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A improvement of Becker's condition of univalence

Mugur Acu

Abstract

Let A be the class of all analytic functions f in the unit disc $U = U(0,1)$ normed with the conditions $f(0) = 0$, $f'(0) = 1$. In this paper we give a sufficient condition for univalence which generalize the well known Becker's criterion of univalence.

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1 Introduction

Let A be the class of functions f, which are analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$, with $f(0) = 0$, $f'(0) = 1$.
In this paper we shall find, using the theory of Löwner chains, a sufficient condition for univalence of a class of functions which generalize Becker's univalence criterion.

A function $L(z,t)$, $z \in U$, $t \geq 0$ is called a Löwner chain, or a subordination chain if $L(z,t)$ is analytic and univalent in U for all positive t and, for all s,t with $0 \leq s < t$, $L(z,s) \prec L(z,t)$ (by " $\prec$" we denote the relation of subordination). In addition, $L(z,t)$ must be continuosly differentiable on $[0,\infty]$ for all $z \in U$.

2 Preliminaries

Let $0 < r \leq 1$ and U_r the disc of the complex plane $\{z \in \mathbf{C} : |z| < r\}$.

Theorem 2.1 *(Pommerenke)([4]). Let $r_0 \in (0,1]$ and let $L(z,t) = a_1(t) \cdot z + a_2(t) \cdot z^2 + \cdots$, $a_1(t) \neq 0$, be analytic in U_{r_0} for all $t \geq 0$, locally absolutely continuos in $[0,\infty)$ locally uniform with respect to U_{r_0} . For almost all $t \geq 0$ suppose*

(1) $$z \cdot \frac{\partial L(z,t)}{\partial z} = p(z,t) \cdot \frac{\partial L(z,t)}{\partial t}\,,\; z \in U_{r_0}$$

where $p(z,t)$ is analytic in U and $Re\ p(z,t) > 0\,,\, z \in U\,,\, t \geq 0$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\left\{\frac{L(z,t)}{a_1(t)}\right\}$ forms a normal family in U_{r_0}, then, for each $t \in [0,\infty)$, $L(z,t)$ has an analytic and univalent extension to the whole disc, and is, consequently, a Löwner chain.

Theorem 2.2 *(Becker)([1],[2]). If $f \in A$ and*

(2) $$\left(1 - |z|^2\right) \cdot \left|\frac{z f''(z)}{f'(z)}\right| \leq 1 \; for\ all\ z \in U$$

then f is univalent in U.

3 Main results

Theorem 3.1 *Let $f, g, h \in A$ and let α , β , γ be complex numbers with $|\alpha| + |\beta| + |\gamma| > 0$. If*

(3) $$|\alpha + \beta + \gamma| < 1$$

(4) $$\left| |z|^2 \cdot (\alpha + \beta + \gamma) + \left(1 - |z|^2\right) \cdot \left(\alpha \cdot \frac{z f'(z)}{f(z)} + \beta \cdot \frac{z g'(z)}{g(z)} + \gamma \cdot \frac{z h'(z)}{h(z)}\right) \right| \leq 1\,,\; z \in U$$

then the function

(5) $$F_{\alpha,\beta,\gamma}(z) = \left[(1 + \alpha + \beta + \gamma) \cdot \int_0^z f^{\alpha}(u) \cdot g^{\beta}(u) \cdot h^{\gamma}(u) du\right]^{\frac{1}{\alpha+\beta+\gamma+1}}$$

is analytic and univalent in U.

Proof. The functions $h_1(u) = \frac{f(u)}{u} = 1 + a_1 \cdot u + a_2 \cdot u^2 + \cdots$, $h_2(u) = \frac{g(u)}{u} = 1 + b_1 \cdot u + b_2 \cdot u^2 + \cdots$, $h_3(u) = \frac{h(u)}{u} = 1 + c_1 \cdot u + c_2 \cdot u^2 + \cdots$ are analytic in U and $h_1(0) = h_2(0) = h_3(0) = 1$. Then, we can choose r_0, $0 < r_0 \leq 1$ so that all these functions do not vanish in U_{r_0}. In this case we denote by h_1^*, h_2^*, h_3^*, the uniform branches of $[h_1(u)]^\alpha$, *of* $[h_2(u)]^\beta$, and of $[h_3(u)]^\gamma$, respectively, which are analytic in U_{r_0} and $h_1^*(0) = h_2^*(0) = h_3^*(0) = 1$. Let $h_4(u) = h_1^*(u) \cdot h_2^*(u) \cdot h_3^*(u)$ and

$$h_5(u) = (1 + \alpha + \beta + \gamma) \int_0^{e^{-t}z} h_4(u) \cdot u^{\alpha+\beta+\gamma} du = \left(e^{-t}z\right)^{1+\alpha+\beta+\gamma} + \cdots . \tag{6}$$

