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Solution of a Recursive Sequence of Order Ten [1]

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Abstract

We obtain in this paper the solutions of the following rational non-linear difference equations

$$x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-4}x_{n-9}}, \quad n = 0, 1, ...,$$

where initial values are non zero real numbers.

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1 Introduction

The study of Difference Equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

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Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example: Aloqeili [1] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [3–5] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Cinar et al.[6] studied the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [8] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

Elabbasy et al. [9] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [10] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Karatas et al. [31] obtained the solution of the difference equation

$$x_{n+1} = \frac{a x_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

Simsek et al. [35] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

In [36] Stevic solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}.$$

Other related results on rational difference equations can be found in refs. [2], [7], [11-40].

Our aim in this paper is to investigate the solution of the following nonlinear difference equations

$$x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-4}x_{n-9}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial values $x_{-j},\ (j = 0, 1, ..., 9)$ are arbitrary non zero real numbers.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 *A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if*

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2 *(Periodicity)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2 MAIN RESULTS

2.1 On the Difference Equation $x_{n+1} = \dfrac{x_{n-9}}{1 + x_{n-4}x_{n-9}}$

In this section we give a specific form of the first equation in the form

$$x_{n+1} = \frac{x_{n-9}}{1 + x_{n-4}x_{n-9}}, \quad n = 0, 1, ..., \tag{3}$$

where the initial values are arbitrary non zero real numbers.

Theorem 1 *Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq.(3). Then for $n = 0, 1, ...$*

$$\begin{aligned}
x_{10n-9} &= p_{i=0}^{n-1}\left(\frac{1+2ipe}{1+(2i+1)\,pe}\right), \qquad x_{10n-8} = k_{i=0}^{n-1}\left(\frac{1+2ikd}{1+(2i+1)\,kd}\right), \\
x_{10n-7} &= h_{i=0}^{n-1}\left(\frac{1+2ihc}{1+(2i+1)\,hc}\right), \qquad x_{10n-6} = g_{i=0}^{n-1}\left(\frac{1+2igb}{1+(2i+1)\,gb}\right), \\
x_{10n-5} &= f_{i=0}^{n-1}\left(\frac{1+2ifa}{1+(2i+1)\,fa}\right), \qquad x_{10n-4} = e_{i=0}^{n-1}\left(\frac{1+(2i+1)pe}{1+(2i+2)\,pe}\right),
\end{aligned}$$

$$\begin{aligned}
x_{10n-3} &= d_{i=0}^{n-1}\left(\frac{1+(2i+1)\,kd}{1+(2i+2)\,kd}\right), \qquad x_{10n-2} = c_{i=0}^{n-1}\left(\frac{1+(2i+1)hc}{1+(2i+2)\,hc}\right), \\
x_{10n-1} &= b_{i=0}^{n-1}\left(\frac{1+(2i+1)gb}{1+(2i+2)\,gb}\right), \qquad x_{10n} = a_{i=0}^{n-1}\left(\frac{1+(2i+1)fa}{1+(2i+2)\,fa}\right),
\end{aligned}$$

where $x_{-9} = p,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{10n-19} &= p_{i=0}^{n-2}\left(\frac{1+2ipe}{1+(2i+1)\,pe}\right), \qquad x_{10n-18} = k_{i=0}^{n-2}\left(\frac{1+2ikd}{1+(2i+1)\,kd}\right), \\
x_{10n-17} &= h_{i=0}^{n-2}\left(\frac{1+2ihc}{1+(2i+1)\,hc}\right), \qquad x_{10n-16} = g_{i=0}^{n-2}\left(\frac{1+2igb}{1+(2i+1)\,gb}\right), \\
x_{10n-15} &= f_{i=0}^{n-2}\left(\frac{1+2ifa}{1+(2i+1)\,fa}\right), \qquad x_{10n-14} = e_{i=0}^{n-2}\left(\frac{1+(2i+1)pe}{1+(2i+2)\,pe}\right), \\
x_{10n-13} &= d_{i=0}^{n-2}\left(\frac{1+(2i+1)\,kd}{1+(2i+2)\,kd}\right), \qquad x_{10n-12} = c_{i=0}^{n-2}\left(\frac{1+(2i+1)hc}{1+(2i+2)\,hc}\right), \\
x_{10n-11} &= b_{i=0}^{n-2}\left(\frac{1+(2i+1)gb}{1+(2i+2)\,gb}\right), \qquad x_{10n-10} = a_{i=0}^{n-2}\left(\frac{1+(2i+1)fa}{1+(2i+2)\,fa}\right).
\end{aligned}$$

