\alpha, \beta)$ were studied by Owa [3] and $\mathcal{M}^\star(\alpha, \beta, n, 1) \equiv \mathcal{M}^\star(\alpha, \beta, n)$, the class studied by Mahzoon and Latha [2] .

Theorem 2 *If $f \in \mathcal{A}_p$, then $\mathcal{M}^\star(\alpha_1, \beta, n, p) \subseteq \mathcal{M}^\star(\alpha_2, \beta, n, p)$ for some $\alpha_1, \alpha_2 (\alpha_1 \leq \alpha_2 \leq 0)$.*

Proof. For $\alpha_1 \leq \alpha_2 \leq 0$, we have

$$\sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha_2(m-1)\}|a_m|$$
$$\leq \sum_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha_1(m-1)\}|a_m|$$

and also,

$$\{|(1+\beta-p)| - |(p-\beta+1)| + 2\alpha_1(p-1)\} \tag{7}$$

$$\leq \{|(1+\beta-p)| - |(p-\beta+1)| + 2\alpha_2(p-1)\} \tag{8}$$

Therefore, if $f \in \mathcal{M}(\alpha_1, \beta, n, p)$, then $f \in \mathcal{M}^\star(\alpha_2, \beta, n, p)$.

4 Convolution of the class $\mathcal{M}^\star(\alpha,\beta,n,p)$

For analytic functions f_1 and f_2 given by

$$(9)\qquad f_1(z)=z^p+\sum_{m=p+1}^{\infty} a_{m,1}\ z^m \ \textit{and}\ f_2(z)=z^p+\sum_{m=p+1}^{\infty} a_{m,2}\ z^m$$

the Hadamard product $(\ f_1 * f_2)(z)$ of f_1 and f_2 is defined by

$$(10)\qquad (\ f_1 * f_2)(z)=z^p+\sum_{m=p+1}^{\infty} a_{m,1}\ a_{m,2}z^m$$

Theorem 3 *If* $f_1\in\mathcal{M}^\star(\alpha,\beta_1,n,p)$ *and* $f_2\in\mathcal{M}^\star(\alpha,\beta_2,n,p)$ *for some* α $(\alpha\le 2-\sqrt{5})$, $\left(p\neq 1:\alpha>\dfrac{\beta_1-p}{1-p}\ \textit{and}\,\alpha>\dfrac{\beta_2-p}{1-p}\right)$, $\beta_1,\ \beta_2\ \ (p<\beta_1,\beta_2\le p+1)$ *and* $n\in\mathbb{N}_0$*, then* $(\ f_1 * f_2\ )\in\mathcal{M}^\star(\alpha,\beta,n,p)$*, where* $\beta=1+\dfrac{Q+R}{S}$ *with*

$$\begin{aligned} Q &= p^n[(\beta_1-p)+\alpha(p-1)][(\beta_2-p)+\alpha(p-1)][p(1-\alpha)+1]\\ R &= (p+1)^n(p-1)(1-\alpha)[p(1-\alpha)+(1-\beta_1)][p(1-\alpha)+(1-\beta_2)]\\ S &= p^n[(\beta_1-p)+\alpha(p-1)][(\beta_2-p)+\alpha(p-1)]+\\ &\quad +(p+1)^n[p(1-\alpha)+(1-\beta_1)][p(1-\alpha)+(1-\beta_2)]. \end{aligned}$$

Proof. From (6), for $f\in\mathcal{M}^\star(\alpha,\beta,n,p)$ with $p<\beta\le p+1$ and $\alpha>\dfrac{\beta-p}{1-p}$, $(p\neq 1)$, we have

$$(11)\qquad \sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^n\{(m-\beta+1)+(m-\beta-1)-2\alpha(m-1)\}|a_m|$$

$$\le \sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^n\{(m-\beta+1)+|m-\beta-1|-2\alpha(m-1)\}|a_m|\le 2p^n[(\beta-p)+\alpha(p-1)].$$

