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REGULAR FRÉCHET–LIE GROUPS OF INVERTIBLE ELEMENTS IN SOME INVERSE LIMITS OF UNITAL INVOLUTIVE BANACH ALGEBRAS

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Abstract. We consider a wide class of unital involutive topological algebras provided with a C^*-norm and which are inverse limits of sequences of unital involutive Banach algebras; these algebras are taking a prominent position in noncommutative differential geometry, where they are often called unital smooth algebras. In this paper we prove that the group of invertible elements of such a unital solution smooth algebra and the subgroup of its unitary elements are regular analytic Fréchet–Lie groups of Campbell–Baker–Hausdorff type and fulfill a nice infinite-dimensional version of Lie's second fundamental theorem.

Introduction

Recent developments in the noncommutative differential geometry originated by A. Connes, particularly in its metric aspects (see [1],[2] and the references therein) and its applications for diverse models in particle physics ([2], [3], [4]), have placed in a prominent position a wide class of unital involutive algebras $\mathbb{A}$ which carry a C^*-norm and which are inverse limits of suitable sequences of unital involutive Banach algebras; they consitute the noncommutative version of inverse limits of suitable sequences of Banach spaces.

The main goal of this work is to prove that the group $\mathrm{GL}(\mathbb{A})$ of invertible elements in $\mathbb{A}$ and the subgroup $\mathcal{U}(\mathbb{A})$ of its unitary elements (playing an important role in the noncommutative geometry picture of Yang–Mills theory ([2],[3])) have a canonical structure of regular analytic Fréchet–Lie group of Campbell–Baker–Hausdorff type fulfilling a nice version of Lie's second fundamental theorem.

The present paper is organized as follows:

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(a) In Section 1 we recall what is the LB-machinery (ILB-chains and strong ILB–Lie groups) initiated by H. Omori ([5] and [6]), as well as their main properties, and we initiate the class of unital involutive ILB-algebras. In the setting of the classical differential geometry a typical example of unital involutive ILB-algebra is given by considering spaces of the type $C^\infty(M; A)$ in which M is a compact smooth manifold and A is a unital C^*-algebra.

(b) Section 2 is devoted to fundamental examples in the context of noncommutative geometry: starting from a unital involutive algebra $\mathbb{A}$ provided with a C^*-norm $\| \|$ and using the construction initiated by J. Cuntz in [7], we prove that there exists a unital involutive ILB-algebra $\mathbb{A}'$ such that $\mathbb{A} \subseteq \mathbb{A}' \subseteq \mathbb{A}_0$, where $\mathbb{A}_0$ is the C^*-algebra completion of $\mathbb{A}$ with respect to $\| \|$, the inclusions being continuous $*$ homomorphisms with dense range.

(c) Section 3 is related to some aspects of differentiability and analyticity on Fréchet spaces according to the ideas of J. Leslie, J. Bochniack, J. Siciak, and some others (see, for example, [8], [9], [6], [10], [5]) and to the notion of generalized Campbell–Baker–Hausdorff Lie group (for shortness: CBH–Lie group) initiated by J. Milnor in [11], which are not necessarily known by nonspecialists in infinite-dimensional analysis and which we shall use in the next sections.

(d) In Section 4 we prove (Theorem 1) that the group $\mathrm{GL}(\mathbb{A})$ of invertible elements of any unital involutive ILB-algebra $\mathbb{A}$ has a natural analytic regular Campbell–Baker–Hausdorff Fréchet–Lie group structure with Lie algebra $\mathbb{A}$.

(e) In Section 5 we quote that the group $\mathrm{GL}(\mathbb{A})$ fulfills a nice infinite-dimensional version of Lie's second fundamental theorem: any closed Lie subalgebra of $\mathbb{A}$ is Lie algebra of a unique connected CBH–Lie group embedded in $\mathrm{GL}(\mathbb{A})$ as a Lie subgroup (Theorem 2).

Although this assertion could be most likely deduced from some results of J. Leslie in [10], the type of proof given here is interesting in itself: the proofs generally used for this theorem in the infinite-dimensional context require appropriate versions of the implicit functions theorem and of Frobenius's theorem (see, for example, [1] for the Banach case, and [10] for more general cases with the use of a bornological machinery); in contrast, our proof is only based on some properties of analytic foliations and on the use of the Campbell–Hausdorff formula.

(f) Let $\mathbb{A}$ be any unital involutive topological algebra. The group of its unitary elements

$$\mathcal{U}(\mathbb{A}) = \{u \in \mathrm{GL}(\mathbb{A}) | u.u^* = u^*.u = \mathbb{1}\}$$

provided with the induced topology of $\mathbb{A}$ is a topological group. This group plays an important part in the "noncommutative geometry version" of the Yang–Mills theory (see, for example, [2]) and has energy representations associated with the image of its "abstract Maurer–Cartan cocycle" by a suitable K-cycle over $\mathbb{A}$, even when it is not a Lie group ([12]).

In Section 6 we prove that when $\mathbb{A}$ is a unital involutive ILB-algebra, $\mathcal{U}(\mathbb{A})$ is a regular Fréchet–CBH–Lie subgroup of $\mathrm{GL}(\mathbb{A})$ with Lie algebra $\mathcal{K}(\mathbb{A}) = \{v \in \mathbb{A} | v + v^* = 0\}$.

1. The Class of Initial Involutive ILB-Algebras

(a) In [5], and especially in [6] to which we refer, H. Omori et al. have indicated what we could call the "the inverse limit Banach-machinery" (to make short: ILB machinery). More precisely (see [6], §6):

Definition 1. (1) A system $\{A, A_k, k \in \mathbb{N}\}$ is called an ILB-chain if A_k is a Banach space and $A_{k+1} \subseteq A_k$ for any integer $k \geq 0$, the inclusion being continuous and with dense image, and if $A = \cap_k A_k$ is provided with the inverse limit topology.

(2) A group Γ with unit element e is called a strong ILB–Lie group modelled on the ILB-chain $\{A, A_k, k \in \mathbb{N}\}$ if the following conditions are fulfilled for any integer $k \geq 0$:

(N1): there exists a mapping Exp from A into Γ and for any element k in $\mathbb{N}$ a convex open neighborhood U_k^0 of zero in A_k such that Exp is a bijective mapping from $U_k^0 \cap A$ onto an open neighborhood W_k^0 of e in Γ; we denote by Log the inverse mapping;

(N2): there exists a convex open neighborhood U_k^1 of zero in A_k such that

$$\mathrm{Exp}(U_k^1 \cap A). \mathrm{Exp}(U_k^1 \cap A) \subseteq \mathrm{Exp}(U_k^0 \cap A)$$
$$\text{and} \quad \mathrm{Exp}(U_k^1 \cap A)^{-1} \subseteq \mathrm{Exp}(U_k^0 \cap A);$$

(N3): let h be the mapping from $(U_k^1 \cap A) \times (U_k^1 \cap A)$ into $U_k^0 \cap A$ defined by

$$h(v, w) = \mathrm{Log}(\mathrm{Exp}(v). \mathrm{Exp}(w));$$

then h can be extended to a continuous mapping (denoted by the same notation) from $U_k^1 \times U_k^1$ into U_k^0;

(N4): for any w in $U_k^1 \cap A$ the mapping h_w defined by $h_w(v) = h(v, w)$ is a C^∞-mapping from $U_k^1 \cap A$ into $U_k^0 \cap A$;

(N5): set $\theta(u, v, w) = (dh_w)_v(w)$; for any pair (j, k) of elements in $\mathbb{N}$ the mapping θ can be extended to a C^j-mapping from $A_{k+j} \times (U_k^1 \cap A_{k+j}) \times (U_k^1 \cap A_k)$ into A_k;

(N6): the mapping c from $U_k^1 \cap A$ into $U_k^0 \cap A$ defined by $c(v) = \mathrm{Log}((\mathrm{Exp}\ \ v)^{-1})$ can be extended to a continuous mapping from U_k^1 into U_k^0;

(N7): for any element γ in Γ there exists a neighborhood V_k of zero in A_k such that

$$\gamma^{-1}. \mathrm{Exp}(V_k \cap A). \gamma \subseteq \mathrm{Exp}(U_k^0 \cap A)$$

and such that the mapping $v \mapsto \mathrm{Log}(\gamma^{-1}.\,\mathrm{Exp}\ v.\gamma)$ can be extended to a C^∞-mapping from V_k into U_k^0.