Now, it follows from Eq.(3) that

$$\begin{aligned}
x_{10n-9} &= \frac{x_{10n-19}}{1+x_{10n-14}x_{10n-19}} \\
&= \frac{p_{i=0}^{n-2}\left(\dfrac{1+2ipe}{1+(2i+1)\,pe}\right)}{1+e_{i=0}^{n-2}\left(\dfrac{1+(2i+1)pe}{1+(2i+2)\,pe}\right)p_{i=0}^{n-2}\left(\dfrac{1+2ipe}{1+(2i+1)\,pe}\right)} \\
&= \frac{p_{i=0}^{n-2}\left(\dfrac{1+2ipe}{1+(2i+1)\,pe}\right)}{1+\left(\dfrac{pe}{1+(2n-2)\,pe}\right)} = \frac{p_{i=0}^{n-2}\left(\dfrac{1+2ipe}{1+(2i+1)\,pe}\right)}{\left(\dfrac{1+(2n-1)\,pe}{1+(2n-2)\,pe}\right)}.
\end{aligned}$$

Hence, we have

$$x_{10n-9} = p_{i=0}^{n-1}\left(\frac{1+2ipe}{1+(2i+1)\,pe}\right).$$

Similarly

$$\begin{aligned}
x_{10n-5} &= \frac{x_{10n-15}}{1+x_{10n-10}x_{10n-15}} \\
&= \frac{f_{i=0}^{n-2}\left(\dfrac{1+2ifa}{1+(2i+1)\,fa}\right)}{1+a_{i=0}^{n-2}\left(\dfrac{1+(2i+1)fa}{1+(2i+2)\,fa}\right)f_{i=0}^{n-2}\left(\dfrac{1+2ifa}{1+(2i+1)\,fa}\right)} \\
&= \frac{f_{i=0}^{n-2}\left(\dfrac{1+2ifa}{1+(2i+1)\,fa}\right)}{1+\left(\dfrac{fa}{1+(2n-2)\,fa}\right)}\left(\frac{1+(2n-2)\,fa}{1+(2n-2)\,fa}\right) \\
&= f_{i=0}^{n-2}\left(\frac{1+2ifa}{1+(2i+1)\,fa}\right)\frac{(1+(2n-2)\,fa)}{(1+(2n-1)\,fa)}.
\end{aligned}$$

Hence, we have

$$x_{10n-5} = f_{i=0}^{n-1}\left(\frac{1+2ifa}{1+(2i+1)\,fa}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2 *Eq.(3) has one equilibrium point which is the zero.*

Proof: For the equilibrium points of Eq.(3), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^2}.$$

Then we have

$$\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}^3 = 0.$$

Thus the equilibrium point of Eq.(3) is $\overline{x} = 0$.

Theorem 3 *Every positive solution of Eq.(3) is bounded.*

Proof: Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq.(3). It follows from Eq.(3) that

$$x_{n+1} = \frac{x_{n-9}}{1 + x_{n-4}x_{n-9}} \leq x_{n-9}.$$

Then

$$x_{n+1} \leq x_{n-9} \quad \text{for all} \quad n \geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so are bounded from above by $M = \max\{x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (3).

Example 1. We consider $x_{-9} = 1.2,\ x_{-8} = 11,\ x_{-7} = 6,\ x_{-6} = 8,\ x_{-5} = 0.4,\ x_{-4} = 0.2, x_{-3} = 13,\ x_{-2} = 9,\ x_{-1} = 7,\ x_0 = 5$ See Fig. 1.

