GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 3, 1995, 291-297

ON A GENERALIZATION OF THE KELDYSH THEOREM

G. JAIANI

Abstract. The Keldysh theorem for an elliptic equation with characteristic parabolic degeneration is generalized for the case of an elliptic equation of the second-order canonical form with order and type degeneration. The criteria under which the Dirichlet or Keldysh problems are correct are given in a one-sided neighborhood of the degeneration segment, enabling one to write the criteria in a single form. Moreover, some cases are pointed out in which it is even nessesary to give a criterion in the neighborhood because it is impossible to establish it on the segment of degeneracy of the equation.

Let us consider the equation

$$L(u) \stackrel{\text{def}}{=} y^m \frac{\partial^2 u}{\partial x^2} + y^n \frac{\partial^2 u}{\partial y^2} + a(x,y)\frac{\partial u}{\partial x} + \\ +b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0, \quad m,n = \text{const} \geq 0, \tag{1}$$

in a domain Ω bounded by a sufficiently smooth arc σ lying in the upper half-plane $y \geq 0$ and by a segment $\overline{AB}$ of the x-axis;

$$a, b, c \in \mathfrak{A}(\overline{\Omega}), \quad c \leq 0 \quad \text{in} \quad \overline{\Omega}, \tag{2}$$

where $\mathfrak{A}(\overline{\Omega})$ is the class of functions analytic in $\overline{\Omega}$ with respect to x, y, and two boundary value problems:

Dirichlet Problem. Find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ in Ω from the prescribed continuous values of $L(u)$ in Ω and of u on the boundary$\partial\Omega$.

Keldysh Problem. Find bounded $u \in C^2(\Omega) \cap C(\Omega \cup \sigma)$ in Ω from prescribed continuous values of $L(u)$ in Ω and of u on σ.

$C(\overline{\Omega})$ is a set of functions continuous in closure of Ω. $C^2(\Omega)$ is a set of functions with continuous derivatives of orders ≤ 2 in Ω.

1991 *Mathematics Subject Classification.* 35J70.

Key words and phrases. Elliptic equation with order and (or) type degeneration, characteristic parabolic degeneration, Dirichlet problem, Keldysh problem.

Let

$$I_\delta \stackrel{\text{def}}{=} \{(x,y)\in\Omega:\ 0<y<\delta,\quad \delta=\text{const}>0\}.$$

Theorem. *If $n<1$, or $n\geq 1$ and*

$$b(x,y)<y^{n-1}\quad \textit{in}\quad \overline{I}_\delta, \tag{3}$$

the Dirichlet problem is correct while the Keldysh problem has an infinite number of solutions. If $n\geq 1$,

$$b(x,y)\geq y^{n-1}\quad \textit{in}\quad I_\delta, \tag{4}$$

and

$$a(x,y)=O(y^m),\quad y\to 0+ \tag{5}$$

(O is the Landau symbol), the Keldysh problem is correct while the Dirichlet problem, in general, has no solutions.

Proof. In [1] (see pp. 187-194) it is shown for equation (1) that
if there exists $W\in C^2(\Omega)$ such that:

$$W>0\quad \text{in}\quad \Omega\cup\sigma,$$
$$\lim_{y\to 0+}W(x,y)=+\infty$$

uniformly with respect to x,

$$L(W)<0\quad \text{in}\quad \Omega,$$

then the Keldysh problem is correct,
if for any point $(x_0,0)$, $x_0\in\overline{AB}$, there exists (barrier) $v\in C^2(\omega^\delta_{x_0})$, where

$$\omega^\delta_{x_0}\stackrel{\text{def}}{=}\{(x,y)\in\Omega:\ (x-x_0)^2+y^2<\delta,\quad \delta=\text{const}>0\},$$

such that

$$v\in C(\overline{\omega^\delta_{x_0}}),$$
$$v(x_0,0)=0,$$
$$v>0\quad \text{in}\quad \overline{\omega^\delta_{x_0}}\backslash\{(x_0,0)\},$$
$$L(v)<\eta=\text{const}<0\quad \text{in}\quad \omega^\delta_{x_0},$$

then the Dirichlet problem is correct.
Let us show that by (3) the function

$$v(x,y)=(-\ln y)^{-1}+(x-x_0)^2$$

can serve as a barrier function.