A first important result is that any strong ILB-group is a Fréchet–Lie group ([6], Theorem 6.9).

Let us recall now another important result.

Let Γ be a strong ILB–Lie group with unit $\mathbb{1}$ and with Lie algebra γ; let $C(I;\gamma)$ be the set of continuous mappings from $I=[0,1]$ into γ, and let $C^{1,e}(I;\Gamma)$ be the set of C^1-mappings g from I into Γ such that $g(0)=\mathbb{1}$.

We provide $C(I;\gamma)$ with the uniform convergence topology and with the algebraic structures pointwise defined from that of γ so that it becomes a Fréchet–Lie alebra; likewise, we provide $C^{1,e}(I;\Gamma)$ with the pointwise group-operations and with the C^1-uniform convergence topology so that it becomes a FL–Lie group whose Lie algebra consists of the space $C^{1,0}(I;\gamma)$ of C^1-mappings σ from I into γ such that $s(0)=0$ ([6], Lemma 5.2).

It is proved in ([6], Theorems 4.1, 5.1 and 6.9) that

Lemma 1. *Any strong ILB–Lie group Γ with unit $\mathbb{1}$ and with Lie algebra γ is a regular Fréchet–Lie group in the following sense: for any element s in $C(I;\gamma)$ there exists an element g_s in $C^{1,e}(I;\Gamma)$ satisfying the following equation:*

$$\frac{dg_s}{dt}(t) = s(t).g(t) \quad \textit{with} \quad g(0)=\mathbb{1}$$

Moreover, the assignment $s \mapsto g_s$ is a C^∞-diffeomorphism from $C(I;\gamma)$ onto $C^{1,e}(I;\Gamma)$.

(b) Let us initiate now the ILB-version for unital involutive algebras.

Throughout this paper $\mathbb{K}$ denotes indiscriminately one of the fields of numbers $\mathbb{R}$ or $\mathbb{C}$.

A unital involutive algebra B (always assumed to be associative) over $\mathbb{K}$ being given, as usual we shall denote by $*$ its involution, by $\mathbb{1}$ its unit element, by $\mathrm{GL}(B)$ the group of its invertible elements, and by $\mathcal{U}(B)$ the subgroup of its unitary elements

$$\mathcal{U}(B) = \{u \in \mathrm{GL}(B) | u^*.u = u.u^* = \mathbb{1}\}.$$

By C^*-norm (resp.: C^*-seminorm) on B is meant any algebra norm (resp.: algebra seminorm) $\|\ \|$ on B satisfying $\|v^*.v\| = \|v\|^2$, $v \in B$.

Definition 2. Let $\mathbb{A}$ be a unital involutive topological algebra over $\mathbb{K}$ that we assume to be equipped with a C^*-norm $\|\ \|_0$. We shall say that $\mathbb{A}$ is a unital involutive ILB-algebra if there exists a sequence $\{(\mathbb{A}_k, \|\ \|_k)\}_{k\in\mathbb{N}}$ of unital involutive Banach algebras satisfying the following properties:

(R1): $(\mathbb{A}_0, \|\ \|_0)$ is that C^*-algebra consisting of the completion of $\mathbb{A}_k$ with respect to $\|\ \|_0$;

(R2): for any integer $k \geq 0$, $\mathbb{A}_{k+1}$ is a unital involutive subalgebra of $\mathbb{A}$, and the topology of $\mathbb{A}_{k+1}$ given by $\| \|_{k+1}$ is stronger than the one induced by $(\mathbb{A}_k, \| \|_k)$;

(R3): the system $(\mathbb{A}, \mathbb{A}_k, k \in \mathbb{N})$ is an ILB-chain that we shall call the *ILBA-chain* of $\mathbb{A}$.

We note that a unital involutive ILB-algebra $\mathbb{A}$ as a locally convex topological vector space is a Fréchet vector space. The topology of $\mathbb{A}$ being the superior limit of the topologies on $\mathbb{A}$ induced by $\mathbb{A}_k$, one easily deduces that the multiplication m from $\mathbb{A} \times \mathbb{A}$ into $\mathbb{A}$ and the involution on $\mathbb{A}$ are continuous and then

Lemma 2. *Any unital involutive ILB-algebra is a unital involutive Fréchet algebra.*

(c) Of course any unital C^*-algebra $(\mathbb{A}, \| \|)$ is a unital involutive ILB-algebra by taking the ILBA-chain $\{(\mathbb{A}, \mathbb{A}_k, \| \|_k)\}_{k \in \mathbb{N}}$ with $\mathbb{A}_k = \mathbb{A}$ and $\| \|_k = \| \|$ for any k in $\mathbb{N}$.

A typical example of unital involutive ILB-algebra is the set $\mathbb{A} = C^\infty(M; A)$ of smooth mappings from a smooth compact manifold M into a unital C^*-algebra A, provided with the C^∞-uniform convergence topology (see, for example, [13]) and with involution and algebraic structure pointwise defined from that of A.

In this case, for any $k \geq 1$ the unital involutive Banach algebra $\mathbb{A}_k$ is the algebra $C^k(M; A)$ of mappings of class C^k from M into A provided with the C^k-uniform convergence topology, and $\mathbb{A}_0$ is the unital C^*-algebra of continuous mappings from M into A provided with the uniform convergence topology. More generally, one can easily see that the space $C^\infty(B)$ of smooth sections of a smooth bundle B over M of unital C^*-algebras has a natural structure of unital involutive ILB-algebra.

2. Unital Involutive ILB-Algebras in Noncommutative Geometry

We want to discuss now a version of unital involutive ILB-algebras in the context of noncommutative geometry in which the notion of smooth algebra takes an important place (see, for example, [14]): in noncommutative geometry, by smooth algebra is meant a pair $(\mathbb{A}, \mathbb{A}_0)$ in which $\mathbb{A}_0$ is a unital C^*-algebra and $\mathbb{A}$ a unital involutive Fréchet dense subalgebra. A unital involutive ILB-algebra $\mathbb{A}$ with its ILBA-chain $\{(\mathbb{A}, \mathbb{A}_k, \| \|)\}_{k \in \mathbb{N}}$ being given, it follows from Lemma 2 that the pair $(\mathbb{A}, \mathbb{A}_0)$ is a smooth algebra.

(a) Let us refer to [7] and the reference therein for detailed construction and results described below.

