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On Lorentzian β-Kenmotsu manifolds [1]

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Abstract

The present paper deals with Lorentzian β-Kenmotsu manifold with conformally flat and quasi conformally flat curvature tensor. It is proved that in both cases, the manifold is locally isometric with a sphere $S^{2n+1}(c)$. Further it is shown that an Lorentzian β-Kenmotsu manifold with $R(X,Y).C = 0$ is an η-Einstein manifold.

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1 Introduction

In [12], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He

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showed that they can be divided into three classes:
(1) homogeneous normal contact Riemannian manifolds with $c > 0$,
(2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and
(3) a warped product space $\mathbf{R} \times_f \mathbf{C}$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [7] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [7]. In the Gray-Hervella classification of almost Hermitian manifolds [6], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [5]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [11] if the product manifold $M\mathbf{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ [8], [9] coincides with the class of the trans-Sasakian structures of type (α, β). In fact, in [9], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [4], β-Kenmotsu [7] and α-Sasakian [7] respectively. In [13] it is proved that trans-Sasakian structures are generalized quasi-Sasakian [10]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [11] if $(M \times \mathbf{R}, J, G)$ belongs to the class W_4 [6], where J is the almost complex structure on $M\mathbf{R}$ defined by

$$J(X, fd/dt) = (\phi X - f\xi, \eta(X)fd/dt)$$

for all vector fields X on M and smooth functions f on $M \times \mathbf{R}$, and G is the product metric on $M \times \mathbf{R}$. This may be expressed by the condition [3]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

Theorem 1 *A trans-sasakian structure of type (α, β) with β a nonzero constant is always β-Kenmotsu*

In this case β becomes a constant. If $\beta = 1$, then β-Kenmotsu manifold is Kenmotsu.

In this paper, we investigate Lorentzian β-Kenmotsu manifolds in which

$$C = 0 \tag{1}$$

where C is the Weyl conformal curvature tensor. Then we study Lorentzian β-Kenmotsu manifolds in which

$$\widetilde{C} = 0 \tag{2}$$

where $\widetilde{C}$ is the quasi conformal curvature tensor. In both cases, it is shown that Lorentzian β-Kenmotsu is isometric with a sphere $S^{2n+1}(c)$, where $c = \alpha^2$. Finally Lorentzian β-Kenmotsu manifolds with

$$R(X, Y).C = 0 \tag{3}$$

has been considered,where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold of tangent vectors X, Y. It is shown that Lorentzian β-Kenmotsu manifold is a η-Einstein.

2 Preliminaries

A differentiable manifold M of dimension $(2n + 1)$ is called Lorentzian β-Kenmotsu manifold if it admits a $(1, 1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy [1], [2],

$$(a)\ \eta\xi = -1, \quad (b)\ \phi\xi = 0, \quad (c)\ \eta(\phi X) = 0, \tag{4}$$

$$(a)\ \phi^2 X = X + \eta(X)\xi, \quad (b)\ g(X, \xi) = \eta(X), \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

for all $X, Y \in TM$.
Also Lorentzian β-Kenmotsu manifold M is satisfying

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{7}$$

$$(\nabla_X \eta)(Y) = \beta[g(X, Y) - \eta(X)\eta(Y)], \tag{8}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
Further, on Lorentzian β-Kenmotsu manifold M the following relations hold

$$\eta(R(X, Y)Z) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{9}$$

$$R(\xi, X)Y = \beta^2(\eta(Y)X - g(X, Y)\xi), \tag{10}$$

$$R(X, Y)\xi = \beta^2(\eta(X)Y - \eta(Y)X), \tag{11}$$

$$S(X, \xi) = -2n\beta^2\eta(X), \tag{12}$$

$$Q\xi = -2n\beta^2\xi, \tag{13}$$

$$S(\xi, \xi) = 2n\beta^2. \tag{14}$$

3 Lorentzian β-Kenmotsu manifolds with $C = 0$

The conformal curvature tensor C on M is defined as

$$C(X,Y)Z = R(X,Y)Z + \left[\frac{1}{2n-1}\right][S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QZ] - \left[\frac{r}{2n(2n-1)}\right][g(X,Z)Y - g(Y,Z)X], \tag{15}$$

where $S(X, Y) = g(QX, Y)$.
Using (1) we get from (15)

$$R(X,Y)Z = \left[\frac{1}{2n-1}\right][S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \left[\frac{r}{2n(2n-1)}\right][g(X,Z)Y - g(Y,Z)X]. \tag{16}$$

