General Mathematics Vol. 19, No. 1 (2011), 59–74

Uniqueness theorems of entire and meromorphic functions sharing small function [1]

Hong-Yan Xu

Abstract

In this paper, we deal with some uniqueness theorems of two transcendental meromorphic functions with their non-linear differential polynomials sharing a small function. These results in this paper improve those given by of Fang and Hong [M.L.Fang and W.Hong,A unicity theorem for entire functions concerning differential polynomials,Indian J.Pure Appl.Math.32.(2001),No.9,1343-1348.], I.Lahiri and N.Mandal [I.Lahiri and N. Mandal, Uniqueness of nonlinear differential polynomials sharing simple and double 1-points, International Journal of Mathematics and Mathematical Sciences, vol.2005 (2005), no.12, pp.1933-1942.].

2000 Mathematics Subject Classification: 30D30, 30D35.
Key words and phrases: Small function; Differential polynomials; Value shared; Weight shared.

1 Introduction and Main Results

In this paper, we use the standard notations and terms in the value distribution theory[11]. For any nonconstant meromorphic function $f(z)$ on the complex plane $\mathbb{C}$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly for a set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to f if $T(r, a) = S(r, f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f. Set $E(a(z), f) = \{z|f(z)-a(z) = 0\}, a(z) \in$

[1] *Received 17 April, 2009*
Accepted for publication (in revised form) 27 October, 2009

$S(f)$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\overline{E}(a(z), f)$. Let k be a positive integer. Set $E_k(a(z), f) = \{z : f(z) - a(z) = 0\}$, where a zero point with multiplicity $m \leq k$ is counted m times and multiplicity $m > k$ is counted $k + 1$ times in the set.

Let f and g be two transcendental meromorphic functions, $a(z) \in S(f) \cap S(g)$. If $E(a(z), f) = E(a(z), g)$, then we say that f and g share the function $a(z)CM$, especially, we say that f and g have the same fixed-points when $a(z) = z$; if $\overline{E}(a(z), f) = \overline{E}(a(z), g)$, then we say that f and g share the function $a(z)$ IM; If $E_k(a(z), f) = E_k(a(z), g)$, we say that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the multiplicities$\leq k$.

In addition, we also use the following notations.

We denote by $N_{k)}(r, f)$ the counting function for poles of f with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, f)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, f)$ be the counting function for poles of f with multiplicity $\geq k$, and by $\overline{N}_{(k}(r, f)$ be the corresponding one for which multiplicity is not counted. Set $N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \cdots + \overline{N}_{(k}(r, f)$.

Similarly, we have the notations;

$$N_{k)}(r, 1/f), \overline{N}_{k)}(r, 1/f), N_{(k}(r, 1/f), \overline{N}_{(k}(r, 1/f), N_k(r, 1/f).$$

Let f and g be two nonconstant meromorphic functions and $\overline{E}(1, f) = \overline{E}(1, g)$. We denote by $\overline{N}_L(r, 1/(f - 1))$ the counting function for 1-points of both f and g about which f has larger multiplicity than g, with multiplicity not being counted, and denote by $N_{11}(r, 1/(f - 1))$ the counting function for common simple 1-points of both f and g where multiplicity is not counted. Similarly, we have the notation $\overline{N}_L(r, 1/(g - 1))$.

In 1929, Nevanlinna proved the following well-known result, which is the so-called Nevanlinna four-value theorem.

Theorem A [9] Let f and g be two non-constant meromorphic functions. If f and g share four distinct values CM, then f is a Möbius transformation of g.

In 1979, G.Gundersen proved the following result, which is an improvement of Theorem A.

Theorem B [4] Let f and g be two non-constant meromorphic functions. If f and g share three distinct values CM and a fourth value IM, then f is a Möbius transformation of g.

In 1997, Li and Yang proved the following two results, which generalize Theorem A and B to small functions.

Theorem C [8] Let f and g be two non-constant meromorphic functions, and let $a_j(j = 1, \dots, 4)$ be distinct small functions of f and g. If f and g share $a_j(j = 1, \dots, 4)CM^*$, then f is a quasi-Möbius transformation of g.
Theorem D [8] Let f and g be two non-constant meromorphic functions, and let $a_j(j = 1, \dots, 4)$ be distinct small functions of f and g. If f and g share $a_j(j = 1, \dots, 3)CM^*$ and $a_4(z)IM$, then f is a quasi-Möbius transformation of g.

Recently, some papers studied the uniqueness of meromorphic functions and differential polynomials, and obtained some results as followed.

In 2001, Fang and Hong [2] proved the following theorem.
Theorem E [2] Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{1}{n+1}$ and $n \geq 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.

In 2005, I.Lahiri and N.Mandal [5] proved the following results, which improved the Theorem E.
Theorem F [5] Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{1}{n+1}$ and let $n(\geq 17)$ be an integer. $E_{2)}(1, f^n(f-1)f') = E_{2)}(1, g^n(g-1)g')$, then $f \equiv g$.