That is, if $f\in\mathcal{M}^\star(\alpha,\beta,n,p)$, then

$$(12)\qquad \sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^n\frac{[m(1-\alpha)+(\alpha-\beta)]}{[(\beta-p)+\alpha(p-1)]}|a_m|\le 1.$$

Conversely, if f satisfies

$$(13)\qquad \sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^n\frac{[m(1-\alpha)+(1-\beta+\alpha)]}{[(\beta-p)+\alpha(p-1)]}|a_m|\le 1,$$

then $f \in \mathcal{M}^{\star}(\alpha,\beta,n,p)$. From (12), if $f_1 \in \mathcal{M}(\alpha,\beta_1,n,p)$, then

$$\sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{n}\frac{[m(1-\alpha)+(\alpha-\beta_1)]}{[(\beta_1-p)+\alpha(p-1)]}|a_{m,1}| \leq 1 \tag{14}$$

and also if $f_2 \in \mathcal{M}^{\star}(\alpha,\beta_2,n,p)$, then

$$\sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{n}\frac{[m(1-\alpha)+(\alpha-\beta_2)]}{[(\beta_2-p)+\alpha(p-1)]}|a_{m,2}| \leq 1. \tag{15}$$

Applying the Schwarz's inequality, we have the following inequality

$$\sum_{m=p+1}^{\infty}\sqrt{\left(\frac{m}{p}\right)^{2n}\frac{[m(1-\alpha)+(\alpha-\beta_1)][m(1-\alpha)+(\alpha-\beta_2)]}{[(\beta_1-p)+\alpha(p-1)][(\beta_2-p)+\alpha(p-1)]}}\sqrt{|a_{m,1}||a_{m,2}|} \leq 1 \tag{16}$$

by (14) and (15). From (13) and (16), if the following inequality

$$\sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{n}\frac{[m(1-\alpha)+(1-\beta+\alpha)]}{[(\beta-p)+\alpha(p-1)]}|a_{m,1}||a_{m,2}|$$

$$\leq \sum_{m=p+1}^{\infty}\sqrt{\left(\frac{m}{p}\right)^{2n}\frac{[m(1-\alpha)+(\alpha-\beta_1)][m(1-\alpha)+(\alpha-\beta_2)]}{[\beta_1-p)+\alpha(p-1)][(\beta_2-p)+\alpha(p-1)]}}\sqrt{|a_{m,1}||a_{m,2}|} \tag{17}$$

is satisfied, then we say that $(f_1 * f_2) \in \mathcal{M}^{\star}(\alpha,\beta,n,p)$ this inequality holds true if

$$\left(\frac{m}{p}\right)^{n}\frac{[m(1-\alpha)+(1-\beta+\alpha)]}{[(\beta-p)+\alpha(p-1)]}\sqrt{|a_{m,1}||a_{m,2}|}$$

$$\leq \sqrt{\left(\frac{m}{p}\right)^{2n}\frac{[m(1-\alpha)+(\alpha-\beta_1)][m(1-\alpha)+(\alpha-\beta_2)]}{[(\beta_1-p)+\alpha(p-1)][(\beta_2-p)+\alpha(p-1)]}} \tag{18}$$

for all $m \geq p+1$. Therefore, we have

$$\left(\frac{m}{p}\right)^{n}\frac{[m(1-\alpha)+(1-\beta+\alpha)]}{[(\beta-p)+\alpha(p-1)]}$$

$$\leq \left(\frac{m}{p}\right)^{2n}\frac{[m(1-\alpha)+(\alpha-\beta_1)][m(1-\alpha)+(\alpha-\beta_2)]}{[\beta_1-p)+\alpha(p-1)][(\beta_2-p)+\alpha(p-1)]} \tag{19}$$

which is equivalent to $\beta = 1 + \dfrac{Q+R}{S}$ where

$$\begin{aligned}
Q &= p^n[(\beta_1 - p) + \alpha(p-1)][(\beta_2 - p) + \alpha(p-1)][p(1-\alpha) + 1]\\
R &= (p+1)^n(p-1)(1-\alpha)[p(1-\alpha) + (1-\beta_1)][p(1-\alpha) + (1-\beta_2)]\\
S &= p^n[(\beta_1 - p) + \alpha(p-1)][(\beta_2 - p) + \alpha(p-1)]+\\
&\quad +(p+1)^n[p(1-\alpha) + (1-\beta_1)][p(1-\alpha) + (1-\beta_2)].
\end{aligned}$$

for all $m \geq p+1$.