Taking $Z = \xi$ in (16) and using (5), (11) and (12), we find

$$[\eta(X)QY - \eta(Y)QX] = \left[2n\beta^2 + \frac{r}{2n} - (2n-1)\beta^2\right][\eta(X)Y - \eta(Y)X]. \tag{17}$$

Taking $Y = \xi$ in (17) and using (4), we get

$$QX = \left[\frac{r}{2n} + \beta^2\right] X + \left[\frac{r}{2n} + \beta^2 + 2n\beta^2\right]\eta(X)\xi. \tag{18}$$

Contracting (18), we get

$$r = -2n(2n+1)\beta^2. \tag{19}$$

Using (19) in (18), we find

$$QX = -2n\beta^2 X. \tag{20}$$

Using (20) in (16) and simplifying we get

$$R(X,Y)Z = \beta^2[g(X,Z)Y - g(Y,Z)X]. \tag{21}$$

Therefore the manifold is of constant scalar curvature β^2. Hence we can state:

Theorem 2 *A conformally flat Lorentzian β-Kenmotsu manifold is locally isometric to a sphere* $S^{2n+1}(c)$*, where* $c = \beta^2$.

4 Lorentzian β-Kenmotsu manifolds with $\widetilde{C} = 0$

The quasi conformal curvature tensor $\widetilde{C}$ on M is defined as

$$\begin{aligned}\widetilde{C}(X,Y)Z = {} & aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & - g(X,Z)QY] - \left[\frac{r}{2n+1}\right]\left[\frac{a}{2n} + 2b\right][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{22}$$

where a, b are constants such that $a, b \neq 0$ and $S(Y,Z) = g(QY,Z)$.
Using (3), we find from (22) that

$$\begin{aligned}R(X,Y)Z = {} & \frac{b}{a}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] \\ & + \left[\frac{r}{(2n+1)a}\right]\left[\frac{a}{2n} + 2b\right][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{23}$$

Taking $Z = \xi$ in (23) and using (5), (11) and (12), we get

(24)
$$[\eta(Y)QX - \eta(X)QY] = \left[\frac{r}{(2n+1)b}\left(\frac{a}{2n} + 2b\right) + 2n\beta^2 + \frac{a}{b}\beta^2\right][\eta(Y)X - \eta(X)Y].$$

Taking $Y = \xi$ in (24) and applying (4), we have

$$\begin{aligned}(25) \qquad QX &= \left[\frac{r}{(2n+1)b}\left(\frac{a}{2n} + 2b\right) + 2n\beta^2 + \frac{a}{b}\beta^2\right] X \\ &+ \left[\frac{r}{(2n+1)b}\left(\frac{a}{2n} + 2b\right) + 4n\beta^2 + \frac{a}{b}\beta^2\right] \eta(X)\xi.\end{aligned}$$

Contracting (25), we get

$$(26) \qquad r = -2n(2n+1)\beta^2.$$

Using (26) in (25), we find

$$(27) \qquad QX = -2n\beta^2 X.$$

Using (27) in (23),we get

$$(28) \qquad R(X,Y)Z = \beta^2[g(X,Z)Y - g(Y,Z)X].$$

Thus we can state

Theorem 3 *A quasi conformally flat Lorentzian β-Kenmotsu manifold is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \beta^2$.*

5 Lorentzian β-Kenmotsu manifold satisfying $R(X,Y).C = 0$

In view of (5) and (9), we obtained from (15)

$$\begin{aligned}(29)\eta(C(X,Y)Z) = &\left[-\beta^2 + \frac{2n\beta^2}{2n-1} + \frac{r}{2n(2n-1)}\right][g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)] - \left[\frac{1}{2n-1}\right][S(Y,Z)\eta(X) - S(X,Z)\eta(Y)].\end{aligned}$$

Putting $Z=\xi$ in (29) and using (5) and (12), we get

$$\eta(C(X,Y)\xi)=0. \tag{30}$$

Again taking $X=\xi$ in (29) we have

$$\eta(C(\xi,Y)Z)=\left[\beta^2-\frac{2n\beta^2}{2n-1}-\frac{r}{2n(2n-1)}\right][g(Y,Z)+\eta(Y)\eta(Z)] + \left[\frac{1}{2n-1}\right][S(Y,Z)-2n\beta^2\eta(Y)\eta(Z)]. \tag{31}$$