Question 1.1 *Is it possible that the value 1 can be replaced by a small function $a(z)$ in Theorem E and Theorem F?*

Question 1.2 *Is it possible to relax the nature of sharing a small function $a(z)$ and if possible how far?*

In this paper we answer the above questions and obtain the following results:

Theorem 1.1 *Let f and g be two transcendental meromorphic functions and $n \geq 12, k \geq 3$ be two positive integers. If $E_k(z, f^n(f-1)f') = E_k(z, g^n(g-1)g')$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

Theorem 1.2 *Let f and g be two transcendental meromorphic functions and $n(\geq 14)$ be a positive integer. If $E_2(z, f^n(f-1)f') = E_2(z, g^n(g-1)g')$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

Theorem 1.3 *Let f and g be two transcendental meromorphic functions and $n(\geq 22)$ be a positive integer. If $E_1(z, f^n(f-1)f') = E_1(z, g^n(g-1)g')$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

Theorem 1.4 *Let f and g be two transcendental meromorphic functions and $n(\geq 27)$ be a positive integer.If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z IM and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

When f and g are two transcendental entire functions, similarly we can get the following results.

Theorem 1.5 *Let f and g be two transcendental entire functions and $n \geq 8, k \geq 3$ be two positive integers. If $E_k(z, f^n(f-1)f') = E_k(z, g^n(g-1)g')$, then $f \equiv g$.*

Theorem 1.6 *Let f and g be two transcendental entire functions and $n \geq 11$ be a positive integer. If $E_2(z, f^n(f-1)f') = E_2(z, g^n(g-1)g')$, then $f \equiv g$.*

Theorem 1.7 *Let f and g be two transcendental entire functions and $n \geq 18$ be a positive integer. If $E_1(z, f^n(f-1)f') = E_1(z, g^n(g-1)g')$, then $f \equiv g$.*

Theorem 1.8 *Let f and g be two transcendental entire functions and $n \geq 22$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z IM, then $f \equiv g$.*

2 Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 [10] *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \cdots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2 [12] *Let f and g be two meromorphic functions, and let k be a positive integer, then*

$$N(r, 1/f^{(k)}) \leq N(r, 1/f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.3 [7] *Let f be a nonconstant meromorphic function and k be a positive integer. Then*

$$N_2(r, 1/f^{(k)}) \leq k\overline{N}(r, f) + N_{2+k}(r, 1/f) + S(r, f).$$

Lemma 2.4 *Let f and g be two transcendental meromorphic functions. Then $f^n(f-1)f'g^n(g-1)g' \not\equiv z^2$, where $n \geq 5$ is a positive integer.*

Proof: If possible let $f^n(f-1)f'g^n(g-1)g' \equiv z^2$. Let $z_0(\neq 0,\infty)$ be an 1-point of f with multiplicity $p(\geq 1)$. Then z_0 is a pole of g with multiplicity $q(\geq 1)$ such that $p+p-1 = nq+q+q+1$ and so $p \geq \frac{n+4}{2}$.

Let $z_1(\neq 0,\infty)$ be a zero of f with multiplicity $p(\geq 1)$ and it be a pole of g with multiplicity $q(\geq 1)$. Then $np+p-1 = nq+q+q+1$ *i.e.*,$q \geq n-1$. So $(n+1)p = (n+2)q+2$,*i.e.*, $p \geq n$.

Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we get

$$\begin{aligned} \overline{N}(r,f) &\leq \overline{N}(r,1/g) + \overline{N}(r,1/(g-1)) + \overline{N}_0(r,1/g') \\ &\leq \tfrac{1}{n}N(r,1/g) + \tfrac{2}{n+4}N(r,1/(g-1)) + \overline{N}_0(r,1/g') \\ &\leq (\tfrac{1}{n} + \tfrac{2}{n+4})T(r,g) + \overline{N}_0(r,1/g'), \end{aligned}$$

where $\overline{N}_0(r,1/g')$ is the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

By the second fundamental theorem we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,1/f) + \overline{N}(r,f) + \overline{N}(r,1/(f-1)) - \overline{N}_0(r,1/f') + S(r,f) \\ &\leq \tfrac{1}{n}N(r,1/f) + \tfrac{2}{n+4}N(r,1/(f-1)) + (\tfrac{1}{n} + \tfrac{2}{n+4})T(r,g) \\ &\quad + \overline{N}_0(r,1/g') - \overline{N}_0(r,1/f') + 2\log r + S(r,f). \end{aligned}$$

So

$$\begin{aligned} (1 - \tfrac{1}{n} - \tfrac{2}{n+4})T(r,f) &\leq (\tfrac{1}{n} + \tfrac{2}{n+4})T(r,g) + \overline{N}_0(r,1/g') \\ &\quad - \overline{N}_0(r,1/f') + 2\log r + S(r,f). \end{aligned} \tag{1}$$

Similarly we get

$$\begin{aligned} (1 - \tfrac{1}{n} - \tfrac{2}{n+4})T(r,g) &\leq (\tfrac{1}{n} + \tfrac{2}{n+4})T(r,f) + \overline{N}_0(r,1/f') \\ &\quad - \overline{N}_0(r,1/g') + 2\log r + S(r,g). \end{aligned} \tag{2}$$

Adding (1) and (2) we get

$$(1 - \frac{2}{n} - \frac{4}{n+4})\{T(r,f) + T(r,g)\} \leq 4\log r + S(r,f) + S(r,g),$$

which is a contradiction. This proves this lemma.