It is clear that, if $z \in U_{r_0}$, then $e^{-t}z \in U_{r_0}$, and, from the analycity of h_4 in U_{r_0}, we have that $h_5(z,t)$ is also analytic in U_{r_0} for all $t \geq 0$ and:

$$h_5(z,t) = \left(e^{-t}z\right)^{1+\alpha+\beta+\gamma} \cdot h_6(z,t) \; where \tag{7}$$

$$h_6(z,t) = 1 + \cdots . \tag{8}$$

If we put

$$h_7(z,t) = h_6(z,t) + \left(e^{2t} - 1\right) \cdot h_4\left(e^{-t}z\right) \tag{9}$$

we have that $h_7(0,t) = e^{2t} \neq 0$ for all $t \geq 0$. Then, we can choose r_1, $0 < r_1 \leq r_0$ so that h_7 does not vanish in U_{r_1} $(t \geq 0)$.
Now, denote by $h_8(z,t)$ the uniform branch of $[h_7(z,t)]^{\frac{1}{1+\alpha+\beta+\gamma}}$, which is analytic in U_{r_1} and $h_8(0,t) = e^{\frac{2t}{1+\alpha+\beta+\gamma}}$. It follows that the function

$$L(z,t) = e^{-t}z \cdot h_8(z,t) \tag{10}$$

is analytic in U_{r_1} and $L(0,t) = 0$ for all $t \geq 0$. It also clear that $e^{-t} \cdot h_8(0,t) = e^{\frac{1-(\alpha+\beta+\gamma)}{1+(\alpha+\beta+\gamma)} \cdot t}$. Now, we can formally write (using (6), (7), (8), (9), (10)):

$$L(z,t) = \left[(1+\alpha+\beta+\gamma) \cdot \int_0^{e^{-t}z} f^\alpha(u) \cdot g^\beta(u) \cdot h^\gamma(u)\, du + \left(e^{2t}-1\right) e^{-t}z \cdot f^\alpha(e^{-t}z) \cdot g^\beta(e^{-t}z) \cdot h^\gamma(e^{-t}z)\right]^{\frac{1}{1+\alpha+\beta+\gamma}} = \tag{11}$$

$$= e^{\frac{1-(\alpha+\beta+\gamma)}{1+(\alpha+\beta+\gamma)} \cdot t} \cdot z + \cdots = a_1(t) \cdot z + \cdots .$$

From (3) we have that $Re\, \frac{1-(\alpha+\beta+\gamma)}{1+(\alpha+\beta+\gamma)} > 0$ and then:

$$\lim_{t\to\infty} |a_1(t)| = \lim_{t\to\infty} \left| e^{\frac{1-(\alpha+\beta+\gamma)}{1+(\alpha+\beta+\gamma)} \cdot t} \right| = \lim_{t\to\infty} e^{t \cdot Re\, \frac{1-(\alpha+\beta+\gamma)}{1+(\alpha+\beta+\gamma)}} = \infty .$$

$\frac{L(z,t)}{a_1(t)}$ is analytic in U_{r_1} for all $t \geq 0$ and then, it follows that $\left\{\frac{L(z,t)}{a_1(t)}\right\}$ is uniformly bounded in $U_{\frac{r_1}{2}}$.
Applying Montel's theorem, we have that $\left\{\frac{L(z,t)}{a_1(t)}\right\}$ forms a normal family in $U_{\frac{r_1}{2}}$. Using (9) and (10) we have:

$$\frac{\partial L(z,t)}{\partial t} = e^{-t}z \cdot \left[\frac{1}{1+\alpha+\beta+\gamma} \cdot (h_7(z,t))^{\frac{-\alpha-\beta-\gamma}{1+\alpha+\beta+\gamma}} \cdot \frac{\partial h_7(z,t)}{\partial t} - (h_7(z,t))^{\frac{1}{1+\alpha+\beta+\gamma}}\right] \tag{12}$$