Now

$$(R(X,Y)C)(U,V)Z = R(X,Y)C(U,V)Z-C(R(X,Y)U,V)Z -C(U,R(X,Y)V)Z-C(U,V)R(X,Y)Z. \tag{32}$$

By virtue of $R(X,Y).C=0$, we have

$$R(X,Y)C(U,V)Z–C(R(X,Y)U,V)Z–C(U,R(X,Y)V)Z–C(U,V)R(X,Y)Z=0.$$

Therefore, $g[R(\xi,Y)C(U,V)Z,\xi]-g[C(R(\xi,Y)U,V)Z,\xi]$
$-g[C(U,R(\xi,Y)V)Z,\xi]-g[C(U,V)R(\xi,Y)Z,\xi]=0.$

From this it follows that

$$\beta^2\grave{C}(U,V,Z,Y)+\beta^2\eta(Y)\eta(C(U,V)Z)-\beta^2\eta(U)\eta(C(Y,V)Z) +\beta^2 g(U,Y)\eta(C(\xi,V)Z)-\beta^2\eta(V)\eta(C(U,Y)Z) +\beta^2 g(Y,V)\eta(C(U,\xi)Z)-\beta^2\eta(Z)\eta(C(U,V)Y)=0. \tag{33}$$

where $\grave{C}(U,V,Z,Y)=g(C(U,V)Z,Y)$.
Putting $Y=U$ in (33), we get

$$\grave{C}(U,V,Z,U) + g(U,U)\eta(C(\xi,V)Z)+g(U,V)\eta(C(U,\xi)Z) - \eta(V)\eta(C(U,U)Z)-\eta(Z)\eta(C(U,V)U)=0. \tag{34}$$

Let $\{e_i\}$, $i=1,2,...,(2n+1)$ be an orthonormal basis of the tangent space at any point. Then the sum for $1\le i\le(2n+1)$ of the relation (34) for $U=e_i$, yields

$$\eta(C(\xi,V)Z) = \left[\frac{1}{(2n+1)(2n-1)}\right] S(V,Z) + \left[\frac{\beta^2}{2n+1} - \frac{2n\beta^2}{(2n+1)(2n-1)} - \frac{r}{2n(2n+1)(2n-1)}\right] g(V,Z) + \left[\frac{\beta^2}{2n+1} - \frac{4n\beta^2}{(2n+1)(2n-1)} - \frac{r}{2n(2n+1)(2n-1)}\right] \eta(V)\eta(Z). \tag{35}$$

From (31) and (35), we have

$$S(V,Z) = \left[\frac{r}{2n} + \beta^2\right] g(V,Z) + \left[\frac{r}{2n} + \beta^2 + 2n\beta^2\right] \eta(V)\eta(Z). \tag{36}$$

Hence we can state the following:

Theorem 4 *In a Lorentzian β-Kenmotsu manifold M, if the relation $R(X,Y)\cdot C = 0$ holds then the manifold is η-Einstein.*

References

[1] C.S. Bagewadi, D.G. Prakasha and N.S. Basavarajappa, *On Lorentzian β-Kenmotsu manifolds,* Int. Jour. Math. Analysis., 19(2), 2008, 919-927.

[2] C.S. Bagewadi and E. Girish Kumar, *Note on Trans-Sasakian manifolds,* Tensor.N.S., 65(1), 2004, 80-88.

[3] D.E. Blair and J.A. Oubina, *Conformal and related changes of metric on the product of two almost contact metric manifolds,* Publications Matematiques, 34, 1990, 199-207.

[4] D.E. Blair, *Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics,* Springer-Verlag, Berlin, 509, 1976, 146.

[5] S. Dragomir and L. Ornea, *Locally conformal Kaehler geometry,* Progress in mathematics 155, Birkhauser Boston, Inc., MA, 1998.

[6] A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants,* Ann. Mat. Pura Appl., 123(4), 1980, 35-58.

[7] K. Kenmotsu, *A class of almost contact Riemannian manifolds,* Tohoku Math.J., 24, 1972, 93-103.

[8] J.C. Marrero, *The local structure of trans-Sasakian manifolds,* Ann.Mat.Pura Appl., 162(4), 1992, 77-86.

[9] J.C. Marrero and D. Chinea, *On trans-Sasakian manifolds,* Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Vol. I-III(Spanish)(Puerto de la Cruz, 1989), 655-659, Univ. La Laguna, La Laguna, 1990.

[10] R.S. Mishra, *Almost contact metric manifolds, Monograph 1,* Tensor Society of India, Lucknow, 1991.

[11] J.A. Oubina, *New classes of contact metric structures,* Publ.Math.Debrecen, 32(3- 4), 1985, 187-193.

[12] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds,* Tohoku Math. J, 21, 1969, 21-38.

[13] M.M. Tripathi, *Trans-Sasakian manifolds are generalized quasi-Sasakian,* Nepali Math.Sci.Rep., 18(1-2), 1999-2000, 11-14.

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