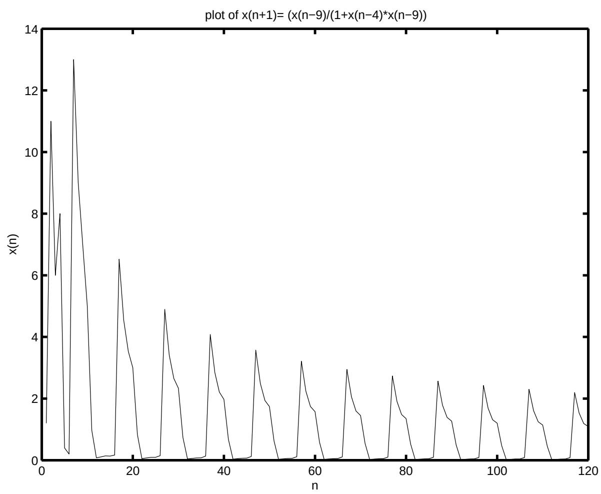

Figure 1.

Example 2. See Fig. 2, since $x_{-9} = 9,\ x_{-8} = 7,\ x_{-7} = 6,\ x_{-6} = 0.3,\ x_{-5} = 4,\ x_{-4} = -1.7,\ x_{-3} = -3,\ x_{-2} = -1.9,\ x_{-1} = 9,\ x_0 = -3.$

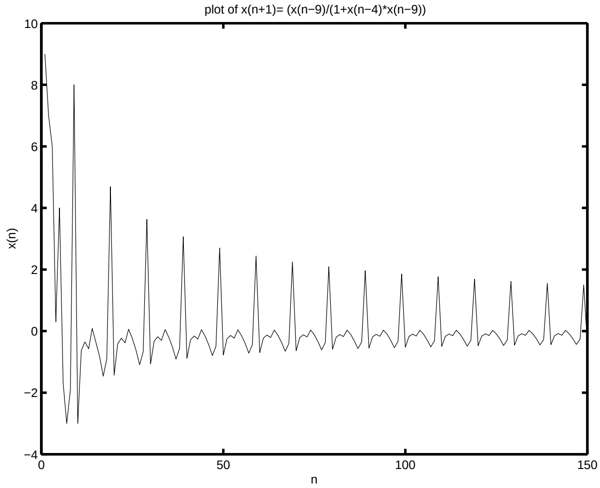

Figure 2.

2.2 On the Difference Equation $x_{n+1} = \dfrac{x_{n-9}}{-1 + x_{n-4}x_{n-9}}$

In this section we obtain the solution of the second equation in the form

$$x_{n+1} = \frac{x_{n-9}}{-1 + x_{n-4}x_{n-9}}, \quad n = 0, 1, ..., \tag{4}$$

where the initial values are arbitrary non zero real numbers with $x_{-9}x_{-4} \neq 1,\ x_{-8}x_{-3} \neq 1,\ x_{-7}x_{-2} \neq 1,\ x_{-6}x_{-1} \neq 1,\ x_{-5}x_0 \neq 1.$

Theorem 4 *Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq.(4). Then for $n = 0, 1, ...$*

$$\begin{aligned}
x_{10n-9} &= \frac{p}{(-1+pe)^n}, \qquad x_{10n-8} = \frac{k}{(-1+kd)^n}, \\
x_{10n-7} &= \frac{h}{(-1+hc)^n}, \qquad x_{10n-6} = \frac{g}{(-1+gb)^n}, \\
x_{10n-5} &= \frac{f}{(-1+fa)^n}, \qquad x_{10n-4} = e\left(-1+pe\right)^n, \\
x_{10n-3} &= d\left(-1+kd\right)^n, \qquad x_{10n-2} = c\left(-1+hc\right)^n, \\
x_{10n-1} &= b\left(-1+gb\right)^n, \qquad x_{10n} = a\left(-1+fa\right)^n,
\end{aligned}$$

where $x_{-9} = p,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{10n-19} &= \frac{p}{(-1+pe)^{n-1}}, \qquad x_{10n-18} = \frac{k}{(-1+kd)^{n-1}}, \\
x_{10n-17} &= \frac{h}{(-1+hc)^{n-1}}, \qquad x_{10n-16} = \frac{g}{(-1+gb)^{n-1}}, \\
x_{10n-15} &= \frac{f}{(-1+fa)^{n-1}}, \qquad x_{10n-14} = e\left(-1+pe\right)^{n-1}, \\
x_{10n-13} &= d\left(-1+kd\right)^{n-1}, \qquad x_{10n-12} = c\left(-1+hc\right)^{n-1}, \\
x_{10n-11} &= b\left(-1+gb\right)^{n-1}, \qquad x_{10n-10} = a\left(-1+fa\right)^{n-1}.
\end{aligned}$$