Lemma 2.5 *Let f and g be two transcendental meromorphic functions, $F = \frac{f^n(f-1)f'}{z}$ and $G = \frac{g^n(g-1)g'}{z}$, where $n(\geq 4)$ is a positive integer. If $F \equiv G$ and*

$$\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n+1},$$

then $f \equiv g$.

Proof: If $F \equiv G$, that is

$$F^* \equiv G^* + c \tag{3}$$

where c is a constant,

$$F^* = \frac{1}{n+2} f^{n+2} - \frac{1}{n+1} f^{n+1} \quad and \quad G^* = \frac{1}{n+2} g^{n+2} - \frac{1}{n+1} g^{n+1}.$$

If follows that

$$T(r, f) = T(r, g) + S(r, f). \tag{4}$$

Suppose that $c \neq 0$. By the second fundamental theorem,from (3) and (4) we have

$$\begin{aligned}(n+2)T(r,g) &= T(r,G^*) < \overline{N}(r,\tfrac{1}{G^*}) + \overline{N}(r,\tfrac{1}{G^*+c}) + \overline{N}(r,G^*) + S(r,g)\\ &\leq \overline{N}(r,\tfrac{1}{g}) + \overline{N}(r,\tfrac{1}{g-(n+2)/(n+1)}) + \overline{N}(r,g) + \overline{N}(r,\tfrac{1}{f})\\ &\quad + \overline{N}(r,\tfrac{1}{f-(n+2)/(n+1)}) + S(r,f) \leq 5T(r,f) + S(r,f),\end{aligned}$$

which contradicts the condition. Therefore $F^* \equiv G^*$, that is

$$f^{n+1}(\frac{1}{n+2} f - \frac{1}{n+1}) = g^{n+1}(\frac{1}{n+2} g - \frac{1}{n+1}).$$

We consider the following two case.

Case 1. Let $h = f/g$ be a constant. If $h \equiv 1$, that is $f \equiv g$. If $h \not\equiv 1$, we deduce that

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})} \qquad and \qquad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}.$$

This is a contradiction because f, g are nonconstant.

Case 2. Let $h = f/g$ be not a constant. Thus we get

$$g = \frac{n+2}{n+1}(\frac{h^{n+1}}{1+h+h^2+\cdots+h^{n+1}} - 1).$$

then we obtain by Nevanlinnas first fundamental theorem and Lemma 2.1,

$$\begin{aligned}T(r,g) &= T(r, \textstyle\sum_{j=0}^{n+1} \frac{1}{h^j}) + S(r,h) = (n+1)T(r,1/h) + S(r,h)\\ &= (n+1)T(r,h) + S(r,h).\end{aligned}$$

Now we note that a pole of h is not a pole of $[(n+2)/(n+1)][h^{n+1}/(1+h+h^2+\cdots+h^{n+1}) - 1]$. So we can get

$$\sum_{j=0}^{n+1} \overline{N}(r, \frac{1}{h-u_k}) \leq \overline{N}(r,g),$$

where $u_k = exp(2k\pi i/n)$ for $k = 1, 2, \ldots, n+1$. By the second fundamental theorem we get

$$\begin{aligned}(n-1)T(r,h) &\leq \textstyle\sum_{k=1}^{n+1} \overline{N}(r, \frac{1}{h-u_k}) + S(r,h)\\ &\leq \overline{N}(r,\infty; g) + S(r,h)\\ &< (1-\Theta(\infty,g)+\varepsilon)T(r,g) + S(r,h)\\ &= (n+1)(1-\Theta(\infty,g)+\varepsilon)T(r,h) + S(r,h),\end{aligned}$$

where $\varepsilon > 0$.

Again putting $h_1 = 1/h$, noting that $T(r,h) = T(r,h_1) + O(1)$ and proceeding as above we get

$$(n-1)T(r,h) \leq (n+1)(1-\Theta(\infty,f)+\varepsilon)T(r,h) + S(r,h),$$

where $\varepsilon > 0$. Since $\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n+1}$, there exists a $\delta(>0)$ such that $\Theta(\infty,f) + \Theta(\infty,g) > \delta + \frac{4}{n+1}$.Then we can get in view of the given condition

$$\begin{aligned}2(n-1)T(r,h) &\leq (n+1)(2-\Theta(\infty,f)-\Theta(\infty,g)+2\varepsilon)T(r,h) + S(r,h)\\ &< (n+1)(2-\tfrac{4}{n+1}-\delta+2\varepsilon)T(r,h) + S(r,h),\end{aligned}$$

and so $(\delta - 2\varepsilon)T(r,h) \leq S(r,h)$, which is a contradiction for any $\varepsilon(0 < 2\varepsilon < \delta)$. Therefore,$f \equiv g$ and so the lemma is proved.