Let $\mathbb{A}$ be a unital involutive algebra over $\mathbb{K}$, and let $\mathcal{D}\,\mathbb{A}$ be the universal algebra generated by elements $p_i(v)$, $i \in \mathbb{N}$, $v \in \mathbb{A}$, which are linear in v and fulfill the following relations: for any k in $\mathbb{N}$, and any pair (v, w) of elements in $\mathbb{A}$:

$$p_k(v, w) = \sum_{i+j=k} p_i(v).p_j(w).$$

$\mathcal{D}\mathbb{A}$ is a $\mathbb{N}$-graded involutive algebra, the degree of $p_i(v)$ being the integer i, and the involution being given by

$$p_i(v)^* = -p_i(v^*);$$

moreover, $\mathcal{D}\mathbb{A}$ carries a natural derivation δ such that

$$\delta(p_{i_1}(v_1)\dots p_{i_m}(v_m)) = \sum_{k=1}^{k=m} (i_k+1)p_{i_1}(v_1)\cdots p_{i_k+1}(v_k)\cdots p_{i_m}(v_m).$$

For any integer $i \geq 0$ we have $p_i(v) = \frac{1}{i!}\delta^i(v)$ so that, identifying p_0 with the identity on $\mathbb{A}$, $\mathcal{D}\mathbb{A}$ is generated by $\mathbb{A}$ and all the $\delta^i(\mathbb{A})$, $i \geq 1$; one obtains then a formal homomorphism e^δ from $\mathbb{A}$ into $\mathcal{D}\mathbb{A}$ by taking

$$e^\delta(v) = \sum_{i=0}^{i=\infty} p_i(v).$$

For any integer $k \geq 0$ let $J_k\mathbb{A}$ be the ideal of $\mathcal{D}\mathbb{A}$ generated by all elements with degree $\geq k$; then e^δ leads to an actual homomorphism of algebras from $\mathbb{A}$ into $\mathcal{D}\mathbb{A}/J_k\mathbb{A}$ which allows one to consider $\mathbb{A}$ as a subalgebra of $\mathcal{D}\mathbb{A}/J_k\mathbb{A}$.

Let us assume moreover that $\mathbb{A}$ is equipped with a C^*-norm $\| \ \|$; for any integer $k \geq 1$ we provide $\mathcal{D}\mathbb{A}/J_k\mathbb{A}$ with the Hausdorff locally convex topological algebra structure by taking the topology given by all algebra seminorms which are bounded by some multiple of $\| \ \|$ on $\mathbb{A}$ and for which the projection onto the subspace of elements with degree k is bounded.

Let us denote by $\mathbb{A}^{0,k}$ the completion of $e^\delta(\mathbb{A})$ in $\mathcal{D}\mathbb{A}/J_k\mathbb{A}$, by $\mathbb{A}^0_k$ the unital involutive topological algebra $\mathbb{A}^{0,k}/(\mathbb{A}^{0,k} \cap J_1\mathbb{A})$, and by $\mathbb{A}_0$ the completion of $\mathbb{A}$ with respect to the C^*-norm $\| \ \|$. We have the dense inclusions of unital involutive Hausdorff locally convex complete topological algebras containing $\mathbb{A}$:

$$\cdots \subseteq \mathbb{A}^0_{k+1} \subseteq \mathbb{A}^0_k \subseteq \cdots \subseteq \mathbb{A}_0,$$

and δ extends to a continuous derivation sending $\mathbb{A}^0_{k+1}$ into $\mathbb{A}^0_k$, $k \in \mathbb{N}$.

(b) We want now to show how the above construction may be connected with unital involutive ILB-algebras. To see this, let us define recursively a norm $\| \ \|_{(k)}$ on $\mathbb{A}^0_k$ by $\| \ \|_{(0)} = \| \ \|$, and for $k \geq 1$ by

$$\|a\|_{(k)} = \|a\|_{(k-1)} + \|\delta a\|_{(k-1)}, \quad a \in \mathbb{A}^0_k,$$

so that for any integer $n \geq 0$ and any a in $\mathbb{A}_n^0$ one has

$$\|a\|_{(k)} = \sum_{k=0}^{k=n} \frac{n!}{k!(n-k)!} \|\delta^k a\|.$$

Let $\mathbb{A}_k$ be the completion of $\mathbb{A}_k^0$ with respect to the norm $\| \ \|_{(k)}$, $k \geq 1$, and let $\mathbb{A}' = \bigcap_{k \in \mathbb{N}} \mathbb{A}_k$.

Lemma 3. $\mathbb{A}'$ *is a unital involutive ILB-algebra with the ILBA-chain* $(\mathbb{A}', \mathbb{A}_k, k \in \mathbb{N})$.

Proof. By definition $(\mathbb{A}_0, \| \ \|_0 = \| \ \|)$ is a unital C^*-algebra, and then a unital Banach algebra; moreover, without loss of generality we may assume that for any pair (u, v) of elements in $\mathbb{A}_0$ one has $\|u.v\| \leq \|u\|.\|v\|$; of course, we have also $\|u^*\| = \|u\|$.

Let u and v be in $\mathbb{A}_1$. We have

$$\begin{aligned} \|u.v\|_1 = \|u.v\| + \|\delta(u.v)\| &= \|u.v\| + \|\delta u.v + u.\delta v\| \leq \\ &\leq \|u\|.\|v\| + \|\delta u\|.\|v\| + \|u\|.\|\delta v\| \leq \\ &\leq (\|u\| + \|\delta u\|).(\|v\| + \|\delta v\|) \leq \|u\|_1.\|v\|_1 \end{aligned}$$

and $\|u^*\|_1 = \|u^*\| + \|\delta u^*\| = \|u^*\| + \|-(\delta u)^*\| = \|u\| + \|\delta u\| = \|u\|_1$, which proves that $(\mathbb{A}_1, \| \ \|_1)$ is a unital involutive Banach algebra; similarly, by induction on k, one proves that for any integer $k \geq 0$ and for any pair (u, v) of elements in $\mathbb{A}_k$, one has

$$\|u.v\|_{(k)} \leq \|u\|_{(k)}.\|v\|_{(k)} \quad \text{and} \quad \|u^*\|_{(k)} = \|u\|_{(k)}.$$

The inclusions $\mathbb{A} \cdots \subseteq \mathbb{A}_{k+1}^0 \subseteq \mathbb{A}_k^0 \subseteq \cdots \subseteq \mathbb{A}_0$ imply the inclusions

$$\mathbb{A} \cdots \subseteq \mathbb{A}_{k+1} \subseteq \mathbb{A}_k \subseteq \cdots \subseteq \mathbb{A}_0.$$

Moreover, the equality $\|a\|_{(k)} = \|a\|_{(k-1)} + \|\delta a\|_{(k-1)}$, $a \in \mathbb{A}_k^0$, the continuity of δ, and the density of $\mathbb{A}_{k+1}^0$ in $\mathbb{A}_k^0$ imply that the inclusion of $\mathbb{A}_{k+1}$ in $\mathbb{A}_k$ is continuous and that it is with dense range. The assertion is now obvious. □

3. On Differentiable and Analytic Mappings in Fréchet Spaces; Campbell–Baker–Hausdorff Lie Groups

(a) In infinite-dimensional Hausdorff locally convex vector spaces there are several notions of differentiability and of analyticity that are generally nonequivalent (see, for example, [6,8,9]), although they coincide with the usual ones in the case of finite-dimensional spaces.

The notion of differentiability used here for Fréchet spaces is that initiated by J. Leslie and which has been taken up in [6], §1, to which we refer: E and F being two Fréchet spaces, and U being some nonempty open subset in E, a mapping f from U into F is said to be C^0 on U if it is continuous on U. It is called of class C^k on U, $k \geq 1$, if it is of class C^{k-1} on U, if there exists a continuous mapping $D^k f$ from $U \times E^k$ into F with the following properties:

— for any x in U the mapping $D^k f(x)$ is a symmetric k-linear mapping from E^k into F;

— let ϕ be the mapping defined on E by

$$\phi(v) = f(x+v) - f(x) - (Df(x))(v) - \cdots - (D^k f(x))(v, \cdots, v);$$

then the mapping from $\mathbb{R} \times E$ into F defined by $R(t,v) = \begin{cases} \frac{\phi(tv)}{t^k} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0 \end{cases}$ is continuous on some neighborhood of $(0,0)$ in $\mathbb{R} \times E$.

Although on infinite-dimensional Banach spaces this notion of differentiability is weaker than the usual definition, it fulfills the chain rule and a nice version of Taylor's expansion.