Now, it follows from Eq.(4) that

$$\begin{aligned}
x_{10n-9} &= \frac{x_{10n-19}}{-1+x_{10n-14}x_{10n-19}} = \frac{\frac{p}{(-1+pe)^{n-1}}}{-1+e\left(-1+pe\right)^{n-1}\frac{p}{(-1+pe)^{n-1}}} \\
&= \frac{p}{(-1+pe)^{n-1}\left(-1+pe\right)}.
\end{aligned}$$

Hence, we have

$$x_{10n-9} = \frac{p}{(-1+pe)^{n}}.$$

Similarly

$$\begin{aligned}
x_{10n-3} &= \frac{x_{10n-13}}{-1+x_{10n-8}x_{10n-13}} = \frac{d\left(-1+kd\right)^{n-1}}{-1+\frac{k}{(-1+kd)^{n}}d\left(-1+kd\right)^{n-1}} \\
&= \frac{d\left(-1+kd\right)^{n-1}}{-1+\frac{kd}{(-1+kd)}} = \frac{d\left(-1+kd\right)^{n}}{-1\left(-1+kd\right)+kd}.
\end{aligned}$$

Hence, we have

$$x_{10n-3} = d\left(-1+kd\right)^{n}.$$

Similarly, one can easily prove the other relations. Thus, the proof is completed.

Theorem 5 *Eq.(4) has three equilibrium points which are* $0, \sqrt{2}, -\sqrt{2}$.

Proof: For the equilibrium points of Eq.(4), we can write

$$\overline{x} = \frac{\overline{x}}{-1+\overline{x}^2}.$$

Then we have

$$-\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}(\overline{x}^2 - 2) = 0.$$

Thus the equilibrium points of Eq.(4) are $0, \sqrt{2}, -\sqrt{2}$.

Lemma 1 It is easy to see that every solution of Eq.(4) is unbounded except in the following case.

Theorem 6 *Eq.(4) has a periodic solutions of period ten iff* $pe = kd = hc = gb = fa = 2$ *and will be take the form* $\{p, k, h, g, f, e, d, c, b, a, p, k, h, g, f, e, d, c, b, a, ...\}$.

Proof: First suppose that there exists a prime period ten solution

$$p, k, h, g, f, e, d, c, b, a, p, k, h, g, f, e, d, c, b, a, ...,$$

of Eq.(4), we see from the form of solution of Eq.(4) that

$$\begin{aligned}
p &= \frac{p}{(-1+pe)^n}, \qquad k = \frac{k}{(-1+kd)^n},\\
h &= \frac{h}{(-1+hc)^n}, \qquad g = \frac{g}{(-1+gb)^n},\\
f &= \frac{f}{(-1+fa)^n}, \qquad e = e\left(-1+pe\right)^n,\\
d &= d\left(-1+kd\right)^n, \qquad c = c\left(-1+hc\right)^n,\\
b &= b\left(-1+gb\right)^n, \qquad a = a\left(-1+fa\right)^n,
\end{aligned}$$

or,

$$\begin{aligned}
(-1+pe)^n &= 1, \qquad (-1+kd)^n = 1,\\
(-1+hc)^n &= 1, \qquad (-1+gb)^n = 1,\\
(-1+fa)^n &= 1.
\end{aligned}$$

Then

$$pe = kd = hc = gb = fa = 2.$$

Second suppose $pe = kd = hc = gb = fa = 2$. Then we see from Eq.(4) that

$$\begin{aligned} x_{10n-9} &= p, \quad x_{10n-8} = k, \quad x_{10n-7} = h, \quad x_{10n-6} = g, \quad x_{10n-5} = f, \\ x_{10n-4} &= e, \quad x_{10n-3} = d, \quad x_{10n-2} = c, \quad x_{10n-1} = b, \quad x_{10n} = a. \end{aligned}$$

Thus we have a period ten solution and the proof is complete.