Lemma 2.6 [1] *Let f and g be two meromorphic functions, and let k be a positive integer. If $E_k(1,f) = E_k(1,g)$, then one of the following cases must occur:*

(i)

$$\begin{aligned}T(r,f)+T(r,g) &\leq \overline{N}_2(r,f) + \overline{N}_2(r,1/f) + \overline{N}_2(r,g) + \overline{N}_2(r,1/g)\\ &\quad +\overline{N}(r,1/(f-1)) + \overline{N}(r,1/(g-1))\\ &\quad -N_{11}(r,1/(f-1)) + \overline{N}_{(k+1}(r,1/(f-1))\\ &\quad +\overline{N}_{(k+1}(r,1/(g-1)) + S(r,f) + S(r,g);\end{aligned}$$

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where $a(\neq 0), b$ are two constants.*

Lemma 2.7 [3] *Let f and g be two meromorphic functions. If f and g share 1 IM, then one of the following cases must occur:*

(i)

$$\begin{aligned}T(r,f)+T(r,g) &\leq 2[\overline{N}_2(r,f) + \overline{N}_2(r,1/f) + \overline{N}_2(r,g) + \overline{N}_2(r,1/g)]\\ &\quad +3\overline{N}_L(r,1/(f-1)) + 3\overline{N}_L(r,1/(g-1))\\ &\quad +S(r,f) + S(r,g);\end{aligned}$$

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where* $a(\neq 0), b$ *are two constants.*

Lemma 2.8 *Let* f *and* g *be two transcendental meromorphic functions,* $n \geq 7$ *be a positive integer, and let* $F = \frac{f^n(f-1)f'}{z}$ *and* $G = \frac{g^n(g-1)g'}{z}$, *If*

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{5}$$

where $a(\neq 0), b$ *are two constants and* $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$,*then* $f \equiv g$.

Proof: By lemma 2.1 we know

$$\begin{aligned} T(r,F) &= T(r, \tfrac{f^n(f-1)f'}{z}) \\ &\leq T(r, f^n(f-1)) + T(r.f') + \log r \\ &\leq (n+1)T(r,f) + 2T(r,f) + \log r + S(r,f) \\ &= (n+3)T(r,f) + \log r + S(r,f). \end{aligned} \tag{6}$$

(7)

$$\begin{aligned} (n+1)T(r,f) &= T(r, f^n(f-1)) + S(r,f) \\ &= N(r, f^n(f-1)) + m(r, f^n(f-1)) + S(r,f) \\ &\leq N(r, \tfrac{f^n(f-1)f'}{z}) - N(r,f') + m(r, \tfrac{f^n(f-1)f'}{z}) + m(r, 1/f') \\ &\quad + \log r + S(r,f) \\ &\leq T(r, \tfrac{f^n(f-1)f'}{z}) + T(r,f') - N(r,f') - N(r,1/f') \\ &\quad + \log r + S(r,f) \\ &\leq T(r,F) + T(r,f) - N(r,f) - N(r,1/f') + \log r + S(r,f). \end{aligned}$$

So

$$T(r,F) \geq nT(r,f) + N(r,f) + N(r,1/f') + \log r + S(r,f). \tag{8}$$

Thus, by (6),(8) and $n \geq 7$, we get $S(r,F) = S(r,f)$. Similarly, we get

$$T(r,G) \geq nT(r,g) + N(r,g) + N(r,1/g') + \log r + S(r,g). \tag{9}$$

Without loss of generality, we suppose that $T(r,f) \leq T(r,g), r \in I$, where I is a set with infinite measure. Next, we consider three cases.

Case 1. $b \neq 0, -1$, If $a - b - 1 \neq 0$, then by (5) we know

$$\overline{N}(r, \frac{1}{G + \frac{a-b-1}{b+1}}) = \overline{N}(r, \frac{1}{F}).$$

By the Nevanlinna second fundamental theorem and lemma 2.2 we have

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,\tfrac{1}{G+\frac{a-b-1}{b+1}})+S(r,G)\\
&= \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,\tfrac{1}{F})+S(r,g)\\
&\leq \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+T(r,g)+\overline{N}(r,\tfrac{1}{g'})+\log r\\
&\quad +\overline{N}(r,\tfrac{1}{f})+T(r,f)+N(r,\tfrac{1}{f})+\overline{N}(r,f)+\log r+S(r,g)\\
&\leq 2T(r,g)+\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g'})+\log r+2N(r,\tfrac{1}{f})\\
&\quad +T(r,f)+\overline{N}(r,f)+\log r+S(r,g)\\
&\leq 6T(r,g)+\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g'})+2\log r+S(r,g).
\end{aligned}$$

Hence, by $n\geq 7$ and (9), we know $T(r,g)\leq S(r,g), r\in I$, This is impossible.

If $a-b-1=0$, then by (5) we know $F=((b+1)G)/(bG+1)$. Obviously,

$$\overline{N}(r,\frac{1}{G+\frac{1}{b}})=\overline{N}(r,F).$$

By the Nevanlinna second fundamental theorem and lemma 2.2 we have

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,\tfrac{1}{G+\frac{1}{b}})+S(r,G)\\
&= \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,F)+S(r,g)\\
&\leq \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g})+T(r,g)+\overline{N}(r,\tfrac{1}{g'})+\log r+\overline{N}(r,f)\\
&\quad +\log r+S(r,g)\\
&\leq 2T(r,g)+\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g'})+T(r,f)+2\log r+S(r,g)\\
&\leq 3T(r,g)+\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{g'})+2\log r+S(r,g).
\end{aligned}$$

Then by $n\geq 7$ and (9), we know $T(r,g)\leq S(r,g), r\in I$, a contradiction.