(b) For the corresponding notion of analyticity we refer to [9]. More precisely, let E and F be two Fréchet spaces and let U be a nonempty subset of E; an integer $n \geq 0$ being given, a continuous mapping p from E into F is said to be a continuous homogeneous polynomial with degree n if there exists an n-linear mapping $\widehat{p}$ from the Cartesian product E^n into F such that

$$p(v) = \widehat{p}(v, \ldots, v)$$

for any element v in E. We shall denote by $S(E;F)$ the space of normal series of the form

$$s = \sum_{n=0}^{n=\infty} p_n,$$

where for each integer $n \geq 0$ p_n is a continuous homogeneous polynomial with degree n from E into F. In this context a continuous mapping f from U into F is said to be analytic on U if for any element n in U there exists a formal series

$$s_x = \sum_{n=0}^{n=\infty} p_{n,x}$$

in $S(E,F)$ and an open neighborhood V_x of E such that for any v in V_x

$$f(x+v) = \sum_{n=0}^{n=\infty} p_{n,x}(v).$$

As any Fréchet space is a Banach space, it follows from Theorem 5.2 in [9] that

Lemma 4. *If the formal series $s = \sum p_n$ in $S(E;F)$ is convergent on some nonempty open subset U of E, the mapping $v \mapsto s(v) = \sum p_n(v)$ is analytic on U.*

(c) According to J. Milnor's idea, by Campbell–Baker–Hausdorff Lie group (in short, CBH–Lie group) is meant any analytic Fréchet–Lie group Γ with Lie algebra γ and unit $\mathbb{1}$ such that the exponential mapping exp from γ into Γ fulfills the following property: there exists an open neighborhood U of zero in γ and an open neighborhood W of $\mathbb{1}$ in Γ such that the restriction of exp to U is an analytic diffeomorphism from U onto W so that $(W,\ \log = \exp^{-1})$ is a canonical system of analytic local coordinates for Γ near $\mathbb{1}$ (Cf. [11]). As a consequence, the group law near $\mathbb{1}$ is given by the well-known Campbell–Hausdorff formula.

Let B be a unital Banach algebra with Banach algebra norm denoted by $\| \ \|$ and with unit $\mathbb{1}$, and let $\mathrm{GL}(B)$ be the group of its invertible elements. B has a canonical Banach–Lie algebra structure, the Lie bracket being given by $[v,w] = v.w - w.v$, $(v,w) \in B \times B$; moreover, without loss of generality, we may assume that there exists a constant $C > 0$ such that for any pair (v,w) of elements in $B : \|v.w\| \leq C\|v\|.\|w\|$ and $\|[v,w]\| \leq \|v\|.\|w\|$.

In the next lemma we summarize, without proof, the well-known results about the group $\mathrm{GL}(B)$; we refer to [15], Chap. II, §6 for the proof of these assertions:

Lemma 5. $\mathrm{GL}(B)$ *has a canonical structure of Banach–Lie group with Lie algebra B, its underlying topology being the induced topology of B. Let* Exp *be the mapping from B into* $\mathrm{GL}(B)$ *defined by*

$$\mathrm{Exp}\ v \sum_{n \in N} \frac{v^n}{n!}, \quad v \in B.$$

Then, Exp *is the exponential mapping of* $\mathrm{GL}(B)$*, and there exist an open neighborhood U of zero in B and an open neighborhood W of $\mathbb{1}$ in* $\mathrm{GL}(B)$ *such that* Exp *is an analytic diffeomorphism from U onto W so that* $\mathrm{GL}(B)$ *is an analytic Lie group; we shall denote by* Log *the inverse diffeomorphism. As a consequence* (U, Log) *is a canonical local chart of* $\mathrm{GL}(B)$ *near $\mathbb{1}$, and the group law in* $\mathrm{GL}(B)$ *near $\mathbb{1}$ is given by the Campbell–Hausdorff formula*

$$\mathrm{Exp}(v).\,\mathrm{Exp}(w) = h(v,w) = v + w + \frac{1}{2}[v,w] + \\ + \frac{1}{12}([v,[v,w]] - [w,[v,w]]) + \cdots .$$

We observe in particular that by Lemma 5 $\mathrm{GL}(B)$ is a CBH–Lie group for any unital Banach algebra. In contrast, it is known that the Fréchet–Lie group $\mathrm{Diff}\,(M)$ of all diffeomorphisms of a compact smooth manifold is not a CBH–Lie group (see, for example, [11], §9).

4. The Lie Group GL($\mathbb{A}$), $\mathbb{A}$ Being a Unital Involutive ILB-Algebra

(a) Let $\{\mathbb{A}, (\mathbb{A}_k, \| \|_k),\ k \in \mathbb{N}\}$ be the ILBA-chain of a unital involute ILB-algebra $\mathbb{A}$. The inverse limit topology on $\mathbb{A}$ is given by all the algebra norms $\| \|_k$ restricted to $\mathbb{A}$; taking into account Lemma 5 the group $\mathrm{GL}(\mathbb{A})$ appears as the inverse limit of the analytic Banach–Lie groups $\mathrm{GL}(\mathbb{A}_k)$, $k \in \mathbb{N}$, and, moreover, Exp_k denoting the exponential mapping of $\mathbb{A}_k$, it is clear that its restriction to $\mathbb{A}$ (still denoted by Exp) maps $\mathbb{A}$ into $\mathrm{GL}(\mathbb{A})$. Taking into account this observation one easily deduces that

Lemma 6. *The mapping* Exp *from* $\mathbb{A}$ *into* $\mathrm{GL}(\mathbb{A})$ *is a local homeomophism, i.e., restricts to a homeomorphism from some open neighborhood* U *of zero in* $\mathbb{A}$ *onto some open neighborhood of* $\mathbb{1}$ *in* $\mathrm{GL}(\mathbb{A})$.

For any u in $\mathrm{GL}(\mathbb{A})$ let L_u and R_u be respectively the left and the right multiplication by u in $\mathbb{A}$ so that $L_u(v) = u.v$ and $R_u(v) = v.u$ for any v in $\mathbb{A}$; we shall denote by $\mathrm{Ad}\, u$ the automorphism $L_u \circ R_{u^{-1}}$ so that $\mathrm{Ad}\, u(v) = u.v.u^{-1}$ for any v in $\mathbb{A}$.

Lemma 7. *For any* u *in* $\mathrm{GL}(\mathbb{A})$ *the mappings* L_u, R_u, *and* $\mathrm{Ad}\, u$ *are analytic* $\mathbb{K}$*-linear automorphisms of* $\mathbb{A}$.

Proof. For any u in $\mathrm{GL}(\mathbb{A})$, L_u, R_u, and $\mathrm{Ad}\, u$ are clearly algebraic $\mathbb{K}$-linear automorphisms of $\mathbb{A}$; moreover, the equalities $(L_u)^{-1} = L_{u^{-1}}$, $(R_u)^{-1} = R_{u^{-1}}$, and $(\mathrm{Ad}\, u)^{-1} = \mathrm{Ad}\, u^{-1}$ show that it suffices to prove the analyticity of the mappings L_u and R_u.

A trivial computation shows that for any v and any w in $\mathbb{A}$ the corresponding Câteaux derivative is given by

$$(L_u)'_v(w) = \lim_{t\to 0} \frac{1}{t}(L_u(v + tw) - L_u(v)) = u.w = L_u(w);$$

the continuity of the multiplication and the $\mathbb{K}$-linearity of L_u imply then that L_u, as the mapping from $\mathbb{A}$ into itself, is analytic; the proof is similar for R_u. $\square$

Theorem 1. *Let* $\mathbb{A}$ *be a unital involutive ILB-algebra; the group* $\mathrm{GL}(\mathbb{A})$ *has a canonical structure of regular analytic Fréchet–CBH–Lie group, with Lie algebra* $\mathbb{A}$, *with exponential mapping* Exp *defined by*

$$\mathrm{Exp}\, v = \sum_{n\in\mathbb{N}} \frac{v^n}{n!}, \quad v \in \mathbb{A},$$

and with adjoint representation Ad *of* $\mathrm{GL}(\mathbb{A})$ *into* $\mathbb{A}$ *such that for any* u *in* $\mathrm{GL}(\mathbb{A})$

$$\mathrm{Ad}(u) = \mathrm{Ad}\, u.$$

Proof. It is divided into four steps.