Numerical examples

Example 3. We consider $x_{-9} = 1.2,\ x_{-8} = 0.11,\ x_{-7} = 0.6,\ x_{-6} = 0.8,\ x_{-5} = 0.4,\ x_{-4} = 0.2,\ x_{-3} = 1.3,\ x_{-2} = 0.9,\ x_{-1} = 0.7,\ x_0 = 0.5$. See Fig. 3.

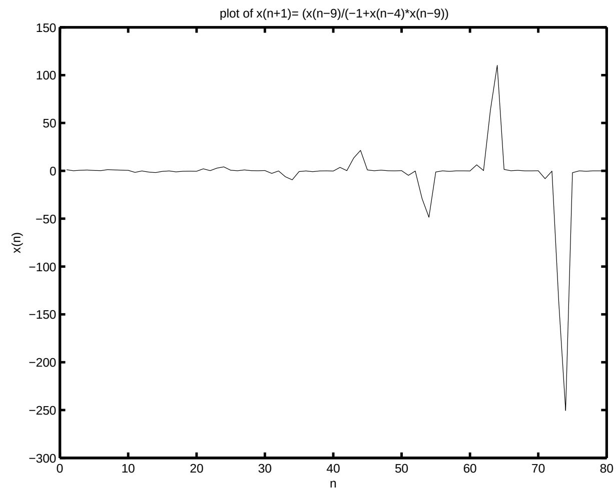

Figure 3.

Example 4. See Fig. 4, since $x_{-9} = 8,\ x_{-8} = -11,\ x_{-7} = 6,\ x_{-6} = -7,\ x_{-5} = 4,\ x_{-4} = 1/4,\ x_{-3} = -2/11,\ x_{-2} = 1/3,\ x_{-1} = -2/7,\ x_0 = 1/2$.

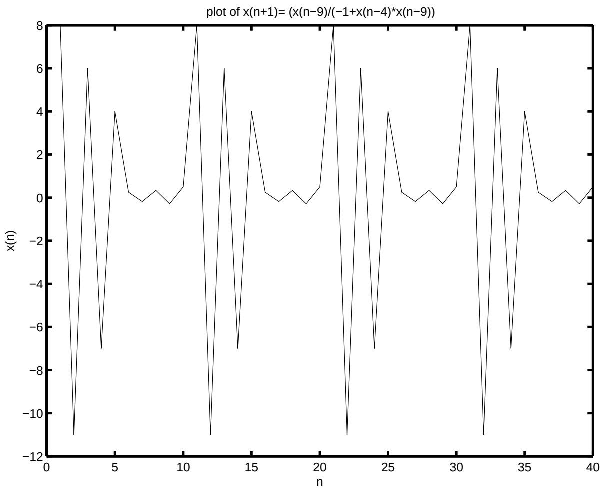

Figure 4.

The following cases can be proved similarly.

2.3 On the Difference Equation $x_{n+1} = \dfrac{x_{n-9}}{1 - x_{n-4}x_{n-9}}$

In this section we get the solution of the third following equation

$$(5) \qquad x_{n+1} = \frac{x_{n-9}}{1 - x_{n-4}x_{n-9}}, \quad n = 0, 1, ...,$$

where the initial values are arbitrary non zero real numbers.

Theorem 7 *Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, ...$*

$$\begin{aligned}
x_{10n-9} &= p\prod_{i=0}^{n-1}\left(\frac{1-2ipe}{1-(2i+1)\,pe}\right), & x_{10n-8} &= k\prod_{i=0}^{n-1}\left(\frac{1-2ikd}{1-(2i+1)\,kd}\right),\\
x_{10n-7} &= h\prod_{i=0}^{n-1}\left(\frac{1-2ihc}{1-(2i+1)\,hc}\right), & x_{10n-6} &= g\prod_{i=0}^{n-1}\left(\frac{1-2igb}{1-(2i+1)\,gb}\right),\\
x_{10n-5} &= f\prod_{i=0}^{n-1}\left(\frac{1-2ifa}{1-(2i+1)\,fa}\right), & x_{10n-4} &= e\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)pe}{1-(2i+2)\,pe}\right),\\
x_{10n-3} &= d\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)\,kd}{1-(2i+2)\,kd}\right), & x_{10n-2} &= c\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)hc}{1-(2i+2)\,hc}\right),\\
x_{10n-1} &= b\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)gb}{1-(2i+2)\,gb}\right), & x_{10n} &= a\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)fa}{1-(2i+2)\,fa}\right),
\end{aligned}$$