Case 2. $b=-1$. Then (5) becomes $F=a/(a+1-G)$.

If $a+1\neq 0$, then $\overline{N}(r,1/(G-a-1))=\overline{N}(r,F)$. Similarly, we can deduce a contradiction as in Case 1.

If $a+1=0$, then $FG\equiv 1$, that is,

$$f^n(f-1)f'g^n(g-1)g'\equiv z^2.$$

Since $n\geq 7$, by lemma 2.4, a contradiction.

Case 3. $b=0$. Then (5) becomes $F=(G+a-1)/a$.

If $a-1\neq 0$, then $\overline{N}(r,1/(G+a-1))=\overline{N}(r,1/F)$. Similarly, we can again deduce a contradiction as in Case 1.

If $a-1=0$, then $F\equiv G$, that is

$$f^n(f-1)f'\equiv g^n(g-1)g'.$$

By the lemma 2.4 and lemma 2.5, we obtain $f\equiv g$.

This completes the proof of this lemma.

3 The Proofs of Theorems

Let F and G be defined as in Lemma 2.8.

The Proof of Theorem 1.1: Since $k \geq 3$, we have

$$\begin{aligned} & \overline{N}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) - N_{11}(r, \tfrac{1}{F-1}) + \overline{N}_{(k+1}(r, \tfrac{1}{F-1}) + \overline{N}_{(k+1}(r, \tfrac{1}{G-1}) \\ \leq & \tfrac{1}{2}N(r, \tfrac{1}{F-1}) + \tfrac{1}{2}N(r, \tfrac{1}{G-1}) \\ \leq & \tfrac{1}{2}T(r, F) + \tfrac{1}{2}T(r, G) + S(r, f) + S(r, g). \end{aligned}$$

Then (i) in Lemma 2.6 becomes

$$T(r,F)+T(r,G) \leq 2\{N_2(r, \frac{1}{F})+N_2(r,F)+N_2(r, \frac{1}{G})+N_2(r,G)\}+S(r,f)+S(r,g).$$

Since

$$\begin{aligned} N_2(r, \tfrac{1}{F}) + N_2(r, F) &= N_2(r, \tfrac{z}{f^n(f-1)f'}) + N_2(r, \tfrac{f^n(f-1)f'}{z}) \\ &\leq 2\overline{N}(r, \tfrac{1}{f}) + N_2(r, \tfrac{1}{f-1}) + N(r, \tfrac{1}{f'}) \\ &\quad +2\overline{N}(r, f) + 2\log r. \end{aligned} \tag{10}$$

Similarly, we obtain

$$\begin{aligned} & N_2(r, \tfrac{1}{G}) + N_2(r, G) \\ \leq & 2\overline{N}(r, \tfrac{1}{g}) + N_2(r, \tfrac{1}{g-1}) + N(r, \tfrac{1}{g'}) + 2\overline{N}(r, g) + 2\log r. \end{aligned} \tag{11}$$

Suppose that

$$\begin{aligned} T(r, F) + T(r, G) \leq & 2\{N_2(r, \tfrac{1}{F}) + N_2(r, F) + N_2(r, \tfrac{1}{G}) \\ & +N_2(r, G)\} + S(r, f) + S(r, g). \end{aligned} \tag{12}$$

By Lemma 2.2-2.3 and (10)-(12), we get

$$\begin{aligned} & T(r, F) + T(r, G) \\ \leq & 4\overline{N}(r, \tfrac{1}{f}) + 2N_2(r, \tfrac{1}{f-1}) + 2N(r, \tfrac{1}{f'}) + 4\overline{N}(r, f) \\ & +4\overline{N}(r, \tfrac{1}{g}) + 2N_2(r, \tfrac{1}{g-1}) + 2N(r, \tfrac{1}{g'}) + 4\overline{N}(r, g) \\ & +8\log r + S(r, f) + S(r, g) \\ \leq & 5N(r, \tfrac{1}{f}) + 2N_2(r, \tfrac{1}{f-1}) + N(r, \tfrac{1}{f'}) + 5\overline{N}(r, f) \\ & +5N(r, \tfrac{1}{g}) + 2N_2(r, \tfrac{1}{g-1}) + N(r, \tfrac{1}{g'}) + 5\overline{N}(r, g) \\ & +8\log r + S(r, f) + S(r, g) \\ \leq & 11T(r, f) + \overline{N}(r, f) + N(r, \tfrac{1}{f'}) + S(r, f) + 11T(r, g) \\ & +\overline{N}(r, g) + N(r, \tfrac{1}{g'}) + 8\log r + S(r, g). \end{aligned} \tag{13}$$

By $n \geq 12$ and (8),(9), we can obtain a contradiction.

Thus, by lemma 2.6, $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a(\neq 0), b$ are two constants. By lemma 2.8, we get $f \equiv g$.

This completes the proof of Theorem 1.1.