Indeed, taking into account (1), we have

$$\begin{aligned} L(v) = 2y^m + 2y^{n-2}(-\ln y)^{-3} - y^{n-2}(-\ln y)^{-2} + 2a\cdot(x-x_0) + \\ + b\cdot y^{-1}(-\ln y)^{-2} + cv = [b(x,y) - y^{n-1}]y^{-1}(-\ln y)^{-2} + \\ + 2y^{n-2}(-\ln y)^{-3} + 2y^m + 2a\cdot(x-x_0) + cv < \\ < \eta < 0 \quad \text{in} \quad w^{\delta}_{x_0}, \end{aligned} \tag{6}$$

since the sign of $L(v)$ when $y \to 0+$ is defined by the first term of (6)

$$[b(x,y) - y^{n-1}]y^{-1}(-\ln y)^{-2}, \tag{7}$$

and for $n \geq 1$, in view of (3),

$$\lim_{y\to 0+} L(v) = -\infty. \tag{8}$$

If $0 \leq n < 1$, we rewrite the first term of (6) as

$$[y^{1-n}b(x,y) - 1]y^{n-2}(-\ln y)^{-2}. \tag{9}$$

Because of (2)

$$\lim_{y\to 0+}[y^{1-n}b(x,y) - 1] = -1.$$

Therefore (8) holds in this case too.

It is easy to see that the other properties of the barrier are also fulfilled.

To prove the second part of the theorem, let us consider the function

$$W(x,y) = -\ln y - (x-\alpha)^l + k,$$

where $x - \alpha > 1$, $\alpha, k = \text{const}$, $l > 2$ is an integer.

Obviously,

$$\begin{aligned} L(W) = -l(l-1)(x-\alpha)^{l-2}y^m + y^{n-2} - la\cdot(x-\alpha)^{l-1} - \frac{b}{y} + cW = \\ = \frac{y^{n-1} - b(x,y)}{y} - \frac{1}{3}l\big[y^m(l-1) + 3a\cdot(x-\alpha)\big](x-\alpha)^{l-2} - \\ - \frac{2}{3}l(l-1)(x-\alpha)^{l-2}y^m + cW. \end{aligned} \tag{10}$$

In view of (5) l can be chosen so that

$$l - 1 > 3\max_{\Omega}(x-\alpha)\sup_{\Omega}\frac{|a|}{y^m} \geq \frac{3|a|(x-\alpha)}{y^m} \quad \text{in} \quad \Omega. \tag{11}$$

On the other hand, by virtue of (4),

$$\frac{y^{n-1} - b(x,y)}{y} \leq 0 \quad \text{in} \quad I_\delta.$$

Hence, taking into account that $1 < (x-\alpha)^{l-2}$ and

$$-\frac{2}{3}l(l-1)y^m > -\frac{2}{3}l(l-1)(x-\alpha)^{l-2}y^m \quad \text{in} \quad \Omega,$$

from (10) we have

$$\begin{aligned} L(W) &< -\frac{2}{3}l(l-1)(x-\alpha)^{l-2}y^m + cW \le \\ &\le -\frac{2}{3}l(l-1)y^m < 0 \quad \text{in} \quad I_\delta, \end{aligned} \tag{12}$$

since $W > 0$ in $\Omega \cup \sigma$ for suitably chosen k. It is clear that there exist $A, l = \text{const}$ such that

$$\frac{y^{n-1} - b(x,y)}{y} < A \quad \text{and} \quad l(l-1) > \frac{3A}{y^m} \quad \text{in} \quad \overline{\Omega \backslash I_\delta}.$$

Further

$$L(W) < A - \frac{2}{3}l(l-1)y^m + cW < -\frac{1}{3}l(l-1)y^m < 0 \quad \text{in} \quad \overline{\Omega \backslash I_\delta}. \tag{13}$$

From (12) and (13) there follows

$$L(W) < 0 \quad \text{in} \quad \Omega.$$

The fulfillment of the other properties of the function W is obvious. $\square$

Remark 1. Condition (5) is not necessary. If $a(x,y) \ge 0$ in I_δ, then

$$L(W) < -l(l-1)(x-\alpha)^{l-2}y^m < -l(l-1)y^m < 0 \quad \text{in} \quad I_\delta,$$

and by virtue of (13), which is valid since (11) holds for $\overline{\Omega \backslash I_\delta}$, the theorem remains true without restriction (5). On the other hand, if (4) is fulfilled in Ω and $c < 0$ in $\overline{\Omega}$ or $b(x,y) > y^{n-1}$ in Ω and $c \le 0$ in $\overline{\Omega}$, then

$$W^* = -\ln y + k$$

can serve as the Keldysh function, since

$$L(W^*) = \frac{y^{n-1} - b(x,y)}{y} + cW^* < 0 \quad \text{in} \quad \Omega,$$

and condition (5) is again unnecessary.