where $x_{-9} = p,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Theorem 8 *Eq.(5) has a unique equilibrium point which is the number zero.*

Example 5. Assume that $x_{-9} = 8,\ x_{-8} = -11,\ x_{-7} = 6,\ x_{-6} = -7,$ $x_{-5} = 4,\ x_{-4} = 0.2,\ x_{-3} = 1.1,\ x_{-2} = 0.6,\ x_{-1} = -2,\ x_0 = 4$ see Fig. 5

Example 6. See Fig. 6. for $x_{-9} = 3,\ x_{-8} = 9,\ x_{-7} = 0.8,\ x_{-6} = 0.7,$ $x_{-5} = 0.4,\ x_{-4} = 2,\ x_{-3} = 13,\ x_{-2} = 6,\ x_{-1} = 0.2,\ x_0 = 4$

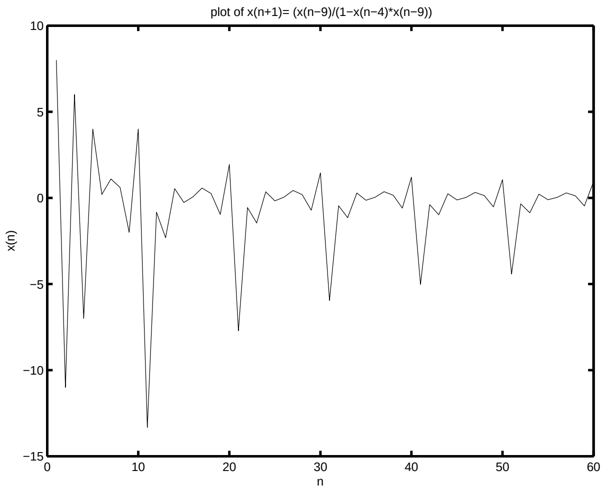

Figure 5.

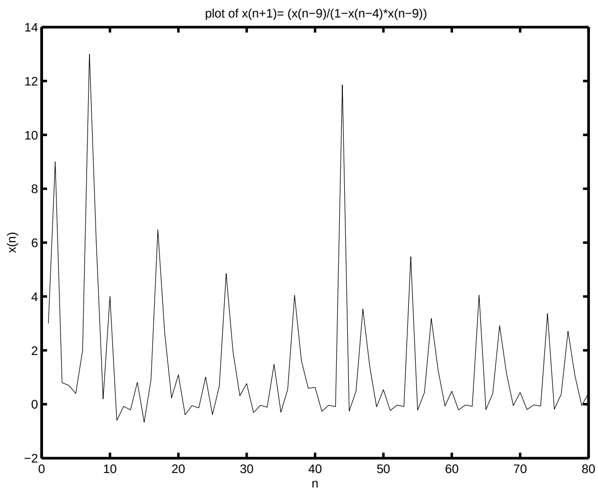

Figure 6.

2.4 On the Difference Equation $x_{n+1}=\dfrac{x_{n-9}}{-1-x_{n-4}x_{n-9}}$

Here we obtain a form of the solutions of the equation

$$x_{n+1}=\frac{x_{n-9}}{-1-x_{n-4}x_{n-9}},\quad n=0,1,..., \tag{6}$$

where the initial values are arbitrary non zero real numbers with $x_{-9}x_{-4}\neq -1,\ x_{-8}x_{-3}\neq -1,\ x_{-7}x_{-2}\neq -1,\ x_{-6}x_{-1}\neq -1,\ x_{-5}x_0\neq -1.$