(1) The notations being as in Lemma 5, let W° be a symmetric open neighhborhood of $\mathbb{1}$ in $\mathrm{GL}(\mathbb{A})$ contained in W and fulfilling $W^\circ.W^\circ \subseteq W$, and let U° be the open neighborhood of zero in A defined by $U^\circ = \mathrm{Log}(W^\circ)$.

For any integer k the family $\{v^n\}_{n\in\mathbb{N}}$ is summable in $\mathbb{A}_k$; one easily deduces from Lemma 4 that one can find an open neighborhood $U^\infty \subseteq U^\circ$ of zero in $\mathbb{A}$ such that for any element v in U^∞ the element $\mathbb{1}-v$ is invertible with

$$(\mathbb{1}-v)^{-1} = \sum_{n\in\mathbb{N}} v^n,$$

so that one can find an open neighborhood $W^\infty \subseteq W^\circ$ of $\mathbb{1}$ in $\mathrm{GL}(\mathbb{A})$ such that the “inverse” mapping $u \mapsto u^{-1}$ is analytic in W^∞; it siffices to take $W^\infty = \mathrm{Exp}(U^\infty)$. By Lemma 6 the mapping Log is a homeomorphism from W^∞ onto U^∞. It follows from Lemma 7 that for any element u in $\mathrm{GL}(\mathbb{A})$ the set $u.W^\infty$ is an open neighborhood of u in $\mathrm{GL}(\mathbb{A})$ and $\mathrm{Log} \circ L_{u^{-1}}$ is a homeomorphism from $u.W^\infty$ onto U^∞. As a consequence $\mathrm{GL}(\mathbb{A})$ is an open subset of $\mathbb{A}$.

Moreover, the families

$$\left\{\frac{v^n}{n!}\right\}_{n\in\mathbb{N}} \quad \text{and} \quad \left\{(-1)^n\frac{(\mathbb{1}-u)^n}{n}\right\}_{n\geq 1}$$

are summable and the corresponding series which converge: the first to $\mathrm{Exp}\, v$ in some neighborhood of ero in $\mathbb{A}$, and the second to $\mathrm{Log}\, u$ in some neighborhood of $\mathbb{1}$ in $\mathrm{GL}(\mathbb{A})$. It follows then from Lemmas 4 and 5 that there exist an open neighborhood Ω of zero in $\mathbb{A}$ and an open neighborhood Λ of $\mathbb{1}$ in $\mathrm{GL}(\mathbb{A})$ such that the mapping

$$\mathrm{Exp} : v \mapsto \mathrm{Exp}\, v = \sum_{n\in\mathbb{N}} \frac{v^n}{n!}$$

is an analytic diffeomorphism from Ω onto Λ, the inverse diffeomorphism Log from L onto W being given by

$$\log u = \sum_{n\geq 1}(-1)^n\frac{(\mathbb{1}-u)^n}{n}.$$

At this stage we have proved that $\mathrm{GL}(\mathbb{A})$ is an analytic manifold modelled on the Fréchet space $\mathbb{A}$, the underlying topology being the topology induced from $\mathbb{A}$, and that for any element u in $\mathrm{GL}(\mathbb{A})$ the pair

$$(u.\Lambda, \log \circ L_{u^{-1}})$$

is an analytic chart near u.

(2) To prove that $\mathrm{GL}(\mathbb{A})$ is an analytic Lie group modelled on $\mathbb{A}$ it remains to prove the analyticity of the mappings $u - u^{-1}$, $u \in \mathrm{GL}(\mathbb{A})$, and of the multiplication $m : (u, u') \mapsto u.u'$ on $\mathrm{GL}(\mathbb{A}) \times \mathrm{GL}(\mathbb{A})$.

Using the analyticity of the mappings $u \mapsto u^{-1}$ on some open neighborhood of $\mathbb{1}$ in $\mathrm{GL}(\mathbb{A})$ and the analyticity of the mappings $L_{u'}$, $u' \in \mathrm{GL}(\mathbb{A})$, implies clearly the analyticity of the mapping $u \mapsto u^{-1}$ on the whole manifold $\mathrm{GL}(\mathbb{A})$.

Let u, u' be elements in $\mathrm{GL}(\mathbb{A})$ and let v, v' be elements in $\mathbb{A}$; an easy computation of the Gâteaux derivative $\mathrm{Dm}(u, u')_{v,v'}$ of m at the point (u, u') following the vector (v, v') gives

$$\mathrm{Dm}(u, u')_{v,v'} = u.v' + v.u' = R_{u'}(v) + L_u(v'),$$

from which one deduces that for any (u, u') in $\mathrm{GL}(\mathbb{A}) \times \mathrm{GL}(\mathbb{A})$ the linear mapping

$$\mathrm{Dm}(u, u') : (v, v') \mapsto \mathrm{Dm}(u, u')_{v,v'} = R_{u'}(v) + L_u(v')$$

is a continuous endomorphism of $\mathbb{A} \times \mathbb{A}$; according to the type of analyticity used here it follows that m is analytic.

(3) According to the general theory of finite or infinite-dimensional manifold the tangent space $T_1 \mathrm{GL}(\mathbb{A})$ of $\mathrm{GL}(\mathbb{A})$ at the point $\mathbb{1}$ is the set of equivalent classes of parametrized smooth paths through $\mathbb{1}$ which are defined as follows: let $P(\mathrm{GL}(\mathbb{A}))$ be the set of smooth mappings p from an open neighborhood of zero in $\mathbb{R}$ with values in $\mathrm{GL}(\mathbb{A})$ such that $p(0) = 1$; then two elements p, q will be equivalent if the following equality is fulfilled:

$$\frac{d}{dt}\big(\operatorname{Log} p(t)\big)_{t=0} = \frac{d}{dt}\big(\operatorname{Log} q(t)\big)_{t=0}.$$

An easy computation shows that $\frac{d}{dt}\big(\operatorname{Log} p(t)\big)_{t=0} = \frac{dp}{dt}(0)$, which belongs to $\mathbb{A}$.

Conversely, for any v in $\mathbb{A}$, let us associate the smooth mapping from $\mathbb{R}$ into $\mathrm{GL}(\mathbb{A})$ defined by

$$p_v(t) = \operatorname{Exp}(tv);$$

a trivial computation gives $\frac{d}{dt}\big(\operatorname{Log} p(t)\big)_{t=0} = v$ so that $T_1 \mathrm{GL}(\mathbb{A}) \cong \mathbb{A}$.

It remains to compute the Lie bracket of $\mathbb{A}$ as Lie algebra of $\mathrm{GL}(\mathbb{A})$. To do this, we have to observe first of all that the adjoint representation Ad of $\mathrm{GL}(\mathbb{A})$ into its Lie algebra $\mathbb{A}$ is clearly given by $\mathrm{Ad}(u) = \mathrm{Ad}\, u = L_u \circ R_{u^{-1}}$, $u \in \mathrm{GL}(\mathbb{A})$, with $\mathrm{Ad}\, u$ analytic by Lemma 7.

Let v, w be elements in $\mathbb{A}$; according to the general theory of Lie groups the Lie bracket $[v, w]$ must be equal to the image of w by the first derivative at the point $u = \mathbb{1}$ in the direction v of the smooth mapping $\eta : u \mapsto \operatorname{Ad} u(w) - w = u.w.u^{-1} - w$ from $\mathrm{GL}(\mathbb{A})$ into $\mathbb{A}$; an easy computation gives

$$[v,w] = h'(\mathbb{1},v)(w) = \lim_{t\to 0}\Big(\frac{(\mathbb{1}+tv).w.(\mathbb{1}+tv)^{-1} - w}{t}\Big) =$$
$$= \lim_{t\to 0}\Big(\frac{1}{t}\Big((\mathbb{1}+tv).w.\Big(\sum_{n\geq 0}(-1)^n t^n v^n\Big) - w\Big)\Big) =$$
$$= \lim_{t\to 0}\Big(\frac{1}{t}\Big((\mathbb{1}+tv).w.\Big(\mathbb{1} + t\sum_{n\geq 1}(-1)^n t^{n-1} v^n\Big) - w\Big)\Big) = v.w - w.v,$$

so that $\mathbb{A}$ is the Lie algebra of $\mathrm{GL}(\mathbb{A})$ provided with its canonical Lie bracket.