Let $R(m)$ be the right hand side of the last inequality. Further, let us define $S(m)$ by numerator of $R^{'}(m)$. Then, $S(m)$ gives us that,

$$\begin{aligned}
&S(m) = p^n[(\beta_1 - p) + \alpha(p-1)][(\beta_2 - p) + \alpha(p-1)]\{p^n[(\beta_1 - p)\\
&+\alpha(p-1)][(\beta_2 - p) + \alpha(p-1)](1-\alpha) + 2m^n(p-1)(1-\alpha)^2[m(1-\alpha)+\alpha]\\
&+m^{n-1}(1-\alpha)[n(p-1)-m][m(1-\alpha)+\alpha]^2 - nm^{n-1}[m(1-\alpha)+\alpha]^3\\
&-nm^{n-1}(p-1)(1-\alpha)[m(1-\alpha)+\alpha]\beta_1 - nm^{n-1}(p-1)(1-\alpha)[m(1-\alpha)+\alpha]\beta_2\\
&+nm^{n-1}[m(1-\alpha)+\alpha]^2\beta_1 + nm^{n-1}[m(1-\alpha)+\alpha]^2\beta_2 - nm^{n-1}[m(1-\alpha)+\alpha]\beta_1\beta_2\\
&+m^{n-1}(1-\alpha)[m+n(p-1)]\beta_1\beta_2 - m^n(p-1)(1-\alpha)^2\beta_1 - m^n(p-1)(1-\alpha)^2\beta_2\}\\
&\leq p^n[(\beta_1 - p) + \alpha(p-1)][(\beta_2 - p) + \alpha(p-1)]\{p^n(1-\alpha)\\
&+2m^n(p-1)(1-\alpha)^2[m(1-\alpha)+\alpha] + m^{n-1}(1-\alpha)[n(p-1)-m][m(1-\alpha)+\alpha]^2\\
&-nm^{n-1}[m(1-\alpha)+\alpha]^3 - 2nm^{n-1}(p-1)(1-\alpha)[m(1-\alpha)+\alpha]p\\
&+2nm^{n-1}[m(1-\alpha)+\alpha]^2(p+1) - nm^{n-1}[m(1-\alpha)+\alpha]p^2\\
&+m^{n-1}(1-\alpha)[m+n(p-1)](p+1)^2 - 2m^n(p-1)(1-\alpha)^2p\} \leq 0
\end{aligned}$$

for $\alpha \leq 2-\sqrt{5}$ and $\left(p \neq 1:\ \alpha > \dfrac{\beta_1 - p}{1-p} \text{ and } \alpha > \dfrac{\beta_2 - p}{1-p}\right)$ which shows that $R(m)$ is decreasing for $m \geq p+1$, $\alpha \leq 2-\sqrt{5}$, $\left(p \neq 1:\ \alpha > \dfrac{\beta_1 - p}{1-p} \text{ and } \alpha > \dfrac{\beta_2 - p}{1-p}\right)$ and for $n \in \mathbb{N}_0$, $p \in \mathbb{N}$.

Thus, $R(p+1)$ is the maximum of $R(m)$ for α given by

$$\alpha \leq 2-\sqrt{5}, \left(p \neq 1 : \alpha > \frac{\beta_1 - p}{1-p} \text{ and } \alpha > \frac{\beta_2 - p}{1-p}\right)$$

and for $n \in \mathbb{N}_0,\ p \in \mathbb{N}$.

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