The Proof of Theorem 1.2: Obviously, we have

$$\begin{aligned} & \overline{N}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) - N_{11}(r, \tfrac{1}{F-1}) + \tfrac{1}{2}\overline{N}_{(3}(r, \tfrac{1}{F-1}) + \tfrac{1}{2}\overline{N}_{(3}(r, \tfrac{1}{G-1}) \\ \leq\ & \tfrac{1}{2}N(r, \tfrac{1}{F-1}) + \tfrac{1}{2}N(r, \tfrac{1}{G-1}) \\ \leq\ & \tfrac{1}{2}T(r, F) + \tfrac{1}{2}T(r, G) + S(r, f) + S(r, g). \end{aligned}$$

Considering

$$\begin{aligned} \overline{N}_{(3}(r, \tfrac{1}{F-1}) &\leq \tfrac{1}{2}N(r, \tfrac{F}{F'}) = \tfrac{1}{2}N(r, \tfrac{F'}{F}) + S(r, f) \\ &\leq \tfrac{1}{2}\overline{N}(r, F) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{F}) + S(r, f) \\ &\leq \tfrac{1}{2}[\overline{N}(r, \tfrac{1}{f}) + \overline{N}(r, \tfrac{1}{f-1}) + N(r, \tfrac{1}{f'}) + \overline{N}(r, f)] \\ &\quad + \log r + S(r, f) \\ &\leq \tfrac{5}{2}T(r, f) + \log r + S(r, f). \end{aligned} \tag{14}$$

Then (i) in Lemma 2.6 becomes

$$\begin{aligned} T(r, F) + T(r, G) \leq\ & 2\{N_2(r, \tfrac{1}{F}) + N_2(r, F) + N_2(r, \tfrac{1}{G}) + N_2(r, G)\} \\ & + \overline{N}_{(3}(r, \tfrac{1}{F-1}) + \overline{N}_{(3}(r, \tfrac{1}{G-1}) + S(r, f) + S(r, g). \end{aligned}$$

Similarly, we get

$$\overline{N}_{(3}(r, \frac{1}{G-1}) \leq \frac{5}{2}T(r, g) + \log r + S(r, g). \tag{15}$$

Suppose that

$$\begin{aligned} & T(r, F) + T(r, G) \\ \leq\ & 2\{N_2(r, \tfrac{1}{F}) + N_2(r, F) + N_2(r, \tfrac{1}{G}) + N_2(r, G)\} + \overline{N}_{(3}(r, \tfrac{1}{F-1}) \\ & + \overline{N}_{(3}(r, \tfrac{1}{G-1}) + S(r, f) + S(r, g). \end{aligned} \tag{16}$$

Combining (10),(11) and (14)-(16), we can get

$$\begin{aligned} T(r, F) + T(r, G) \leq\ & \tfrac{27}{2}T(r, f) + \overline{N}(r, f) + N(r, \tfrac{1}{f'}) + S(r, f) + \tfrac{27}{2}T(r, g) \\ & + \overline{N}(r, g) + N(r, \tfrac{1}{g'}) + 10\log r + S(r, g). \end{aligned}$$

From $n \geq 14$ and (8),(9), we can get a contradiction.

By Lemma 2.6, we obtain $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a(\neq 0), b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.2.

The Proof of Theorem 1.3: Similarly, we get

$$\begin{aligned} & \overline{N}(r, \tfrac{1}{F-1}) + \overline{N}(r, \tfrac{1}{G-1}) - N_{11}(r, \tfrac{1}{F-1}) \\ \leq\ & \tfrac{1}{2}N(r, \tfrac{1}{F-1}) + \tfrac{1}{2}N(r, \tfrac{1}{G-1}) \\ \leq\ & \tfrac{1}{2}T(r, F) + \tfrac{1}{2}T(r, G) + S(r, f) + S(r, g). \end{aligned}$$

Then (i) in Lemma 2.6 becomes

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & 2\{N_2(r,\tfrac{1}{F})+N_2(r,F)+N_2(r,\tfrac{1}{G})+N_2(r,G)+ \\ & \overline{N}_{(2}(r,\tfrac{1}{F-1})+\overline{N}_{(2}(r,\tfrac{1}{G-1})\}+S(r,f)+S(r,g). \end{aligned}$$

Considering

$$\begin{aligned} \overline{N}_{(2}(r,\tfrac{1}{F-1}) &\leq N(r,\tfrac{F}{F'})=N(r,\tfrac{F'}{F})+S(r,f) \\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+S(r,f) \\ &\leq 5T(r,f)+2\log r+S(r,f). \end{aligned} \tag{17}$$

Similarly, we have

$$\overline{N}_{(2}(r,\frac{1}{G-1}) \leq 5T(r,g)+2\log r+S(r,g). \tag{18}$$

Suppose that

$$\begin{aligned} & T(r,F)+T(r,G) \\ \leq\ & 2\{N_2(r,\tfrac{1}{F})+N_2(r,F)+N_2(r,\tfrac{1}{G})+N_2(r,G)+\overline{N}_{(2}(r,\tfrac{1}{F-1}) \\ & +\overline{N}_{(2}(r,\tfrac{1}{G-1})\}+S(r,f)+S(r,g) \end{aligned} \tag{19}$$

Considering (10),(11),(13) and (17)-(19), we know

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & 21T(r,f)+\overline{N}(r,f)+N(r,\tfrac{1}{f'})+S(r,f)+21T(r,g) \\ & +\overline{N}(r,g)+N(r,\tfrac{1}{g'})+12\log r+S(r,g). \end{aligned}$$

By $n \geq 22$ and (8),(9), we get a contradiction.