(4) As for integer $k \geq 0$ the Banach Lie group $\mathrm{GL}(\mathbb{A}_k)$ is a CBH–Lie group, and its Campbell–Hausdorff series is convergent (and then is analytic by Lemma 4) on some neighborhood of $(0,0)$ in $\mathbb{A}_k \times \mathbb{A}_k$; one easily deduces that the formal Campbell–Hausdorff series, as element of $S(\mathbb{A}\times\mathbb{A};\mathbb{A})$, is convergent on some open neighborhood Ξ of $(0,0)$ in $\mathbb{A}\times\mathbb{A}$.

$\mathbb{A}$ being Fréchet space, it follows that $\mathbb{A}\times\mathbb{A}$ is a Baire space, and then the analyticity of the Campbell–Hausdorff function of $\mathrm{GL}(\mathbb{A})$ on Ξ is fulfilled by Lemma 4. As a consequence $\mathrm{GL}(\mathbb{A})$ is an analytic Fréchet–CBH–Lie group.

The proof that $\mathrm{GL}(\mathbb{A})$ fulfills the properties (N1)–(N7) described in Definition 1 is now straightforward and easy so that $\mathrm{GL}(\mathbb{A})$ is also a strong ILB–Lie group, and then a regular Fréchet–Lie group by Lemma 1. We summarize these properties by saying that $\mathrm{GL}(\mathbb{A})$ is a regular analytic Fréchet–CBH–Lie group. □

5. Lie's Second Fundamental Theorem for GL($\mathbb{A}$)

(a) For any real or complex finite-dimensional Lie group G with Lie algebra $\mathcal{G}$ we have the following result known as Lie's second fundamental theorem:

"Let $\mathcal{H}$ be a Lie subalgebra of $\mathcal{G}$; then $\mathcal{H}$ is integrable, that is to say, there exists a unique connected Lie group H with Lie algebra $\mathcal{H}$ which can be immersed as Lie subgroup of G."

A generalization of this assertion in the infinite-dimensional context forces one, first of all, to consider only closed Lie subalgebras: for closed Lie subalgebras Lie's second fundamental theorem remains true for Banach–Lie groups; we refer to [10] for a detailed discussion of this theorem in the infinite-dimensional case.

Another point is that the property for a mapping M from an infinite-dimensional manifold M into another infinite-dimensional manifond N to be an immersion is a very strong property; a weaker property for π is to be an embedding, that is to say, a C^1-mapping on M such that for any x in M the differential $(d\pi)_x$ is one-to-one.

Moreover, even with a stronger hypothesis (by considering only closed Lie subalgebras) and with a weaker result (embedding Lie subgroups instead of immersed Lie subgroups), the proof for an extension of Lie's second theorem meets, for non-Banach Lie groups, with major difficulties or obstructions due to the fact that there is no "nice" theory of differential equations, and a fortiori no "good" Frobenius theorem for the most part of non-Banach Hausdorff locally convex topological vector spaces in spite of some efforts using the bornological machinery (see, for example, [10]).

(b) **Theorem 2.** *Let* $\mathbb{A}$ *be a unital involutive ILB-algebra provided with its Lie algebra structure of the analytic CBH–Lie group* $\mathrm{GL}(\mathbb{A})$*. Any closed Lie subalgebra* $\mathcal{H}$ *of* $\mathbb{A}$ *is the Lie algebra of a unique connected CBH–Lie group which can be embedded as a Lie subgroup* H *in* $\mathrm{GL}(\mathbb{A})$*.*

Proof. (1) Let $\mathbb{A}$ be a unital involutive algebra. It follows from Theorem 1 that one can find an open neighborhood Λ of $\mathbb{1}$ in the analytic CBH–Lie group $\mathrm{GL}(\mathbb{A})$ such that Exp is an analytic diffeomorphism from Ω onto Λ so that Log is an analytic diffeomorphism from Λ onto Ω; we consider Λ provided with the analytic manifold structure induced by that of $\mathrm{GL}(\mathbb{A})$ (or equivalently by that of $\mathbb{A}$!).

Let $\mathcal{H}$ be a closed Lie subalgebra of the Lie algebra $\mathbb{A}$ so that, as a topological vector subspace of $\mathbb{A}$, it is a Fréchet space. $\Omega \cap \mathcal{H}$ is an open neighborhood of zero on $\mathcal{H}$ that we equip with the induced topology of $\mathcal{H}$. Let us consider now the set

$$H_1 = \mathrm{Exp}(\Omega \cap \mathcal{H})$$

provided with the topology carried from that of $\Omega \cap \mathcal{H}$ by Exp; owing to the analyticity of Exp one easily deduces that H_1 is an analytic manifold modelled on the Fréchet space $\mathcal{H}$ and regularly embedded in the analytic manifold Λ. Moreover, the mapping

$$(a, a') \mapsto a.a' = \mathrm{Exp}(\mathrm{Log}\, a).\, \mathrm{Exp}(\mathrm{Log}\, a') = \mathrm{Exp}\left(h(\mathrm{Log}\, a, \mathrm{Log}\, a')\right),$$

where h denotes the analytic mapping defined by the Campbell–Hausdorff formula, is analytic (with respect to the analytic structure of $H_1 \times H_1$ on some nonempty open neighborhood of $(\mathbb{1}, \mathbb{1})$ in $H_1 \times H_1$.

(2) Let u be any element in $\mathrm{GL}(\mathbb{A})$; the analyticity of the mappings L_u implies that $u.H_1$ is an analytic manifold regularly embedded in $u.\Lambda$ and analytically diffeomorphic to H_1; the mapping $\widehat{u}$ defined for any element $u.a$ in $u.H_1$ by $\widehat{u}(u.a) = \mathrm{Log}\ a$ is an analytic diffeomorphism from the open neighborhood $u.(\Lambda \cap H_1)$ onto $\Omega \cap \mathcal{H}$.

Let $\mathcal{L}$ be the union over $\mathrm{GL}(\mathbb{A})$ of all the analytic manifolds $u.\mathcal{H}$, $u \in \mathrm{GL}(\mathbb{A})$. $\mathcal{L}$ is both an analytic vector bundle over $\mathrm{GL}(\mathbb{A})$ with fibers of type $\mathcal{H}$, and the invariant distribution of left translation of the Lie subalgebra $\mathcal{H}$.

Let us prove that $u.H_1$, when provided with the chart $\{u.(\Lambda \cap H_1); \widehat{u}\}$, is an integral manifold of $\mathcal{L}$ in a neighborhood of u.

Let $a \in u.H_1 \cap H_1$ so that $b = u^{-1}.a$ belongs to H_1; one can find an open neighborhood V of $\mathbb{1}$ in H_1 such that $b.V$ is an open neighborhood of b in H_1 and $a.V$ an open neighborhood of a in H_1. It follows that $u.(b.V) = u.(u^{-1}.a.V) = a.V$, which is an open subset in $u.H_1$ and in H_1, is then an open neighborhood of a in $u.H_1 \cap H_1$.

(3) We have then proved that the $u.H_1$, $u \in \mathrm{GL}(\mathbb{A})$, are integral manifolds and that the family of open subsets of the $u.H_1$ provides a basis for a topology Θ on $\mathrm{GL}(\mathbb{A})$ which is stronger than the induced topology of that of $\mathbb{A}$ (which is the underlying topology of its structure of analytic Lie group).