Theorem 9 *Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq.(6). Then for $n=0,1,...$*

$$\begin{aligned}
x_{10n-9} &= \frac{(-1)^n p}{(1+pe)^n}, & x_{10n-8} &= \frac{(-1)^n k}{(1+kd)^n},\\
x_{10n-7} &= \frac{(-1)^n h}{(1+hc)^n}, & x_{10n-6} &= \frac{(-1)^n g}{(1+gb)^n},\\
x_{10n-5} &= \frac{(-1)^n f}{(1+fa)^n}, & x_{10n-4} &= (-1)^n e\,(1+pe)^n\,,\\
x_{10n-3} &= d\,(-1)^n\,(1+kd)^n\,, & x_{10n-2} &= c\,(-1)^n\,(1+hc)^n\,,\\
x_{10n-1} &= b\,(-1)^n\,(1+gb)^n\,, & x_{10n} &= a\,(-1)^n\,(1+fa)^n\,,
\end{aligned}$$

where $x_{-9}=p,\ x_{-8}=k,\ x_{-7}=h,\ x_{-6}=g,\ x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a.$

Theorem 10 *Eq.(6) has a unique equilibrium point which is the number zero.*

Lemma 2 It is easy to see that every solution of Eq.(6) is unbounded except in the following case.

Theorem 11 *Eq.(6) has a periodic solutions of period ten iff $pe=kd=hc=gb=fa=-2$ and will be take the form $\{p,k,h,g,f,e,d,c,b,a,p,k,h,g,f,e,d,c,b,a,...\}$.*

Example 7. Consider $x_{-9}=13,\ x_{-8}=9,\ x_{-7}=1.8,\ x_{-6}=0.7,\ x_{-5}=0.4,\ x_{-4}=0.2,\ x_{-3}=1.3,\ x_{-2}=6,\ x_{-1}=0.2,\ x_0=4$ see Fig. 7

Example 8. Fig. 8. shows the solutions when $x_{-9}=13,\ x_{-8}=-3,\ x_{-7}=0.11,\ x_{-6}=4,\ x_{-5}=0.14,\ x_{-4}=-2/13,\ x_{-3}=2/3,\ x_{-2}=-20/11,\ x_{-1}=-1/2,\ x_0=-10/7.$

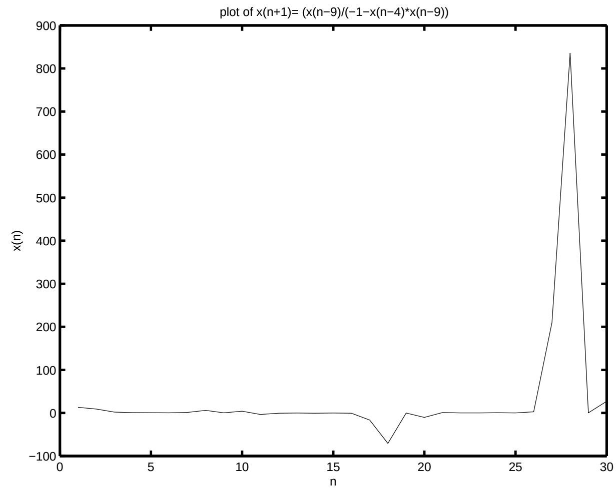

Figure 7.

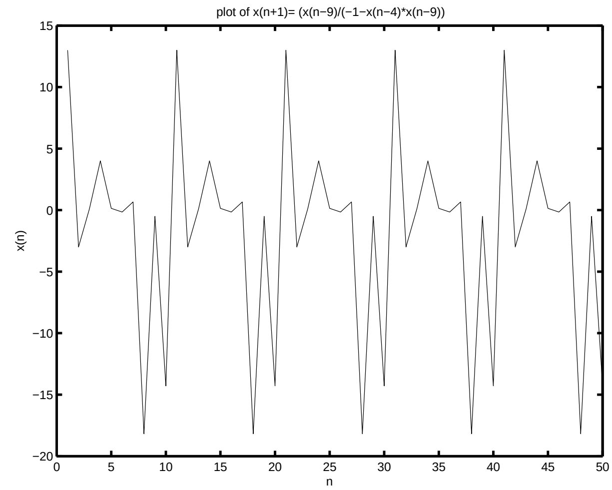

Figure 8.

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