Applying Lemma 2.6, we know $F=((b+1)G+(a-b-1))/(bG+(a-b))$, where $a(\neq 0), b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.3.

The Proof of Theorem 1.4: Since

$$\begin{aligned} \overline{N}_{L}(r,\tfrac{1}{F-1}) &\leq N(r,\tfrac{F}{F'})=N(r,\tfrac{F'}{F})+S(r,f) \\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+S(r,f) \\ &\leq 5T(r,f)+2\log r+S(r,f). \end{aligned} \tag{20}$$

Similarly, we have

$$\overline{N}_{L}(r,\frac{1}{G-1}) \leq 5T(r,g)+2\log r+S(r,g). \tag{21}$$

Suppose that F and G satisfied (i) in Lemma 2.7, then we get

$$\begin{aligned} & T(r,F)+T(r,G) \\ \leq\ & 2\{N_2(r,\tfrac{1}{F})+N_2(r,F)+N_2(r,\tfrac{1}{G})+N_2(r,G)\}+3\overline{N}_{L}(r,\tfrac{1}{F-1}) \\ & +3\overline{N}_{L}(r,\tfrac{1}{G-1})+S(r,f)+S(r,g). \end{aligned} \tag{22}$$

Considering (10),(11),(13) and (20)-(22), we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq 26T(r,f)+\overline{N}(r,f)+N(r,\tfrac{1}{f'})+S(r,f)+26T(r,g) \\ &\quad +\overline{N}(r,g)+N(r,\tfrac{1}{g'})+20\log r+S(r,g). \end{aligned}$$

From $n \geq 27$ and (8),(9), we get a contradiction.

Applying Lemma 2.7, we know $F=((b+1)G+(a-b-1))/(bG+(a-b))$, where $a(\neq 0), b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.4.

The Proof of Theorem 1.5: Since $k \geq 3$, we have

$$\begin{aligned} &\overline{N}(r,\tfrac{1}{F-1})+\overline{N}(r,\tfrac{1}{G-1})-N_{11}(r,\tfrac{1}{F-1})+\overline{N}_{(k+1}(r,\tfrac{1}{F-1})+\overline{N}_{(k+1}(r,\tfrac{1}{G-1}) \\ \leq\ & \tfrac{1}{2}N(r,\tfrac{1}{F-1})+\tfrac{1}{2}N(r,\tfrac{1}{G-1}) \\ \leq\ & \tfrac{1}{2}T(r,F)+\tfrac{1}{2}T(r,G)+S(r,f)+S(r,g). \end{aligned}$$

Since

$$\begin{aligned} N_2(r,\tfrac{1}{F})+N_2(r,F) &= N_2(r,\tfrac{z}{f^n(f-1)f'})+N_2(r,\tfrac{f^n(f-1)f'}{z}) \\ &\leq 2\overline{N}(r,\tfrac{1}{f})+N_2(r,\tfrac{1}{f-1})+N(r,\tfrac{1}{f'})+2\log r. \end{aligned} \tag{23}$$

Similarly, we obtain

$$N_2(r,\frac{1}{G})+N_2(r,G) \leq 2\overline{N}(r,\frac{1}{g})+N_2(r,\frac{1}{g-1})+N(r,\frac{1}{g'})+2\log r. \tag{24}$$

Suppose that F and G satisfied (i) in Lemma 2.6, then we get

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\{N_2(r,\tfrac{1}{F})+N_2(r,F)+N_2(r,\tfrac{1}{G}) \\ &\quad +N_2(r,G)\}+S(r,f)+S(r,g). \end{aligned} \tag{25}$$

By Lemma 2.2-2.3 and (23)-(25), we get

$$\begin{aligned} & T(r,F)+T(r,G) \\ \leq\ & 4\overline{N}(r,\tfrac{1}{f})+2N_2(r,\tfrac{1}{f-1})+2N(r,\tfrac{1}{f'})+4\overline{N}(r,\tfrac{1}{g})+2N_2(r,\tfrac{1}{g-1}) \\ & +2N(r,\tfrac{1}{g'})+8\log r+S(r,f)+S(r,g) \\ \leq\ & 5N(r,\tfrac{1}{f})+2N_2(r,\tfrac{1}{f-1})+N(r,\tfrac{1}{f'})+5N(r,\tfrac{1}{g})+2N_2(r,\tfrac{1}{g-1}) \\ & +N(r,\tfrac{1}{g'})+8\log r+S(r,f)+S(r,g) \\ \leq\ & 7T(r,f)+N(r,\tfrac{1}{f'})+S(r,f)+7T(r,g)+N(r,\tfrac{1}{g'}) \\ & +8\log r+S(r,g). \end{aligned} \tag{26}$$

Noting that

$$T(r,F) \geq nT(r,f)+N(r,1/f')+\log r+S(r,f). \tag{27}$$

$$(28)\qquad T(r,G) \geq nT(r,g) + N(r,1/g') + \log r + S(r,g).$$

By $n \geq 8$ and (27),(28), we can obtain a contradiction.