Let H be the connected component of $\mathbb{1}$ in $\mathrm{GL}(\mathbb{A})$ with respect to Θ. It is clear that H contains H_1; moreover, the regularity of the embedding of H_1 in $\mathrm{GL}(\mathbb{A})$ implies that H is an analytic manifold. It follows that H is an integral manifold of $\mathcal{L}$ and that $\{u.H\}_{u\in \mathrm{GL}(\mathbb{A})}$ is an analytic foliation of $\mathcal{L}$. As a result it follows that H is a subgroup of $\mathrm{GL}(\mathbb{A})$.

(4) We have now to prove that H is a CBH–Lie group.

Let (u°, v°) be an element in $H \times H$, let $w^\circ := u^\circ.v^{\circ -1}$, and let $i(v^\circ)$ be the inner automorphism of $\mathrm{GL}(\mathbb{A})$ defined by $i(v^\circ)(u) = v^\circ.u.v^{\circ -1}$ which is analytic on $\mathrm{GL}(\mathbb{A})$, since it is the restriction of $\mathrm{Ad}\, v^\circ$ which is analytic on $\mathbb{A}$ by Lemma 7.

The restriction $j(v^\circ)$ of $i(v^\circ)$ to H is an inner automorphism of H and we can find an open neighborhood Y of $\mathbb{1}$ in H such that the set $j(v^\circ)(Y) = Y^*$ is contained in $\Lambda \cap H$ so that one easily deduces that $j(v^\circ)$ is analytic (in the sense of the analytic structure of the manifold H) in the neighborhood Y^* of $\mathbb{1}$ in H.

Let us point out that for any (u, v) sufficiently near (u°, v°) in H so that $u^{\circ -1}.u.v^{-1}.v^\circ$ lies in Y^*, easy computation shows that $u.v^{-1} = w^\circ.j(v^\circ)(u^{\circ -1}.u.v^{-1}.v^\circ)$.

Taking into account the analyticity of the left multiplication in H_1 (and then in H) and of $j(v^\circ)$ in Y^*, one deduces that the mapping μ from $H \times H$ into H defined by $\mu(u, v) = u.v^{-1}$ is analytic in some neighborhood of (u°, v°) for any (u°, v°) in $H \times H$. It follows that H is an analytic Lie group embedding in $\mathrm{GL}(\mathbb{A})$ and its Lie algebra is $\mathcal{H}$. As the exponential mapping of H is clearly the restriction of the exponential mapping of $\mathrm{GL}(\mathbb{A})$, one easily deduces that H is a CBH–Lie group.

(5) It remains now to prove the unicity of H up to an isomorphism of Lie groups. Let H' be a connected CBH–Lie group embedded in $\mathrm{GL}(\mathbb{A})$ and with Lie algebra $\mathcal{H}$; we denote by s the embedding from H' into $\mathrm{GL}(\mathbb{A})$ and by s^* the differential mapping of s at the unit which is then a smooth isomorphism of Lie algebras from the tangent space T_1H' of H' at the unit onto the Lie algebra $\mathcal{H}$.

The unicity of a one-parameter subgroup having the same velocity vector at $t = 0$ implies that near the unit one has

$$s(\operatorname{Exp} u) = \operatorname{Exp}(s^*(u)),$$

which proves that s is a local Lie group isomorphism from H' into H.

We achieve the proof by observing that, H and H' being connected Lie groups, we can find open neighborhoods W and W' of the unit in H and H', respectively, so that s maps diffeomorphically W onto W' such that H is the union of all the $W.s.W = W^n$ and H' is the union of all the $W'.\cdots.W' = {W'}^n$, $n \in N$; it follows then that $s(H') = H$ so that H' is actually a CBH–Lie group isomorphic to H via s. □

6. The Fréchet–CBH–Lie Group of Unitary Elements of a Unital Involutive ILB-Algebra

(a) Let B be any unital topological involutive algebra over $\mathbb{K}$, let $\mathrm{GL}(B)$ be the group of its invertible elements, and let $\mathcal{U}(B) := \{u \in B | u^*.u = u.u^* = \mathbb{1}\}$ be the subgroup of its unitary elements.

Lemma 8. *Provided with the induced topology of B the group $\mathcal{U}(B)$ is a topological group closed in B.*

Proof. Let f' and f'' be the mappings from B into B respectively defined for any v in B by

$$f'(v) = v.v^* \quad \text{and} \quad f''(v) = v^*.v.$$

Due to the continuity of the involution and of the multiplication, f' and f'' are continuous mappings; it follows that $\mathcal{U}(B) = {f'}^{-1}(\{\mathbb{1}\}) \cap {f''}^{-1}(\{\mathbb{1}\})$ is closed in B. Moreover, the continuity of the multiplication in B implies its continuity in $\mathcal{U}(B)$, and the continuity of the involution on B implies the continuity of $u \mapsto u^{-1} = u^*$ on $\mathcal{U}(B)$ so that $\mathcal{U}(B)$ is a topological group. □

In the particular case where B is a unital involutive Banach algebra, $\mathrm{GL}(B)$ is a CBH–Banach–Lie group and it is well known that it induces on its subgroup $\mathcal{U}(B)$ a Banach–Lie group structure. Unfortunately, for more general infinite-dimensional Lie groups, due to the lack of local compactness of the underlying space, a closed subgroup of a Lie group is not necessarily a Lie group.

(b) Let $(\mathbb{A}, \mathbb{A}_k\, k \in \mathbb{N})$ be the ILBA-chain of a unital involutive ILB-algebra $\mathbb{A}$, let θ be the involution on $\mathbb{A}$ defined by

$$\theta(v) = -v^*, \quad v \in \mathbb{A},$$

and let us consider its eigenspaces

$$\mathcal{K}(\mathbb{A}) = \big\{v \in \mathbb{A} | \theta(v) = v\big\} \quad \text{and} \quad \mathcal{P}(\mathbb{A}) = \big\{v \in \mathbb{A} | \theta(v) = -v\big\}.$$

Due to the continuity of θ, $\mathcal{K}(\mathbb{A})$ and $\mathcal{P}(\mathbb{A})$ are closed subspaces of $\mathbb{A}$ and we have the direct sum

$$\mathbb{A} = \mathcal{K}(\mathbb{A}) \otimes \mathcal{P}(\mathbb{A}).$$

Let us consider $\mathbb{A}$ with its canonical structure of Lie algebra of $\mathrm{GL}(\mathbb{A})$; for any pair (v, w) of elements of $\mathbb{A}$ one has

$$[v, w]^* = (v.w)^* - (w.v)^* = w^*.v^* - v^*.w^* = -[v^*, w^*].$$

As an obvious consequence of the above discussion, we have the following result:

Lemma 9.
(i) $[\mathcal{K}(\mathbb{A}), \mathcal{K}(\mathbb{A})] \subseteq \mathcal{K}(\mathbb{A})$ *and* $\mathcal{K}(\mathbb{A})$ *is a closed Lie subalgebra of* $\mathbb{A}$*;*
(ii) $[\mathcal{K}(\mathbb{A}), \mathcal{P}(\mathbb{A})] \subseteq \mathcal{P}(\mathbb{A})$*;*
(iii) $[\mathcal{P}(\mathbb{A}), \mathcal{P}(\mathbb{A})] \subseteq \mathcal{K}(\mathbb{A})$.

Remark 1. Let us observe that $[\mathbb{A}, \mathbb{A}] = \mathbb{A}$; by analogy with what happens for finite-dimensional reductive Lie algebras over K, Lemma 5 shows, at least at the algebraic level, a kind of Cartan decomposition of $\mathbb{A}$ with respect to the "Cartan involution" θ.

This statement seems to be all the more justified as we shall see that $\mathcal{K}(\mathbb{A})$ is the Lie algebra of a Lie group which appears as a generalization of compact Lie groups of the type $U(n)$.