Because $h_7(0,t) = e^{2t} \neq 0$, we consider an uniform branch of $(h_7(z,t))^{\frac{-\alpha-\beta-\gamma}{1+\alpha+\beta+\gamma}}$ which is analytic in U_{r_2}, where r_2, $0 < r_2 \leq \frac{r_1}{2}$ is chosen so that the above-mentioned uniform branch, which takes in $(0,t)$ the value $e^{\frac{-2t\cdot(\alpha+\beta+\gamma)}{1+\alpha+\beta+\gamma}}$, does not vanish in U_{r_2}. It is also clear that $\frac{\partial h_7(z,t)}{\partial t}$ is analytic in U_{r_2}, and then, it follows that $\frac{\partial L(z,t)}{\partial t}$ is also. Then $L(z,t)$ is locally absolutely continuous. Let

$$p(z,t) = \frac{z \cdot \dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}. \tag{13}$$

In order to prove that $p(z,t)$ has an analytic extension with positive real part in U, for all $t \geq 0$, it is sufficient to prove that the function:

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \tag{14}$$

is analytic in U for $t \geq 0$ and

$$|w(z,t)| < 1 \tag{15}$$

for all $z \in U$ and $t \geq 0$. Using (14), after simple calculations we obtain:

$$w(z,t) = \left[(\alpha+\beta+\gamma) \cdot h_1(e^{-t}z)h_2(e^{-t}z)h_3(e^{-t}z)\right] \frac{1}{e^{2t} \cdot h_1(e^{-t}z)h_2(e^{-t}z)h_3(e^{-t}z)} + \tag{16}$$

$$\frac{+(e^{2t}-1) \cdot \left[\alpha f'(e^{-t}z)h_2(e^{-t}z)h_3(e^{-t}z) + \beta g'(e^{-t}z)h_1(e^{-t}z)h_3(e^{-t}z) + \gamma h'(e^{-t}z)h_1(e^{-t}z)h_2(e^{-t}z)\right]}{e^{2t} \cdot h_1(e^{-t}z)h_2(e^{-t}z)h_3(e^{-t}z)}$$

Because h_1, h_2 *and* h_3 do not vanish in U_{r_2} and are analytic, it follows that $w(z,t)$ is also analytic in the same disc, for all $t \geq 0$. Then, $w(z,t)$ has an analytic extension in U denoted also by $w(z,t)$.

For $t=0$, $|w(z,0)| = |\alpha+\beta+\gamma| < 1$ from (3). Let now $t>0$. In this case $w(z,t)$ is analytic in $\overline{U}$ because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \overline{U}$. Then

$$|w(z,t)| < max_{|z|=1}\,|w(z,t)| = \left|w(e^{i\theta},t)\right| \;\; with\, \theta\, real. \tag{17}$$

To prove (15) it is sufficient that:

$$\left|w(e^{i\theta},t)\right| \leq 1 \;\; for\; all\, t > 0. \tag{18}$$

Note $u = e^{-t}\cdot e^{i\theta}$, $u \in U$. Then $|u| = e^{-t}$ and from (16) we obtain:

$$\left|w(e^{i\theta},t)\right| = \left||u|^2\cdot(\alpha+\beta+\gamma) + \left(1-|u|^2\right)\cdot\left[\alpha\frac{uf'(u)}{f(u)} + \beta\frac{ug'(u)}{g(u)} + \gamma\frac{uh'(u)}{h(u)}\right]\right| \tag{19}$$

and inequality (18) becomes:

$$\left||u|^2\cdot(\alpha+\beta+\gamma) + \left(1-|u|^2\right)\cdot\left[\alpha\frac{uf'(u)}{f(u)} + \beta\frac{ug'(u)}{g(u)} + \gamma\frac{uh'(u)}{h(u)}\right]\right| \leq 1\,. \tag{20}$$

Because $u \in U$, relation (4) implies (20). Combining (17), (18), (19) and (20), it follows that $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$. Applying Theorem 2.1, we have that $L(z,t)$ is a Löwner chain and, then the function $L(z,0) = F_{\alpha,\beta,\gamma}(z)$, defined by (5), is analytic and univalent in U.

Remark 3.1 *From Theorem 3.1, with $\beta+\gamma = -\alpha$ and $h = g$ we have: If $f, g \in A$ and α is a complex number, $\alpha \neq 0$, and*

$$\left|\left(1-|z|^2\right)\cdot\left[\alpha\frac{zf'(z)}{f(z)} - \alpha\frac{zg'(z)}{g(z)}\right]\right| \leq 1 \tag{21}$$

for all $z \in U$, then the function

$$F(z) = \int_0^z \left[\frac{f(u)}{g(u)}\right]^\alpha du \tag{22}$$

is analytic and univalent in U.
After simple calculations, we have that condition (21) is equivalent to:

$$\left|\left(1-|z|^2\right)\cdot\frac{zF''(z)}{F'(z)}\right| \leq 1\,. \tag{23}$$

It follows that condition (23) implies the univalence of F. This is Becker's criterion of univalence (see Theorem 2.2). Then Theorem 3.1 is a generalization of Becker's criterion of univalence.

Remark 3.2 *It's easy to see that for $\gamma = 0$ in Theorem 3.1 we obtain the results from [3].*

4 Some particular cases

Corollary 4.1 *If* $f \in A$ *and* $\alpha, \beta, \gamma,$ *are complex numbers,* $|\alpha| + |\beta| + |\gamma| > 0$*, satisfying:*

$$|\alpha + \beta + \gamma| < 1 \tag{24}$$

$$\left| |z|^2 \cdot (\alpha + \beta + \gamma) + \left(1 - |z|^2\right) \cdot \left[(\alpha + \beta) \cdot \frac{zf^{'}(z)}{f(z)} + \gamma \cdot \left(\frac{zf"(z)}{f^{'}(z)} + 1 \right) \right] \right| \leq 1 \tag{25}$$

then the function

$$F_{\alpha,\beta,\gamma}(z) = \left[(\alpha + \beta + \gamma + 1) \cdot \int_0^z f^{\alpha+\beta}(u) \cdot u^{\gamma} \cdot \left[f^{'}(u)\right]^{\gamma} du \right]^{\frac{1}{\alpha+\beta+\gamma+1}} \tag{26}$$

is analytic and univalent in U*.*

Proof. Let $h(z) = zf^{'}(z) \in A$ and $g(z) = f(z)$. By applying Theorem 3.1 we obtain the assertion.

Corollary 4.2 *If* $f \in A$ *and* $\alpha, \beta, \gamma,$ *are complex numbers,* $|\alpha| + |\beta| + |\gamma| > 0$*, satisfying:*

$$|\alpha + \beta + \gamma| < 1 \tag{27}$$

$$\left| |z|^2 \cdot (\alpha + \beta + \gamma) + \left(1 - |z|^2\right) \cdot \left[\alpha \cdot \frac{zf^{'}(z)}{f(z)} + (\beta + \gamma) \cdot \left(\frac{zf"(z)}{f^{'}(z)} + 1 \right) \right] \right| \leq 1 \tag{28}$$

then the function

$$F_{\alpha,\beta,\gamma}(z) = \left[(\alpha + \beta + \gamma + 1) \cdot \int_0^z f^{\alpha}(u) \cdot u^{\beta+\gamma} \cdot \left[f^{'}(u)\right]^{\beta+\gamma} du \right]^{\frac{1}{\alpha+\beta+\gamma+1}} \tag{29}$$

is analytic and univalent in U*.*

Proof. Let $g(z) = h(z) = zf^{'}(z) \in A$. By applying Theorem 3.1 we obtain the assertion.

Corollary 4.3 *If* $f \in A$ *and* $c \in U$ *satisfying:*

$$\left| |z|^2 \cdot c + \left(1 - |z|^2\right) \cdot c \cdot \frac{zf^{'}(z)}{f(z)} \right| \leq 1 \tag{30}$$

then the function

$$F_c(z) = \left[(c + 1) \cdot \int_0^z f^c(u) du \right]^{\frac{1}{c+1}} \tag{31}$$

is analytic and univalent in U*.*

Proof. Let $g(z) = h(z) = f(z) \in A$. By applying Theorem 3.1 , with $\alpha + \beta + \gamma = c$, we obtain the assertion.

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Department of Mathematics
Faculty of Sciences
"Lucian Blaga" University of Sibiu
Str. I. Raţiu 5-7
2400 Sibiu, Romania
E-mail: *acu_mugur@yahoo.com*