Remark 2. When $1 < n < 2$, $b(x,0) \le 0$, the sign of $L(v)$ (see (6)) is defined by (7). Since $b \in \mathfrak{A}(\overline{\Omega})$,

$$\begin{aligned} [b(x,y) - y^{n-1}]y^{-1}\ln^{-2} y = [b(x,0) + \frac{\partial b(x,0)}{\partial y} y + O(y^2)]y^{-1}\ln^{-2} y - \\ -y^{n-2}\ln^{-2} y \le \Big[\frac{\partial b(x,0)}{\partial y} - O(y)\Big]\ln^{-2} y - y^{n-2}\ln^{-2} y, \end{aligned}$$

where the first term tends to zero and the second one tends to $-\infty$. Therefore (8) is fulfilled and the Dirichlet problem is correct.

Remark 3. Because of the continuity in $\overline{\Omega}$ of both sides if $n \geq 1$, (4) holds also in $\overline{I}_\delta$.

Remark 4. Because of (2) $b(x,y) \not\equiv y^{n-1}$ in $\overline{\Omega}$ when $\{n\} \neq 0$ ($\{n\}$ is a fractional part of n) since y^{n-1}, $\{n\} \neq 0$, is not analytic in $\overline{\Omega}$.

Remark 5. For $0 \leq n < 1$, because of the boundedness (see (2)) of $b(x,y)$ and $\lim_{y\to 0+} y^{n-1} = +\infty$, (3) is always fulfilled in I_δ and (4) cannot take place in I_δ. In that case from (3) in I_δ there follows

$$y^{1-n}b(x,y) - 1 < 0 \quad \text{in} \quad \overline{I}_\delta,$$

and the correctness of the Dirichlet problem is clear (see the proof of the theorem). Hence we could embrace the case $0 \leq n < 1$ with condition (3). But because of the clearness of the question (for $0 \leq n < 1$ the Dirichlet problem is always correct), this case is considered separately.

Remark 6. When (3) holds we have either $n = 1$, $b(x,0) < 1$ or $n > 1$, $b(x,0) < 0$ and vice versa.

Indeed, when $n = 1$, (3) obviously implies $b(x,0) < 1$, $x \in \overline{AB}$, and from the latter there follows (3) since $[1 - b(x,y)] \in C(\overline{\Omega})$ (see (2)) and has to preserve its sign in closure of some $I_{\delta^*} \subset I_\delta$, $\delta^* < \delta$. If $n > 1$ from (3) there follows $b(x,0) < 0$ and from the latter as above $b(x,y) < 0$ in some I_{δ^*} and therefore there obviously follows (3).

Remark 7. When (4) holds we have either $n = 1$, $b(x,0) \geq 1$ or $1 < n < 2$, $b(x,0) > 0$ or $n \geq 2$, $b(x,0) \geq 0$ for $x \in \overline{AB}$. The reverse motion is not true in general but if $n = 1$, $b(x,0) > 1$ or $n > 1$, $b(x,0) > 0$, $x \in \overline{AB}$, then

$$b(x,y) > y^{n-1} \quad \text{in} \quad I_\delta. \tag{14}$$

In the latter case, there exist b_0 and δ such that $b(x,y) \geq b_0 = \text{const} > 0$ in $\overline{I}_\delta$. Hence (14) will be fulfilled if $\delta = b_0^{\frac{1}{n-1}}$.

For $1 < n < 2$ condition (4) does not exclude the existence of such $x_0 \in \overline{AB}$ where $b(x_0,0) = 0$. But in that case, because of (2),

$$b(x_0,y) = b(x_0,0) + \frac{\partial b(x_0,0)}{\partial y}y + \frac{1}{2}\frac{\partial^2 b(x_0,0)}{\partial y^2}y^2 + \cdots =$$
$$= y\Big[\frac{\partial b(x_0,0)}{\partial y} + \frac{1}{2}\frac{\partial^2 b(x_0,0)}{\partial y^2}y + \ldots\Big] = y \cdot \varkappa(x_0,y), \quad 0 \leq y < \delta,$$

with $\varkappa(x_0,y)$ bounded for $0 \leq y < \delta$ and, in view of (4), we have

$$y\varkappa(x_0,y) \geq y^{n-1}, \quad 0 < y < \delta,$$

i.e.,

$$\varkappa(x_0, y) \geq y^{n-2}, \quad 0 < y < \delta,$$

which means the unboundedness of $\varkappa$. This is a contradiction. Therefore $b(x_0, 0) \neq 0$ and $b(x, 0) > 0$ for all $x \in \overline{AB}$. The other cases are obvious.