Lemma 10. $\mathcal{U}(\mathbb{A})$ *is a closed topological subgroup of the CBH–Lie group* $\mathrm{GL}(\mathbb{A})$ *for the topology induced from that of* $\mathbb{A}$*; moreover, the exponential mapping* Exp *maps the Lie algebra* $\mathcal{K}(\mathbb{A})$ *into* $\mathcal{U}(\mathbb{A})$.

Proof. By Lemma 8 the group $\mathcal{U}(A)$ is a subgroup of $\mathrm{GL}(\mathbb{A})$ which is closed in $\mathbb{A}$ and then in the Lie group $\mathrm{GL}(\mathbb{A})$ which is open in $\mathbb{A}$. Moreover, the continuity of the multiplication in $\mathrm{GL}(\mathbb{A})$ implies its continuity in $\mathcal{U}(\mathbb{A})$, and the continuity of the involution implies the continuity of $u \mapsto u^{-1} = u^*$ on $\mathcal{U}(\mathbb{A})$, and then $\mathcal{U}(\mathbb{A})$ is a closed topological subsgroup of the analytic Lie group $\mathrm{GL}(\mathbb{A})$.

Let us observe that for any element v in $\mathbb{A}$ we have

$$(\operatorname{Exp} v)^* = \Big(\sum_{n\geq 0} \frac{v^n}{n!}\Big)^* = \sum_{n\geq 0} \frac{(v^n)^*}{n!} = \sum_{n\geq 0} \frac{(v^*)^n}{n!} = \operatorname{Exp}(v^*)$$

and then for any element v in $\mathcal{K}(\mathbb{A})$

$$(\operatorname{Exp} v)^*. \operatorname{Exp} v = \operatorname{Exp}(-v). \operatorname{Exp} v = \mathbb{1} = \\ = (\operatorname{Exp} v). \operatorname{Exp}(-v) = (\operatorname{Exp} v).(\operatorname{Exp} v)*,$$

which proves that Exp maps $\mathcal{K}(\mathbb{A})$ into $\mathcal{U}(\mathbb{A})$. $\square$

In the case $\dim(\mathbb{A}) < \infty$, Lemma 10 allows one to claim that $\mathcal{U}(\mathbb{A})$ is a Lie subgroup of the analytic Lie group $\mathrm{GL}(\mathbb{A})$; in the infinite-dimensional case it is not sufficient owing to the lack of local compactness of the underlying space. Fortunately, in our context we have

Theorem 3. *Let $\mathbb{A}$ be a unital involutive ILB algebra. The regular Fréchet–CBH–Lie group* $\mathrm{GL}(\mathbb{A})$ *induces on its closed topological subgroup $\mathcal{U}(\mathbb{A})$ a structure of regular Fréchet–CBH–Lie group with Lie algebra $\mathcal{K}(\mathbb{A})$, the exponential mapping of which is the restriction of* Exp *to $\mathcal{K}(\mathbb{A})$ (which we shall also denote by* Exp*). Moreover, the adjoint representation* Ad *of $\mathcal{U}(\mathbb{A})$ into $\mathcal{K}(\mathbb{A})$ is given for any u in $\mathcal{U}(\mathbb{A})$ by* $\mathrm{Ad}\ u(v) = u.v.u^*$, $v \in \mathcal{K}(\mathbb{A})$.

Proof. (a) Let $(\mathbb{A}, \mathbb{A}_k\ k \in \mathbb{N})$ be the ILBA-chain of the unital involutive ILB-algebra $\mathbb{A}$, and for any k in $\mathbb{N}$ let $\mathcal{U}(\mathbb{A}_k)$ be the group of unitary elements in the unital involutive Banach algebra $\mathbb{A}_k$; one easily sees that $\mathcal{U}(\mathbb{A}_k)$ is a CBH–Banach–Lie subgroup of the Banach–Lie group $\mathrm{GL}(\mathbb{A}_k)$ with Lie algebra $\mathcal{K}(\mathbb{A}_k) = \{v \in \mathbb{A}_k | v + v^* = 0\}$; moreover, as in the proof of Lemma 10, one proves that the exponential mapping from $\mathcal{K}(\mathbb{A}_k)$ into $\mathcal{U}(\mathbb{A}_k)$ is the restriction to $\mathcal{K}(\mathbb{A}_k)$ of the exponential mapping Exp_k from $\mathbb{A}_k$ into $\mathrm{GL}(\mathbb{A}_k)$.

In this context a straightforward check allows one to assert that $\{\mathcal{K}(\mathbb{A}), \mathcal{K}(\mathbb{A}_k), k \in \mathbb{N}\}$ is an ILB-chain and that

$$\mathcal{U}(\mathbb{A}) = \bigcap_{k\in\mathbb{N}} \mathcal{U}(\mathbb{A}_k)$$

is a strong ILB–Lie group, and then a regular Fréchet–Lie subgroup of $\mathrm{GL}(\mathbb{A})$ by Lemma 1.

(b) Let us prove that the tangent space $T_1(\mathcal{U}(\mathbb{A}))$ of $\mathcal{U}(\mathbb{A})$ at $\mathbb{1}$, which is necessarily a vector subspace of the tangent space $\mathbb{A} = T_1(\mathrm{GL}(\mathbb{A}))$ of $\mathrm{GL}(\mathbb{A})$ at $\mathbb{1}$, is exactly $\mathcal{K}(\mathbb{A})$.

According to the general theory of a smooth infinite-dimensional manifold modelled on a Hausdorff locally convex vector space (see, for example, [12], §4), an element of $T_1(\mathcal{U}(\mathbb{A}))$ is an equivalence class of parametrized smooth mappings p from some open neighborhood of zero in $\mathbb{R}$ taking their values in $\mathcal{U}(\mathbb{A})$ and such that $p(0) = 1$. As in the proof of Theorem 1, one easily proves that it is entirely characterized by the value of $\frac{dp}{dt}(0)$.

As such a path p fulfills the equalities $p(t).p(t)^* = 1$ and $\frac{dp^*}{dt}(t) = \left(\frac{dp}{dt}(t)\right)^*$ for all t, one obtains

$$0 = \frac{d}{dt}(p.p^*)(t) = \frac{dp}{dt}(t)p(t)^* + p(t).\left(\frac{dp}{dt}(t)\right)^*$$

for all parameters t so that for $t = 0$

$$\frac{dp}{dt}(0) + \left(\frac{dp}{dt}(0)\right)^* = 0,$$

which proves that $\frac{dp}{dt}(0)$ lies in $\mathcal{K}(\mathbb{A})$, and then, as $\mathcal{U}(\mathbb{A}) \supseteq \mathrm{Exp}(\mathcal{K}(\mathbb{A}))$, one concludes that $T_1(\mathcal{U}(\mathbb{A})) \cong \mathcal{K}(\mathbb{A})$.

The last part of the assertion follows from the fact that the adjoint representation of $\mathcal{U}(\mathbb{A})$ is the restriction to $\mathcal{U}(\mathbb{A})$ of the adjoint restriction of $\mathrm{GL}(\mathbb{A})$ acting on $\mathcal{K}(\mathbb{A})$ and from the fact that for any u in $\mathcal{U}(\mathbb{A})$ and any v in $\mathcal{K}(\mathbb{A}) : \mathrm{Ad}\, u(v) = u.v.u^{-1} = u.v.u^{*}$. □

Remark 2. By Lemma 9, $\mathcal{K}(\mathbb{A})$ is a closed Lie subalgebra of $\mathbb{A}$; it follows from Theorem 2 that $\mathcal{K}(\mathbb{A})$ is the Lie algebra of a connected CBH–Lie subgroup Γ of $\mathrm{GL}(\mathbb{A})$. Due to the unicity of this group one easily deduces that Γ is necessarily the connected component $\mathcal{U}_0(\mathbb{A})$ of the unit in the Lie group $\mathcal{U}(\mathbb{A})$.

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