Thus, by Lemma 2.7, $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a(\neq 0), b$ are two constants. By using the same argument as in Lemma 2.8 combining f and g are two transcendental entire functions, we get $f \equiv g$.This completes the proof of Theorem 1.5.

Similarly,we can use the analogue method of Theorem 1.5 to prove the Theorem 1.6-1.8 easily. Here we omit the details.

4 Remarks

It follows from the proof of Theorem 1.1-1.8 that if "z" is replaced by "$a(z)$" in the Theorem 1.1-1.8, where $a(z)$ is a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r,a) = o\{T(r,f), T(r,g)\}$, then the conclusions of Theorem 1.1-1.8 still hold. So we obtain the following results.

Theorem 4.1 *Let f and g be two transcendental meromorphic functions and $n \geq 12, k \geq 3$ be two positive integers. If $E_k(a(z), f^n(f-1)f') = E_k(a(z), g^n(g-1)g')$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$,then $f \equiv g$.*

Theorem 4.2 *Let f and g be two transcendental meromorphic functions and $n(\geq 14)$ be a positive integer. If $E_2(a(z), f^n(f-1)f') = E_2(a(z), g^n(g-1)g')$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

Theorem 4.3 *Let f and g be two transcendental meromorphic functions and $n(\geq 22)$ be a positive integer. If $E_1(a(z), f^n(f-1)f') = E_1(a(z), g^n(g-1)g')$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

Theorem 4.4 *Let f and g be two transcendental meromorphic functions and $n(\geq 27)$ be a positive integer.If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $a(z)$ IM and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1}$, then $f \equiv g$.*

Theorem 4.5 *Let f and g be two transcendental entire functions and $n \geq 8, k \geq 3$ be two positive integers. If $E_k(a(z), f^n(f-1)f') = E_k(a(z), g^n(g-1)g')$, then $f \equiv g$.*

Theorem 4.6 *Let f and g be two transcendental entire functions and $n \geq 11$ be a positive integer. If $E_2(a(z), f^n(f-1)f') = E_2(a(z), g^n(g-1)g')$, then $f \equiv g$.*

Theorem 4.7 *Let f and g be two transcendental entire functions and $n \geq 18$ be a positive integer. If $E_1(a(z), f^n(f-1)f') = E_1(a(z), g^n(g-1)g')$, then $f \equiv g$.*

Theorem 4.8 *Let f and g be two transcendental entire functions and $n \geq 22$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $a(z)$ IM, then $f \equiv g$.*

Obviously, we can use the analogue method of Theorem 1.1-1.8 to prove the Theorem 4.1-4.8 easily. Here, we omit them.

5 Acknowledgements

The author want to thanks the referee for his/her thorough review. The research was supported by the Technological Research Projects of Jiangxi Province ([2008]147) and Research Projects of Jingdezhen Ceramic Institute ([2009]85).

References

[1] C.-Y. Fang, M.-L. Fang, Uniqueness of meromorphic functions and differential polynomials, Comput.Math.Appl. 44(2002), No.5-6,607-617.

[2] M.-L.Fang, W.Hong, A unicity theorem for entire functions concerning differential polynomials, Indian J.Pure Appl. Math. 32. (2001), No.9, 1343-1348.

[3] M.-L. Fang, H. Guo, On unique range sets for meromorphic or entire functions, Acta Mathematica Sinica, New Series 14(4)(1998), 569-576.

[4] G.G. Gundersen, Meromorphic functions that share three or four values, J.London.Math.Soc. 20(1979),457-466.

[5] I. Lahiri, N. Mandal, Uniqueness of nonlinear differential polynomials sharing simple and double 1-points, International Journal of Mathematics and Mathematical Sciences, vol.2005(2005), no.12, pp.1933-1942.

[6] I. Lahiri, R. Pal, Non-linear differential polynomials sharing 1-points, Bull.Korean.Math.Soc. 43(2006), No.1,161-168.

[7] I. Lahiri, A. Sarkar, Uniqueness of a meromorphic function and its derivative, J.Inequal.Pure Appl.Math. 5(2004), No.1, Article 20 (electronic).

[8] P. Li, C.C. Yang, On two meromorphic functions that share pairs of small functions, Complex Variables 32 (1997), 177-190.

[9] R. Nevanlinna, Le théorème de Picard-Borel et la theéorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929.

[10] C.C. Yang, On deficiencies of differential polynomials *II*, Math.Z. 125 (1972), 107-112.

[11] L. Yang, Value distribution theory.Springer-Verlag. Berlin(1993).

[12] H.-X. Yi, C.C. Yang, Uniqueness theory of meromorphic functions, Science Press, Beijing. (1995).

Hong-Yan Xu
Department of Informatics and Engineering
Jingdezhen Ceramic Institute(XiangHu XiaoQu)
Jingdezhen,Jiangxi 333403, China
e-mail: xhyhhh@126.com