From Remarks 6 and 7 there follows

Remark 8. For $m = 0$, if $b(x, 0) \neq 1$ when $n = 1$, and $b(x, 0) \neq 0$ when $n > 1$, from our theorem there follows the Keldysh theorem [2].

Remark 9. Let us consider in Ω two equations: one with order and type degeneration

$$y^m \frac{\partial^2 u}{\partial x^2} + y^n \frac{\partial^2 u}{\partial y^2} + b(x, y) \frac{\partial u}{\partial y} = 0 \tag{15}$$

and the other with characteristic type degeneration

$$\frac{\partial^2 u}{\partial x^2} + y^n \frac{\partial^2 u}{\partial y^2} + b(x, y) \frac{\partial u}{\partial y} = 0, \tag{16}$$

where

$$b(x, y) = b_0 y^{[n]-1}, \quad b_0 = \text{const}, \quad m \geq [n] - 1, \quad n \geq 2, \tag{17}$$

and $[n]$ is the integral part of n.

In both cases $b(x, 0) = 0$. Hence, in view of the Keldysh theorem [2], the Keldysh problem is correct for (16). Similarly, expecting the correctness of the Keldysh problem for (15), let us check the fulfillment of (4). Condition (4) will be fulfilled for (17) iff

$$b_0 \geq y^{\{n\}} \quad \text{in} \quad I_\delta. \tag{18}$$

The latter will be fulfilled iff

$$b_0 \geq 1 \quad \text{when} \quad \{n\} = 0, \tag{19}$$

$$b_0 > 0 \quad \text{when} \quad \{n\} > 0. \tag{20}$$

(Indeed, if n is an integer, (18) and (19) coincide. When n is not an integer, for any $b_0 > 0$ we can find the neighborhood I_δ, $\delta = b_0^{\frac{1}{\{n\}}}$ where (18) will be fulfilled.) In these cases the correctness of the Keldysh problem follows from our theorem.

If

$$\begin{aligned} &b_0 < 1 \quad \text{when} \quad \{n\} = 0, \\ &b_0 \leq 0 \quad \text{when} \quad \{n\} > 0, \end{aligned} \tag{21}$$

(3) is fulfilled in I_δ but in $\overline{I}_\delta$ the inequality cannot be strong and we are not able to use our theorem. However, after dividing both sides of (15) by $y^{[n]-1}$ in Ω, we obtain the equation

$$y^{m-[n]+1}\,\frac{\partial^2 u}{\partial x^2}+y^{\{n\}+1}\,\frac{\partial^2 u}{\partial y^2}+b_0\frac{\partial u}{\partial y}=0,$$

and now we can apply our theorem, which by (21) asserts the correctness of Dirichlet problem.

Thus, for both equations (15) and (16) $b(x,0)=0$. Nevertheless the Keldysh problem is correct for (16) for any b_0; the Keldysh problem is correct for (15) for some b_0 (see (19), (20)), and the Dirichlet problem is correct for other b_0 (see (21)). It means that for equation (16) with type degeneration the correctness of admissible problems depends on values of $b(x,y)$ on the line of degeneracy of the equation, but for equation (15) with order and type degeneration, the correctness of admissible problems essentially depends on the behavior of $b(x,y)$ in a neighborhood not on the segment of degeneracy of the equation.

Therefore, when $m>0$, $n>2$, $b(x,0)=0$, the well-posedness of the boundary value problems for (1), even under assumptions (2), essentially depends on additional properties of $b(x,y)$ in the neighborhood (see (3),(4)) of line of degeneracy of (1), i.e., it is nessesary to give the criteria in the neighborhood because it is impossible to establish them on the segment of degeneracy of the equation.

References

1. A. V. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow,* 1981.

2. M. V. Keldysh, On some cases of degeneration of an equation of elliptic type on the domain boundary. (Russian) *Dokl. Akad. Nauk SSSR* **77**(1951), No. 2, 181-183.

(Received 18.01.1994)

Author's address:
I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia